Chapter 1

Fixed point theorems

One of the most important instrument to treat (nonlinear) problems with the aid of functional analytic methods is the fixed point approach. This approach is an important part of nonlinear (functional-)analysis and is deeply connected to geometric methods of topology. We consider in this chapter the famous theorems of Banach, Brouwer and Schauder. A more detailed description of the fixed point theory can be found for instance in [1, 2, 6, 8, 9] dar.

1.1 The fixed point approach – preliminaries

Consider the following fixed point equation

$$x = F(x) \tag{1.1}$$

Here, F is a mapping from $X \longrightarrow X$. We assume that X is endowed with the metric d. A point $z \in X$ which satisfies $z = F(z)$ is called a **fixed point** of F. Fixed point theorems guarantee the existence and/or uniqueness when F and X satisfy certain additional conditions. A simple example of a mapping F which doesn't posses a fixed point is the translation in a vector space X:

$$F : X \ni x \longmapsto x + x^0 \in X \text{ where } x^0 \neq \theta\,.$$

Many problems of applied mathematics may be formulated as a fixed point equation or reformulated in this way: Linear equations, differential equations, zeroes of gradients, ... The following examples show how the treatment of nonlinear problems may be translated into fixed point problems:

- Determination of zeros of nonlinear functions $g : \mathbb{R} \longrightarrow \mathbb{R}$:

  $$g(x) = 0$$

  There are different possibilities to translate this equation into a fixed point problem:

  $$\begin{aligned} F(x) &:= x - g(x) \text{ \textit{simplest} method} \\ F(x) &:= x - \omega g(x) \text{ linear relaxation with relaxation parameter } \omega > 0 \\ F(x) &:= x - (g'(x))^{-1} g(x) \text{ Newton method} \end{aligned}$$

- Consider the determination of a zero z of a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Clearly, such a zero is a fixed point of the mapping $F : \mathbb{R}^n \ni x \longmapsto x + f(x) \in \mathbb{R}^n$. The formulation of equations as a fixed point equation is possible in various ways. Consider

$$f : \mathbb{R} \ni x \longmapsto x^3 - 13x + 18 \in \mathbb{R}\,.$$

  z is a zero of f iff z is a fixed point of $g_i, i = 1, 2, 3, 4$, where

$$\begin{array}{rclrcl} g_1(x) & := & x^3 - 12x + 18 & g_2(x) & := & \dfrac{x^3+18}{13} \\ g_3(x) & := & (13x-18)^{\frac{1}{3}} & g_4(x) & := & \dfrac{13x-18}{x^2} \end{array}$$

- Ordinary differential equations

$$y' = f(t, y)\,,\ y(t_0) = y^0 \qquad (1.2)$$

  If the righthandside f is continuous on an appropriate domain $Q \subset \mathbb{R} \times \mathbb{R}^n$ then this initial value problem is equivalent to the following integral equation of fixed point character:

$$y(t) = F(y)(t) := y^0 + \int_{y^0}^{t} f(s, y(s))ds$$

- (Nonlinear) partial differential equations:

$$-\Delta u = f(u), \text{ in } \Omega\,,\ u = \theta \text{ in } \partial\Omega$$

  f is given and the solution u has to be found. Under certain conditions one can define the inverse $(-\Delta)^{-1}$ of Δ on spaces of functions u which vanish on $\partial\Omega$. Then we obtain the following fixed point equation:

$$u = F(u) := (-\Delta)^{-1}(f(u))$$

Suppose we have a mapping $F : X \longrightarrow X$ as given above. We want to consider methods to compute such a fixed point. Here are some preliminary remarks. We define

$$F^0 := \mathrm{id}\,,\ F^{n+1} := F \circ F^n\,,\ n \in \mathbb{N}_0\,.$$

Choosing $x \in X$, the sequence $(F^n(x))_{n\in\mathbb{N}}$ is called the **orbit** of x under F. This orbit is the result of the iteration

$$x^{n+1} := F(x^n)\,,\ n \in \mathbb{N}_0; x^0 := x\,. \qquad (1.3)$$

This iteration process is in the focus of the following chapters. The main question which we will analyze is under which assumptions (concerning F and X) the orbit converges.

1.2 Fixed point theorem of Banach

Consider the fixed point equation (1.1) again. The essential step to prove the existence of a solution of this equation via the iteration

$$x^{n+1} := F(x^n)\,,\ n \in \mathbb{N}_0; x^0 := x\,, \tag{1.4}$$

is to prove that the orbit $(F^n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence (for every $x \in X$). The assumption which guarantees this is the assumption that F is Lipschitz continuous with Lipschitz constant c satisfying $c < 1$. This means

$$\text{There exists } c \in [0,1) \text{ such that } d(F(x),F(y)) \le c d(x,y) \text{ for all } x,y \in X. \tag{1.5}$$

If (1.5) is satisfyied, F is called a **contraction**. Notice that c^n is the Lipschitz constant of F^n.

Lemma 1.1. *Let* (X,d) *be a metric space and let* $F : X \longrightarrow X$ *be a contraction. If* $c \in [0,1)$ *is the Lipschitz constant of* F, *then we have*

$$d(x,y) \le \frac{1}{1-c}(d(x,F(x)) + d(y,F(y)))\,,\ x,y \in X. \tag{1.6}$$

Proof:
Let $x,y \in X$. Using the triangle inequality

$$d(x,y) \le d(x,F(x)) + d(F(x),F(y)) + d(F(y),y)$$

and using $d(F(x),F(y)) \le c d(x,y)$ the result follows. ■

(1.6) is a very strange inequality: it says that we can estimate how far apart any two points x,y are just from knowing how far x is from its image $F(x)$ and how far y is from its image $F(y)$. A first application is

Corollary 1.2. *Let* (X,d) *be a metric space and let* $F : X \longrightarrow X$ *be a contraction. Then* F *has at most one fixed point.*

Proof:
Suppose that x,y are fixed poits of F. Then $d(x,F(x)), d(y,F(y))$ are zero, so by the inequality (1.6) $d(x,y)$ is zero too. ■

Lemma 1.3. *Let* (X,d) *be a metric space and let* $F : X \longrightarrow X$ *be a contraction. Then the orbit* $(F^n(x))_{n\in\mathbb{N}_0}$ *is a Cauchy sequence.*

Proof:
Let $c \in [0,1)$ be the Lipschitz constant of F. From the inequality (1.6) we read off

$$d(F^n(x),F^m(x)) \le \frac{1}{1-c}(d(F^n(x),F^{n+1}(x)) + d(F^m(x),F^{m+1}(x)))\,,\ m,n \in \mathbb{N}.$$

Recalling that c^k is the Lipschitz constant of F^k we get

$$d(F^n(x),F^m(x)) \le \frac{c^n + c^m}{1-c} d(x,F(x))\,,\ m,n \in \mathbb{N}.$$

Since $\lim_k c^k = 0$ we obtain $\lim_{m,n\to\infty} d(F^n(x),F^m(x)) = 0$. ■

Theorem 1.4 (Contraction Theorem, Banach 1922). *Let* (X, d) *be a complete metric space and let* $F : X \longrightarrow X$ *be a contraction. If* $c \in [0, 1)$ *is the Lipschitz constant of* F, *then the following assertions hold:*

(a) There exists a uniquely determined fixed point z *of* F.

(b) The iteration

$$x^{n+1} := F(x^n)\,,\ n \in \mathbb{N}_0\,,\ x^0 := x\,, \tag{1.7}$$

defines a sequence $(x^n)_{n\in\mathbb{N}_0}$ *which converges to* z *for all* $x \in X$.

(c) $d(x^n, z) \leq \frac{c^n}{1-c}\, d(F(x), x)\,,\ n \in \mathbb{N}_0\,.$

(d) $d(x^n, z) \leq \frac{c}{1-c}\, d(x^n, x^{n-1})\,,\ n \in \mathbb{N}\,.$

(e) $d(x^n, z) \leq c\, d(x^{n-1}, z)\,,\ n \in \mathbb{N}\,.$

Proof:
Ad $(a), (b)$
The uniqueness of a fixed point follows from Corollary 1.2. Let $x \in X$. From Lemma (1.3) we obtain that the orbit $(F^n(x))_{n\in\mathbb{N}_0}$ converges to a point z since (X, d) is complete. Since F is continuous due the fact that F is a contraction – see (1.5) – z is a fixed point.
Ad (c)
Let $x \in X$. We may prove with the help of (1.5) inductively

$$d(F^{n+1}(x), F^n(x)) \leq c^n d(F(x), x)\,. \tag{1.8}$$

Then with (1.8)

$$d(F^m(x), F^n(x)) \ \leq\ \sum_{j=n+1}^{m} d(F^j(x), F^{j-1}(x)) \ \leq\ \sum_{j=n+1}^{m} c^{j-1} d(F(x), x) \tag{1.9}$$

$$\leq\ c^n \frac{1 - c^{m-n}}{1-c} d(F(x), x)\,,\ m \geq n\,. \tag{1.10}$$

Since $c \in [0, 1)$ we obtain the assertion (c) by going to the limit $m \to \infty$ in (1.9),(1.10). We obtain from (1.8)

$$d(F^m(x), F(x)) \leq c \frac{1 - c^{m-1}}{1-c} d(F^n(x), F^{n-1}(x))\,,\ m \in \mathbb{N}\,. \tag{1.11}$$

$(d), (e)$ are simple consequences of these inequalities. ∎

Theorem 1.5 (Banach's fixed point theorem). *Let* $(X, \|\cdot\|)$ *be a Banach space, let* V *be a closed subset of* X *and let* $F : V \longrightarrow V$ *be a contraction, that is*

$$\exists\, c \in [0, 1)\ \forall x, y \in V\, (d(F(x), F(y)) \leq c d(x, y))\,. \tag{1.12}$$

Then the following statements hold:

a) There exists a uniquely determined fixed point z *of* F *in* V.

b) The successive iteration

$$x^{n+1} := F(x^n) = F^{n+1}(x)\,,\ n \in \mathbb{N}_0\,,\ x^0 = x\,, \tag{1.13}$$

defines a sequence $(x^n)_{n\in\mathbb{N}_0}$ *which converges to* z *for all* $x \in V$.

c) $\|x^n - z\| \leq \frac{c^n}{1-c}\,\|F(x) - x\|\,,\ n \in \mathbb{N}$.

(d) $\|x^n - z\| \leq \frac{c}{1-c}\,\|x^n - x^{n-1}\|\,,\ n \in \mathbb{N}$.

(e) $\|x^n - z\| \leq c\,\|x^{n-1} - z\|\,,\ n \in \mathbb{N}$.

Proof:
Consider the metric space (V, d) with $d(x, y) := \|x-y\|\,,\ x, y \in V$. This space is complete due to the closedness of V. Now apply Theorem 1.4. ■

Banach's fixed point theorem is a „perfect theorem". It formulates an easy to check assumption under which we have existence and uniqueness and computatibilty of a fixed point is guarateed. Moreover, it presents the order of the convergence of the approximating sequence and gives hints to stop the iteration in finite steps; see below.

Remark 1.6. *The application of the fixed point theorem of Banach to an initial value problem for ordinary differential equations leads to an existence and uniqueness result, the so called* **Picard–Lindelöff theorem***; see [3]. The assumption we need to apply Theorem 1.5 is that the righthand side* f *in* (1.2) *is continuous and Lipschitz continuous with respect to* y *uniformly in* t. □

In many applications, the mapping F in a fixed point equation is dependent on a parameter $p \in P$:

$$x = F_p(x)\ ,\ x \in X, p \in P\,. \tag{1.14}$$

In this situation we have

Corollary 1.7. *Let* (X, d) *be a complete metric space, let* (P, d') *be a metric space and let* $F_p : X \longrightarrow X$ *be a contraction for each* $p \in P$. *Let* $c_p \in [0, 1)$ *be the Lipschitz constant of* F_p. *We assume* $c_p \leq c < 1, p \in P$. *Moreover we assume, that there exists* $p_0 \in P$ *with* $\lim_{p\to p_0} F_p(x) = F_{p_0}(x)$ *for all* $x \in X$. *Then the equation* (1.14) *posesses for each* $p \in P$ *a uniquely determined solution* $x_p \in X$ *and we have* $\lim_{p\to p_0} x_p = x_{p_0}$.

Proof:
Let $x_p \in P$ be the uniquely determined solution of (1.14). This solution exists due to Theorem 1.4. Then

$$\begin{aligned} d(x_p, x_{p_0}) &= d(F_p(x_p), F_{p_0}(x_{p_0})) \\ &\leq d(F_p(x_p), F_p(x_{p_0})) + d(F_p(x_{p_0}), F_{p_0}(x_{p_0})) \\ &\leq c\,d(x_p, x_{p_0}) + d(F_p(x_{p_0}), F_{p_0}(x_{p_0})) \end{aligned}$$

and we conclude

$$d(x_p, x_{p_0}) \leq \frac{1}{1-c}\,d(F_p(x_{p_0}), F_{p_0}(x_{p_0}))\,.$$

By taking the limit $p \to p_0$ we obtain $\lim_{p\to p_0} x_p = x_{p_0}$. ■

Example 1.8. $X = \mathbb{R}$, $\|\cdot\| := |\cdot|$, $M := [0,\infty)$, $F : M \ni t \longmapsto t + \dfrac{1}{1+t} \in M$. *We have*

$$|F(t) - F(s)| < |t - s| .$$

F *posesses no fixed point! Notice that*

$$|F(t) - F(s)| \leq c|t - s| , \quad t, s \in M ,$$

for no $c \in [0,1)$ *is possible.* □

Corollary 1.9. *Let* X *be a Banach space and let* $F : \overline{B}_r \longrightarrow X$ *be a contraction, i. e.*

$$\exists\, L \in [0,1)\, \forall x, y \in \overline{B}_r\, (\|F(x) - F(y)\| \leq L\|x - y\|) . \tag{1.15}$$

If $F(\partial\overline{B}_r) \subset \overline{B}_r$ *then* F *has a uniquely determined fixed point in* $\overline{B}_r$.

Proof:
Define

$$G : \overline{B}_r \ni x \longmapsto \frac{1}{2}(x + F(x)) \in X .$$

Let $x \in \overline{B}_r, x \neq \theta$, and consider $u := rx\|x\|^{-1} \in \partial\overline{B}_r \subset \overline{B}_r$. Then we obtain

$$\|F(x) - F(u)\| \leq L\|x - u\| = L(r - \|x\|)$$

and hence

$$\|F(x)\| \leq \|F(u)\| + \|F(x) - F(u)\| \leq r + L(r - \|x\|) \leq 2r - \|x\|$$

and

$$\|G(x)\| = \frac{1}{2}\|x + F(x)\| \leq \frac{1}{2}(\|x\| + \|F(x)\|) \leq r .$$

Due to the continuity of G we obtain $\|G(\theta)\| \leq r$.
This shows that $G : \overline{B}_r \longrightarrow \overline{B}_r$ and G is a contraction too as we conclude from

$$\|G(x) - G(y)\| \leq \frac{1}{2}(\|x - y\| + L\|x - y\|) = \frac{1}{2}(1 + L)\|x - y\| , \; x, y \in \overline{B}_r .$$

Theorem 1.5 implies that G has a fixed point $\overline{x} \in \overline{B}_r$. This implies $\overline{x} = F(\overline{x})$. Due to the fact that F is a contraction this fixed point is uniquely determined. ■

Remark 1.10. *What is a kind of the inverse of the contraction fixed point theorem? The most elegant result in this direction is due to Bessaga [5].*

Suppose that X *is an arbitrary nonempty set and suppose that* $F : X \longrightarrow X$ *has the property that* F *and each of its iterates* T^n *has a unique fixed point. Then for each* $c \in (0,1)$ *there exists a metric* d_c *on each* X *such that* (X, d_c) *is complete and for which*

$$d_c(F(x), F(y)) \leq c\, d_c(x, y) \text{ for all } x, y \in X .$$

□

Theorem 1.11 (Edelstein, 1961). *Let* X *be a Banach space and let* $V \subset X$. *Suppose that* $F : V \longrightarrow V$ *is contractive, i. e.*

$$\forall x, y \in V, x \neq y, \ (\|F(x) - F(y)\| < \|x - y\|) . \tag{1.16}$$

Then we have

a) F *has at most a fixed point in* V.

b) If V *is compact then* F *posesses exactly one fixed point.*

Proof:
Ad a) Obvious.
Ad b) Consider the mapping $V \ni v \longmapsto \|v - F(v)\| \in \mathbb{R}$. Since V is compact and since the norm is continuous there exists $\overline{x} \in V$ with $\|\overline{x} - F(\overline{x})\| = \inf_{v \in V} \|v - F(v)\|$. Now we must have $\overline{x} = F(\overline{x})$ because otherwise we would have

$$\|F(\overline{x}) - (F \circ F)(\overline{x})\| < \|\overline{x} - F(\overline{x})\| .$$

■

Remark 1.12. *The completeness that we assume in Theorem 1.5 and in the following results is more or less necessary for the existence of fixed points; see for instance [7, 11, 10, 14, 15].* □

Remark 1.13. *An application of the contraction theorem of Banach is the theorem of the inverse mapping; see for instance [4].* □

1.3 Successive approximation for a contraction

Consider

$$x^{n+1} := F(x^n) , \ n \in \mathbb{N}_0 ; \ x^0 := x , \tag{1.17}$$

with a contraction F in a metric space (X, d). From a practical programming point of view, we need a stopping rule for the successive approximation in order to compute the fixed point in a finite number of iteration steps; see above. The goal of a stopping rule is to guarantee for a given accuracy quantity $\varepsilon > 0$ an approximation $x^N \in X$ with

$$d(x^N, z) \leq \varepsilon$$

where $(F^n(x))_{n \in \mathbb{N}}$ ist the orbit generated by the iteration (1.17). To implement such a stopping rule one may pursue two different concepts.

A priori concept

This results from the estimation (c) in Theorem 1.4:

$$d(x^n, z) \leq \frac{c^n}{1 - c} d(F(x), x) , \ n \in \mathbb{N} .$$

A priori stopping rule

If $\delta := d(F(x), x)$ and $N > (\log(\varepsilon) + \log(1-c) - \log(\delta))/\log(c)$ then $d(x^N, z) \le \varepsilon$.
Notice that such a number N exists due to the fact that $0 \le c < 1$. This number can be computed before we compute the orbit $(F^n(x))_{n\in\mathbb{N}_0}$.
Suppose we take $\varepsilon := 10^{-m}$ in our stopping rule inequality. In order to obtain one more decimal digit of precision we have to do (roughly) $1/|\log(c)|$ more iterations steps. Stated a little differently, if we need N iterative steps to get m decimal digits of precision, then we need another N to double the precision to $2m$ digits. This kind of error behavior is called **linear convergence**.

A posteriori stopping concept
The basis for this conception is the estimation in (d) of Theorem 1.4.

A posteriori stopping rule

If $d(x^N, x^{N-1}) \le \varepsilon(1-c)/c$ then $d(x^N, z) \le \varepsilon$.

Algorithm 1.1 Successive approximation – A posteriori stopping rule

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Given a fixed point equation $x = F(x)$ and an accuracy parameter $\varepsilon > 0$. This algorithm computes a sequence $x^0, x^1, , \dots, x^N$ such that $d(x^N, z) \le \varepsilon$. Here x^0 is a given starting value.
Assumptions: see Theorem 1.4

(1) $k := 0$

(2) $u := x_k$

(3) If $u = F(u)$ set $N := 0$ and go to line (7)

(4) $x_{k+1} := F(u)$

(5) If $d(x^{k+1}, x^k) \le \varepsilon(1-c)/c$ set $N := k+1$ and go to line (7).

(6) Set $k := k+1$ and go to line (2).

(7) STOP with an approximation x^N for the fixed point z which satisfies $d(x^N, z) \le \varepsilon$

For the realization of the stopping rules above we need the Lipschitz constant c. It is obvious that an approximation $\tilde{c}$ with $\tilde{c} \in [c, 1)$ is sufficient.

Now, we consider the successive approximation method under errors. We follow mainly [12]. We consider in the complete metric space (X, d) the fixed point equation (1.1)

$$x = F(x) \tag{1.18}$$

and the associated successive approximation

$$x^{k+1} := F(x^k)\,,\ k \in \mathbb{N}_0\ x^0 := x, \tag{1.19}$$

where $F : X \longrightarrow X$ is a given mapping and x^0 is a given starting value of the iteration. This defines an approximating family $(x^k)_{k\in\mathbb{N}}$. We assume that F is a contraction with contraction constant $c \in [0,1)$. Then – by the contraction theorem – $(x^k)_{k\in\mathbb{N}}$ converges to the unique fixed point z of the equation (1.18). Now, instead of $(x^k)_{k\in\mathbb{N}_0}$ we have at hand an approximating sequence $(\tilde{x}^k)_{k\in\mathbb{N}_0}$ only. This sequence may be constructed by an iteration of a Lipschitz continuous mapping $\tilde{F} : X \longrightarrow X$ as follows:

$$\tilde{x}^{k+1} := \tilde{F}(\tilde{x}^k)\,,\; k \in \mathbb{N}_0\,,\; \tilde{x}^0 := x\,. \tag{1.20}$$

In general, we do not have $\lim_{n\in\mathbb{N}} \tilde{x}^k = z$ when $\tilde{F}$ is different from F.

Theorem 1.14. *Let* (X, d) *be a complete metric space and let* $F : X \longrightarrow X$ *be a contraction with Lipschitz constant* $c \in [0,1)$ *and a fixed point* z. *Let* $\delta > 0$ *and suppose that* $(x^k)_{k\in\mathbb{N}_0}$, $(\tilde{x}^k)_{k\in\mathbb{N}_0} = (\tilde{x}(\delta)^k)_{k\in\mathbb{N}_0}$ *are sequences such that the following conditions are satisfied:*

$$x^{n+1} := F(x^n)\,,\; n \in \mathbb{N}_0\,,\; x^0 = x\,. \tag{1.21}$$

$$d(\tilde{x}^{n+1}, F(\tilde{x}^n)) \le \delta\,,\; n \in \mathbb{N}_0\,. \tag{1.22}$$

$$d(\tilde{x}^0, x^0) \le \delta\,. \tag{1.23}$$

Stopping rule*: Suppose there exists* $N(\delta) \in \mathbb{N}$ *with*

$$d(\tilde{x}^{n+1}, \tilde{x}^n) \le d(\tilde{x}^{n+1}, \tilde{x}^n), n = 0, \dots, N(\delta)\,,\; d(\tilde{x}^{N(\delta)+1}, \tilde{x}^{N(\delta)}) > d(\tilde{x}^{N(\delta)}, \tilde{x}^{N(\delta)-1}) \tag{1.24}$$

Then

$$\lim_{\delta\downarrow 0} \tilde{x}^{N(\delta)} = z \tag{1.25}$$

Proof:
Let $m \le N(\delta)$. Then

$$\begin{aligned} d(\tilde{x}^m, x^m) &\le d(\tilde{x}^m, F(x^{m-1}) \le \delta + c d(\tilde{x}^{m-1}, x^{m-1} \\ &\vdots \\ &\le \delta + c\delta + \cdots + c^{m-1}\delta + c^m d(\tilde{x}^0, x^0) \\ &\le \frac{\delta}{1-c} + d(\tilde{x}^0, x^0)\,. \end{aligned}$$

This implies

$$\lim_{\delta\downarrow 0} d(\tilde{x}^m, x^m) = 0\,,\; m = 1, \dots, N(\delta)\,,\; \lim_{\delta\downarrow 0} d(\tilde{x}^{N(\delta)}, x^{N(\delta)}) = 0\,. \tag{1.26}$$

Since

$$d(\tilde{x}^m, z) \le d(\tilde{x}^m, x^m) + d(x^m, z)\,,\; m = 0, \dots, N(\delta), \tag{1.27}$$

we have to show $\lim_{\delta\downarrow 0} d(x^{N(\delta)}, z) = 0$.
Case 1: $\lim_{\delta\downarrow 0} N(\delta) = \infty$.
Since $\lim_n d(x^n, z) = 0$, nothing has to be proved.
Case 2: If $\lim_{\delta\downarrow 0} N(\delta) = \infty$ does not hold then there exist a sequence $(\delta_l)_{l\in\mathbb{N}}$ and $q \in \mathbb{N}$ with $N_l := N(\delta_l) \le q, l \in \mathbb{N}$.
Let $m \le q$. Then

$$d(x^q, x^{q-1}) \le c d(x^{q-1}, x^{q-2}) \le \cdots \le c^{q-m} d(x^m, x^{m-1})\,. \tag{1.28}$$

Moreover, due to $c^{q-m} \le 1$ we have

$$d(x^q, x^{q-1}) \le d(x^{N_l}, \tilde{x}^{N_l}) + d(\tilde{x}^{N_l}, \tilde{x}^{N_l-1}) + d(\tilde{x}^{N_l-1}, x^{N_l-1})\,. \tag{1.29}$$

For $l \to \infty$ the first and the second term in (1.29) converge to zero due to 1.26. From the stopping rule we obtain

$$d(x^{N_l}, x^{N_l-1}) \le d(x^{N_l+1}, x^{N_l})$$

and hence

$$d(\tilde{x}^{N_l}, \tilde{x}^{N_l-1}) \le d(x^{N_l+1}, F(\tilde{x}^{N_l})) + d(F(\tilde{x}^{N_l}), F(\tilde{x}^{N_l-1})) + d(F(\tilde{x}^{N_l-1}), \tilde{x}^{N_l})\,.$$

This implies

$$d(\tilde{x}^{N_l}, \tilde{x}^{N_l-1}) \le \frac{2\delta}{1-c}\,. \tag{1.30}$$

Therefore, the seond term in (1.29) converges to zero too if l goes to infinity. From this we conclude that we must have

$$d(x^q, x^{q-1}) = d(F(x^{q-1}), x^{q-1}) = 0\,.$$

This implies $x^{q-1} = z$ and

$$d(x^n, z) = 0 \text{ for } n \ge q-1\,. \tag{1.31}$$

For $n < q-1$ we have

$$d(x^n, z) = d(x^n, x^{q-1}) \le d(x^n, x^{n+1}) + \cdots d(x^q, x^{q-1})\,.$$

Each term in the sum on the righthand side can be extimated analogous to (1.29) and converges to zero as l goes to infinity. This shows

$$\lim_{\delta \downarrow 0} d(x^{N(\delta)}, z) = 0\,. \tag{1.32}$$

From (1.26) and (1.32) we conclude (1.25). ■

Remark 1.15. *The stopping rule may be formulated a little bit more general with a general distance function* $\tilde{d}$ *which satisfies*

$$|d(x,y) - \tilde{d}(x,y)| \le \varepsilon(\delta), x, y \in X,$$

with $\lim_{\delta \downarrow 0} \varepsilon(\delta) = 0\,.$ □

1.4 Brouwer's fixed point theorem

Probably, the most simplest existence result for a fixed point is the following one:

Every continuous function $f : [a,b] \longrightarrow [a,b]$ has a fixed point.

To prove this, let $g(t) := f(t) - t, t \in [a,b]\,.$ Then, since f is a mapping from $[a,b]$ into $[a,b]$, we must have $g(a) \ge a, g(b) \le b\,.$ By the mean value theorem there exists $z \in [a,b]$ with $g(z) = 0\,.$ Clearly, z is a fixed point of $f\,.$

This result holds in particular for the interval $[-1,1]\,.$ Now, the classical formulation of Brouwer's fixed point theorem is an „analogous“ result in a finite-dimensional normed vector space. For the proof we refer to the literature; see for instance [6, 8, 16].

Theorem 1.16 (Fixed point theorem of Brouwer, 1912; Poincaré, 1883)**.** *Let* $\overline{B}_1$ *be the closed unit ball in the euclidean space* $\mathbb{R}^n$ *and let* $F : \overline{B}_1 \longrightarrow \overline{B}_1$ *be a continuous mapping. Then* F *has a fixed point.*

Let C be a closed convex subset of $\mathbb{R}^n$ with nonempty interior. We may assume that $\theta \in C$. Then C is homeomorph to the closed unit ball $\overline{B}_1$ in $\mathbb{R}^n$. This can easily seen using the Minkovski functional with respect to the ball which is contained in the interior of C. If C is an arbitrary nonempty closed convex subset in $\mathbb{R}^n$ then C has a nonempty interior as a subset of the affine hull $\mathrm{aff}(C)$ of C. Therefore, C is homeomorph to the unit ball $\overline{B}_1$ in $\mathbb{R}^k$ for some $k \le n$. Let $T : \overline{B}_1 \longrightarrow C$ be such a homoemorphism. We set $G := T^{-1} \circ F \circ T : \overline{B}_1 \longrightarrow \overline{B}_1$. Then G is continuous and has a fixed point z. Then $T(z)$ is a fixed point of F. This consideration leads to a more general formulation of Brouwer's fixed point theorem.

Theorem 1.17. *Let* C *be a nonempty convex compact subset of a finite-dimensional normed space and let* $F : C \longrightarrow C$ *be a continuous mapping. Then* F *has a fixed point.*

Proof:
We present a proof different of the considerations above. It is more in the spirit of the main topic in this monograph.
Without loss of generality we may assume that C is a subset of the euclidean space $\mathbb{R}^k$ for some k. Since C is compact it is bounded. Suppose $C \subset \overline{B}_r$. Consider the mapping

$$P_C : \mathbb{R}^k \longrightarrow C \text{ defined by the property } \|x - P_C(x)\| = \inf_{u \in C} \|x - u\| .$$

Such a map is well defined since the function $f : \mathbb{R}^d \ni x \longmapsto \|x - u\| \in \mathbb{R}$ is continuous and the set C is compact; the uniqueness of the minimum of f over C is obvious. We will study this in the next chapter much more detailed. Notice that P_C is the identity on C. The mapping P_C is continuous. To prove this, let $x \in \mathbb{R}^k$ and let $(x^n)_{n\in\mathbb{N}}$ be a sequence with $x = \lim_n x^n$. Since the points $P_C(x^n)$ belong to the compact set C for all $n \in \mathbb{N}$ $(P_C(x^n))_{n\in\mathbb{N}}$ has a cluster point y which is contained in C; $y := \lim_l P_C(x^{n_l}))$. From

$$\|x - y\| = \lim_l \|x^{n_l} - P_C(x^{n_l})\| = \lim_l \mathrm{dist}(x^{n_l}, C) = \mathrm{dist}(x, C)$$

we conclude $P_C(x) = y$ since $P_C(x)$ is uniquely determined. For the fact $\lim_l \mathrm{dist}(x^{n_l}, C) = \mathrm{dist}(x, C)$ we use that $\mathrm{dist}(\cdot, C)$ is Lipschitz-continuous; see Lemma 2.1.
Now consider $R := P_C\big|_{\overline{B}_r}$. Then $G := F \circ R$ is a continuous mapping from $\overline{B}_r$ into $\overline{B}_r$ and has, due to Brouwer's fixed point theorem, a fixed point x. Since $G(x) \in C$ x is in C too. Hence $R(x) = x$ and $F(x) = x$. ∎

Remark 1.18. *Deeply connected to the Brouwer fixed point theorem are the following theorems:*

- *Hairy ball therorem*
  This theorem states that there is no nonvanishing continuous tangent vector field on evendimensional n*-spheres.*

- *„Brötchensatz“*
  This ham sandwich theorem *states that given* n *(measurable) „objects“ in* n*-dimensional space can be divided in half (with respect to their measure) with a single* $(n-1)$*-dimensional hyperplane.*

- *Theorem of Borsuk-Ulam*
  This theorem states that every continuous function from an n*-sphere into an euclidean* n*-space maps some pair of antipodal points to the same point.*

The connection between these results is the following:

Theorem of Borsuk-Ulam	$\Longrightarrow$	*„Brötchensatz"*
Theorem of Borsuk-Ulam	$\Longrightarrow$	*Brouwers fixed point Theorem*
The hairy ball Theorem	$\Longrightarrow$	*Brouwers fixed point Theorem*

□

Example 1.19. *The following example shows that the result does not hold in infinite-dimensional Banach spaces.*
Let $\mathcal{H}$ *be the space of the square-summable sequences,i.e.:* $\mathcal{H} = l_2$ *. This space is a Hilbert space endowed with the inner product*

$$\langle (x_n)_{n\in\mathbb{N}} | (y_n)_{n\in\mathbb{N}} \rangle := \sum_{n\in\mathbb{N}} x_n y_n$$

and norm

$$\|(x^n)_{n\in\mathbb{N}}\| := (\sum_{n\in\mathbb{N}} x_n^2)^{\frac{1}{2}} .$$

We define a mapping F *on* l_2 *by*

$$F((x^n)_{n\in\mathbb{N}}) := (\sqrt{1 - \|(x^n)_{n\in\mathbb{N}}\|^2}, x_1, x_2, \dots) .$$

Clearly, $\|F((x^n)_{n\in\mathbb{N}})\| = 1$ *for all* $(x^n)_{n\in\mathbb{N}} \in l_2$ *and hence* $F : \overline{B}_1 \longrightarrow \overline{B}_1$ *.* F *is continuous since* $\lim_k x^k = x$ *in* l_2 *implies*

$$\lim_k \|F(x^k) - F(x)\|^2 = \lim_k ((\sqrt{1 - \|x^k\|^2} - \sqrt{1 - \|x\|^2})^2 + \|x^k - x\|^2) = 0 .$$

Assume F *has a fixed point* $x = (x_n)_{n\in\mathbb{N}} \in \overline{B}_1$ *then* $\|x\| = 1$ *(see above) and hence* $x_1 = 0, x_2 = x_1 = 0, x_3 = 0, \dots,$ *. Thus,* $\|x\| = 0$*, contradicting the fact that* $\|x\| = 1$ *.* ■

In Example 1.19 we have seen that Brouwer's fixed point theorem in a infinite-dimensional Hilbert space does not hold in general. The following fixed point theorem of Kakutani shows that the generalization of Brouwer's fixed point theorem without additional assumptions is not possible; we omit the proof.

Theorem 1.20. *Let* $\mathcal{H}$ *be an infinite-dimensional Hilbert space. Then there is a continuous mapping* $F : \mathcal{H} \longrightarrow \mathcal{H}$ *which maps* $\overline{B}_1$ *into* $\overline{B}_1$ *and do not have a fixed point.*

1.5 The fixed point property

A topological space X has the **fixed point property**, if every continuous mapping $f : X \longrightarrow X$ has a fixed point. Theorem 1.16 and 1.17 say that the unit ball in the euclidean space $\mathbb{R}^n$ and every nonempty compact convex subset in a finite-dimensional normed space has the fixed point property. Thus, we see that the unit ball in $\mathbb{R}^n$, independently of the norm chosen, has the fixed point property.

In Chapter 7 we shall prove the following fixed point theorem.

Theorem 1.21 (Browder, Göhde, Kirk, 1965)**.** [1] *Let* $\mathcal{X}$ *be a uniformly convex Banach*

[1] This theorem has been proved independently by Browder, Göhde and Kirk in 1965

space and let $C := \overline{B}_1$ *be the closed unit ball in* $\mathcal{X}$. *If* $F : C \longrightarrow C$ *is a nonexpansive mapping then* F *has a fixed point.*

Let us consider Theorem 1.21 in a Hilbert space $\mathcal{H}$. Such a space is is uniformly convex; see Section 4.1. Then the result can be proved by considering the contraction $F_t, F_t(x) := tu + (1-t)F(x)$ for some $u \in C$ and for $t \in [0,1)$.

Theorem 1.22. *Let* $\mathcal{H}$ *be a Hilbert space, let* C *be a bounded closed convex subset of* $\mathcal{H}$ *and let* $F : C \longrightarrow C$ *be a nonexpansive mapping. Then* F *has a fixed point.*

Proof:
Consider for $t \in (0,1]$ the map $F_t : C \ni x \longmapsto tu + (1-t)F(x) \in \mathcal{H}$. Since C is convex the range of F_t is contained in C. We have

$$\|F_t(x) - F_t(y)\| \le (1-t)\|x - y\|\,,\ x, y \in C\,,$$

and we conclude from Banach's contraction theorem that F_t has a uniquely determined fixed point $x_t \in C$: $x_t = tu + (1-t)F(x_t), t \in (0,1]$. Since C is a bounded set the diameter diam(C) is finite. We obtain

$$\|x_t - F(x_t)\| = t\|u - F(x_t)\| \le t\,\mathrm{diam}(C)\,.$$

This shows $\lim_{t\downarrow 0}\|x_t - F(x_t)\| = 0$ and we say that F has approximate fixed points. Consider $\mathrm{Fix}_\varepsilon(F) := \{z \in C : \|F(z) - z\| \le \varepsilon\}$. We conclude from the fact that $\lim_{t\downarrow 0}\|x_t - F(x_t)\| = 0$ that $\mathrm{Fix}_\varepsilon(F) \neq \emptyset$ for all $\varepsilon > 0$. Clearly, $\mathrm{Fix}_\varepsilon(F)$ is closed for all $\varepsilon > 0$ since F is continuous. Moreover, $\mathrm{Fix}_\varepsilon(F)$ is convex for all $\varepsilon > 0$ since C is a subset of a Hilbert space. We shall prove this in Chapter 7 in a more general context; see 7.16. Thus we know that $\mathrm{Fix}_\varepsilon(F)$ is weakly closed for all $\varepsilon > 0$ and therefore weakly compact as a subset of the weakly compact subset C. This implies $\cap_{\varepsilon>0}\mathrm{Fix}_\varepsilon(F) \neq \emptyset$. Clearly, every $x \in \cap_{\varepsilon>0}\mathrm{Fix}_\varepsilon(F)$ is a fixed point of F. ■

The approach which is used in the proof of Theorem 1.22 is a part of a huge theory on the iterative computation of fixed points of nonexpansive mappings. For example, Halpern's iteration

$$x^{n+1} := \alpha_n x^n + (1-\alpha_n)F(x^n)\,,\ n \in \mathbb{N}_0, \tag{1.33}$$

(see Chapter 7) is used to compute a fixed point of F when we choose the sequence $(\alpha_n)_{n\in\mathbb{N}_0}$ in an appropriate way. Surely, $\lim_n \alpha_n = 0$ should be satisfied. We caome back to this iteration method in Chapter 7.

1.6 Fixed point theorem of Schauder

Definition 1.23. *Let* $\mathcal{X}$ *be a Banach space, let* $M \subset \mathcal{X}$ *and let* $F : M \longrightarrow \mathcal{X}$ *be a mapping.*

(a) F *is called* **bounded** *if* $F(M \cap B)$ *is bounded for every bounded subset* B *of* $\mathcal{X}$.

(b) F *is called* **completely continuous** *if* F *maps every weakly convergent sequence (in* M*) into a convergent sequence.*

(c) F *is called* **compact** *if* $\overline{F(M \cap B)}$ *is compact for every bounded subset* B *of* $\mathcal{X}$ *.*

□

Now, we present three versions of Schauder's fixed point theorem. It is sufficient to proof only one version since they are equivalent. We shall proof the second version in Section 5.8 as a consequence of the considerations concerning the metric projection.

Theorem 1.24 (Fixed point Theorem of Schauder, 1930/First Version). *Let* X *be a Banach space and let* M *be a nonempty bounded closed convex subset of* $\mathcal{X}$ *. Then every completely continuous mapping* $F : M \longrightarrow M$ *posesses a fixed point.*

Theorem 1.25 (Fixed point Theorem of Schauder, 1930/Second Version). *Let* X *be a Banach space and let* M *be a nonempty compact convex subset of* $\mathcal{X}$ *. Let* $F : M \longrightarrow M$ *be continuous. Then* F *posesses a fixed point.*

Proof:
See the proof of Theorem 5.14. ■

Theorem 1.26 (Fixed point Theorem of Schauder, 1930/Third Version). *Let* X *be a Banach space and let* M *be a nonempty closed convex subset of* $\mathcal{X}$ *. Let* $F : M \longrightarrow M$ *be a continuous mapping and let* $\overline{F(M)}$ *be compact. Then* F *posesses a fixed point.*

Definition 1.27. *A metric space* (X, d) *is totally bounded if and only if for every* $r > 0$ *there exists a finite collection of open balls in* X *of radius* r *whose union contains* X*.* □

Theorem 1.28. *Let* $\mathcal{X}, \mathcal{Y}$ *be Banach spaces, let* $U \subset X$ *open and let* $F : U \longrightarrow \mathcal{Y}$*. Let* $A \subset U$ *bounded. Then the following conditions are equivalent:*

(a) $F : A \longrightarrow Y$ *is completely continuous.*

(b) *For every* $\varepsilon > 0$ *there exists a finite-dimensional subspace* Y_ε *of* $\mathcal{Y}$ *and a continuous bounded mapping* $F_\varepsilon : A \longrightarrow \mathcal{Y}_\varepsilon$ *such that*

$$\sup_{x \in A} \|F(x) - F_\varepsilon(x)\| \leq \varepsilon \,.$$

F_ε *can be chosen such that* $F_\varepsilon(A) \subset co(F(A))$ *holds.*

Proof:
Ad $(a) \implies (b)$
Since F is compact and A is bounded $\overline{F(A)}$ is compact. Let $\varepsilon > 0$. Then there exist $y^1, \ldots, y^m \in F(A)$ with

$$\overline{F(A)} \subset \cup_{i=1}^m B_\varepsilon(y^i) \,.$$

We define $\phi_j : A \longrightarrow [0,1]$ by

$$\phi_j(x) := \frac{\operatorname{dist}(F(x), \mathcal{Y} \backslash B_\varepsilon(y^j))}{\sum_{i=1}^m \operatorname{dist}(f(x), \mathcal{Y} \backslash B_\varepsilon(y^i))} \,,\ x \in \mathcal{X} \,.$$

We have $\phi_1(x) + \cdots + \phi_m(x) = 1, x \in A$. Set

$$F_\varepsilon := \sum_{j=1}^m \phi_j y^j \,,\ Y_\varepsilon := \operatorname{span}(y^1, \ldots, y^m) \subset \mathcal{Y} \,.$$

Now we have

$$\sup_{x\in A}\|F(x)-F_\varepsilon(x)\|\le\varepsilon\,,\ F_\varepsilon(A)\subset \mathrm{co}(y^1,\ldots,y^m)\subset \mathrm{co}(F(A)),$$

and since $\mathrm{co}(F(A))$ is bounded $F_\varepsilon(A)$ is bounded too.
Since the function $\mathrm{dist}(\cdot,M)$ is Lipschitz continuous in every metric space M (see Lemma 2.1) we conclude due to the continuity of F that every ϕ_j is continuous. Hence F_ε is continuous.
Ad $(b)\implies(a)$
Since F is the uniformly limit of a family of continuos mappings F is continuous. Since in a Banach space each point possesses a contable basis of neighborhoods it is sufficient to show that $\overline{F(A)}$ is sequentially compact. Hence it is sufficient to show for a sequence $(F(x^k))_{k\in\mathbb{N}}$ in $F(A)$: there exists a subsequence $(F(x_{k_l})_{l\in\mathbb{N}}$ which is convergent to a point $y:=\lim_l F(x^{k_l})$ in $\overline{F(A)}$.
$(F_n(x^k))_{k\in\mathbb{N}}$ is for each $n\in\mathbb{N}$ a bounded sequence in a finite-dimensional space. Hence there exist subsequences of $(x^k)_{k\in\mathbb{N}}$ und $y^n\in\mathcal{Y}$ with $y^n=\lim F_n(x^k), n\in\mathbb{N}$. The sequence $(y^n)_{n\in\mathbb{N}}$ is a Cauchy sequence due to

$$\|y^l-y^m\|\le\|y^l-F_l(x^k)\|+\|F_l(x^k)-F_m(x^k)\|+\|F_m(x^k)-y^m\|$$

if one chooses $k=k(l,m)$ appropriate. As a consequence the sequence $(y^n)_{n\in\mathbb{N}}$; congerges. $y:=\lim_n y^n$. From

$$\|F(x^k)-y\|\le\|F(x^k)-F_n(x^k)\|+\|F_n(x^k)-y^n\|+\|y^n-y\|$$

we conclude $y=\lim_k F(x^k)$. Obviously, $y\in\overline{F(A)}$.

■

1.7 Schauder's fixed point theorem and successive approximation

Now, we want to consider the successive approximation of a fixed point under the assumption of the fixed point theorem of Schauder. The idea is to consider the successive approximation in a partial ordered Banach space. This has been done in a rigorous way for the first time by J. Schröder [13].

Definition 1.29. *Let $\mathcal{X}$ be Banach space endowed with a partial order $\preceq$.*
We say that a mapping $T:D_T\longrightarrow\mathcal{X}, D_T\subset\mathcal{X}$, is **syntone** *iff $v\preceq w$ implies $Tv\preceq Tw$.*
We say that a mapping $T:D_T\longrightarrow\mathcal{X}, D_T\subset\mathcal{X}$, is **antitone** *iff $v\preceq w$ implies $Tw\preceq Tv$.*
□

Consider for a continuous map $T:D_T\longrightarrow\mathcal{X}, D_T\subset\mathcal{X}$ convex, an equation

$$x=F(x):=T(x)+y \text{ for } x\in\mathcal{X}\,, \tag{1.34}$$

and suppose that T can be decomposed as follows:

$$T=T_1+T_2 \text{ where } T_1 \text{ is syntone and } T_2 \text{ is antitone}. \tag{1.35}$$

Then we may consider the iteration

$$v^{n+1} := T_1(v^n) + T_2(w^n) + y\,,\; w^{n+1} := T_1(w^n) + T_2(v^n) + y \tag{1.36}$$

with starting values $v^0, w^0 \in D_T$. If we can verify that

$$v^0 \preceq v^1 \preceq w^1 \preceq w^0 \tag{1.37}$$

then the iteration (1.36) can be continued and we obtain sequences $(v^n)_{n\in\mathbb{N}_0}, (w^n)_{n\in\mathbb{N}_0}$ with

$$v^0 \preceq v^1 \preceq v^2 \preceq \cdots \preceq v^n \preceq w^n \preceq \cdots \preceq w^2 \preceq w^1 \preceq w^0\,,\; n \in \mathbb{N}\,. \tag{1.38}$$

Under assumptions which imply that the „interval“ $\{u \in \mathcal{X} : v^0 \preceq u \preceq w^0\}$ is mapped into a relative compact subset of $\mathcal{X}$ there exists a fixed point $x \in \mathcal{X}$ with

$$v^n \preceq x \preceq w^n \text{ for all } n \in \mathbb{N}_0 \tag{1.39}$$

by Schauder's fixed point theorem 1.26. Thus, we may construct approximations v^n, w^n for the fixed point x of F which allow the inclusion

$$v^n \preceq x \preceq w^n \text{ for all } n \in \mathbb{N}_0\,, \tag{1.40}$$

if we have the additional property that $\theta \preceq u' \preceq u \preceq u''$ implies $\|u'' - u\| \le \|u'' - u'\|$.

This approach may be used in various situations. Notice that a matrix $A \in \mathbb{R}^{n,n}$ may be decomposed into $A = A_1 + A_2$ where both $A_1, -A_2$ have nonnegative entries. Then the mapping $u \longmapsto A_1 u$ is syntone and $u \longmapsto A_2 u$ is antitone with respect to the usual partial order in $\mathbb{R}^n$.
Another area of application is the theory of integral equations $x = F(x) = Tx + y$ when T is an integral operator of (Hammerstein-)type:

$$Tu(t) := \int_B \kappa(t,s)\phi(u(s))ds\,,\; t \in B\,,$$

We may decompose the kernel κ and/or ϕ such that the method above may be applied.

Example 1.30. *Consider the equation*

$$x(t) - 1 = T(x)(t) := \int_0^1 |t-s|(x(s) - \frac{1}{2}x(s)^2)ds\,,\; t \in [0,1],$$

in the space $C[0,1]$ *endowed with the supremum-norm. The partial order in* $C[0,1]$ *is given by* $f \preceq g \iff f(t) \le g(t)$ *for all* $t \in [0,1]$. T *may be decomposed as follows:*

$$T_1(u)(t) := \int_0^1 |t-s|u(s)ds\,,\; T_2(u)(t) := -\frac{1}{2}\int_0^1 |t-s|u(s)^2 ds\,,\; t \in [0,1]\,.$$

Clearly, T_1 *is syntone,* T_2 *is antitone. With* $v^0 \equiv 0, w^0 \equiv 2$ *we obtain* $v^1(t) := 2(t - t^2), w^1(t) = 2(1 - t + t^2)$, $t \in [0,1]$. *We can verfify* $v^0 \preceq v^1 \preceq w^1 \preceq w^0$. *Then we obtain a solution* x *of the equation above with*

$$v^1(t) \le x(t) \le w^1(t)\,,\; t \in [0,1]\,.$$

□

1.8 Appendix: Variational principle of Ekeland and applications

Wir beweisen das Variationslemma von Ekeland in einer endlichdimensionalen Version. Der Beweis ist hier elementar im Gegensatz zur Situation in metrischen oder topologischen Räumen. Sei nun in diesem und dem nächsten Abschnitt $|\cdot|$ stets die euklidische Norm in $\mathbb{R}^n$, assoziert zum euklidischen Skalarprodukt $\langle\cdot|\cdot\rangle$.

Theorem 1.31 (Variationslemma von Ekeland, 1972). *Sei* $f:\mathbb{R}^n \longrightarrow \mathbb{R}$ *unterhalbstetig, nach unten beschränkt. Sei* $\varepsilon > 0, x^* \in \mathbb{R}^n$ *mit*

$$f(x^*) \leq \inf_{x\in\mathbb{R}^n} f(x) + \varepsilon . \tag{1.41}$$

Dann gibt es zu jedem $\gamma > 0$ *ein* $\overline{x} \in \mathbb{R}^n$ *mit*

(a) $f(\overline{x}) \leq f(x^*)$;

(b) $|x^* - \overline{x}| \leq \frac{\varepsilon}{\gamma}$;

(c) $f(x) \geq f(\overline{x}) - \gamma|x - \overline{x}|$ *für alle* $x \in \mathbb{R}^n$.

Proof:
O. E. $\inf\{f(x)|x \in \mathbb{R}^n\} = 0$, also $f(x) \geq 0$ für alle $x \in \mathbb{R}^n$. Betrachte die Abbildung

$$F:\mathbb{R}^n \ni x \longmapsto f(x) + \gamma|x - x^*| \in \mathbb{R} .$$

Da F nach unten beschränkt und unterhalbstetig ist, und da die Niveaumengen $N_r(F) := \|x \in \mathbb{R}^n|F(x) \leq r\}$ beschränkt und abgeschlossen sind, also kompakt sind, gibt es ein $\overline{x}$, das ein Minimum von F darstellt (siehe etwa [4], Satz 3.12) Nun haben wir

$$f(\overline{x}) + \gamma|\overline{x} - x^*| \leq f(x) + \gamma|x - x^*| \text{ für alle } x \in \mathbb{R}^n,\ \varepsilon \geq f(x^*) \geq f(\overline{x}) + \gamma|\overline{x} - x^*| .$$

Daraus folgt $(b), (c)$ sofort. ■

Corollary 1.32. *Sei* $f:\mathbb{R}^n \longrightarrow \mathbb{R}$ *Fréchet-differenzierbar und nach unten beschränkt. Sei* $\varepsilon > 0, x^* \in \mathbb{R}^n$ *mit*

$$f(x^*) \leq \inf_{x\in\mathbb{R}^n} f(x) + \varepsilon . \tag{1.42}$$

Dann gibt es ein $\overline{x} \in \mathbb{R}^n$ *mit*

(a) $f(\overline{x}) \leq f(x^*)$;

(b) $|x^* - \overline{x}| \leq \sqrt{\varepsilon}$;

(c) $|\nabla f(\overline{x})| \leq \sqrt{\varepsilon}$.

Proof:
Wende Satz 10.16 an mit $\gamma := \sqrt{\varepsilon}$. Dann gibt es also $\overline{x} \in \mathbb{R}^n$ mit

$$f(\overline{x}) \leq f(x^*), |x^* - \overline{x}| \leq \sqrt{\varepsilon}, \sqrt{\varepsilon}|x - \overline{x}| \geq (f(\overline{x}) - f(x)) \text{ für alle } x \in \mathbb{R}^n .$$

Aus der dritten Ungleichung folgt

$$f(\overline{x} + tw) - f(\overline{x})) \geq -t\sqrt{\varepsilon}|w| \text{ für alle } t > 0, w \in \mathbb{R}^n .$$

Daraus folgt

$$\langle \nabla f(\overline{x}), w\rangle \geq -\sqrt{\varepsilon}|w| \text{ für alle } w \in \mathbb{R}^n\,,$$

was nun (c) impliziert. ■

Für den Beweis des Variationslemmas im Kontext „metrische Räume/unendlichdimensionale Räume“ stellen wir ein allgemeines Prinzip vor.
Erinnerung: Sei X eine Menge. Eine Relation „$\leq$ “heisst **Halbordnung** auf X genau dann, wenn gilt:

i) $x \leq x$ (Reflexivität)

ii) $x \leq y, y \leq x \implies y = x$ (Symmetrie)

iii) $x \leq y, y \leq z \implies x \leq z \quad \forall x, y, z \in X$ (Transitivität)

Wir schreiben dann $(X, \leq)$, vereinbaren die Schreibweise

$$S(x) := \{y \in X | y \geq x\},$$

und können für eine Folge $(x_n)_{n\in\mathbb{N}}$ in X definieren:

$$(x_n)_{n\in\mathbb{N}} \text{ monoton wachsend, falls } x_n \leq x_{n+1} \text{ für alle } n \in \mathbb{N}\,.$$

Theorem 1.33 (Brezis–Browder, 1976). *Sei* $(X, \leq)$ *eine halbgeordnete Menge und* $\varphi : X \longrightarrow \mathbb{R}$ *monoton wachsend, d.h.* $\varphi(x) \leq \varphi(y)$*, falls* $x \leq y$*. Es gelte:*

i) Für jede monoton wachsende Folge $(x_n)_{n\in\mathbb{N}}$ *mit* $\sup\{\varphi(x_n)|n \in \mathbb{N}\} < \infty$ *existiert* $y \in X$ *mit* $x_n \leq y$ *für alle* $n \in \mathbb{N}$.

ii) Für alle $x \in X$ *gibt es* $u \in X$ *mit* $x \leq u, \varphi(x) < \varphi(u)$.

Dann gilt: $\sup\{\varphi(y)|y \in S(x)\} = \infty$ *für alle* $x \in X$.

Proof:
Für $u \in X$ setze $\mu(u) := \sup\{\varphi(y)|y \in S(u)\}$.
Annahme: Es gibt $x \in X$ mit $\mu(x) < \infty$.
Definiere eine Folge $(x_n)_{n\in\mathbb{N}}$ induktiv in folgender Weise: $x_1 := x$; sind $x_1, \ldots, x_n$ bestimmt, wähle $x_{n+1} \in S(x_n)$ mit $\mu(x_n) - 1/n \leq \varphi(x_{n+1})$.
Wegen $\mu(x_n) \leq \mu(x) < \infty$ existiert x_{n+1} und wegen $x_{n+1} \geq x_n$ gilt $\mu(x_{n+1}) \leq \mu(x_n)$. Nun gilt also

$$x_n \geq x, \text{ d. h. } x_n \in S(x), n \in \mathbb{N}; (x_n)_{n\in\mathbb{N}} \text{ ist monoton wachsend; } \varphi(x_n) \leq \mu(x), n \in \mathbb{N}.$$

Wegen i) gibt es $y \in X$ mit $x_n \leq y$ für alle $n \in \mathbb{N}$. Wähle nach ii) $u \in X$ mit $u \in S(y)$ und $\varphi(y) < \varphi(u)$. Nun gilt $x_n \leq u, \varphi(u) \leq \mu(x_n), n \in \mathbb{N}$. Also

$$\varphi(u) \leq \mu(x_n) \leq \varphi(x_{n+1}) + 1/n \leq \varphi(y) + \frac{1}{n}\,, n \in \mathbb{N},$$

d. h. $\varphi(u) \leq \varphi(y) < \varphi(u)$, was ein Widerspruch ist. ■

Corollary 1.34. *Sei* $(X, \leq)$ *eine halbgeordnete Menge,* $\psi : X \longrightarrow \mathbb{R}$ *nach oben beschränkt und es gelte:*

$$\text{Für jede monoton wachsende Folge } (x_n)_{n\in\mathbb{N}} \text{ in } X \text{ existiert } y \in X \text{ mit } x_n \leq y\,,\ n \in \mathbb{N}\,. \tag{1.43}$$

Dann gilt:

(a) *Ist* ψ *monoton wachsend, so folgt:* $\forall u \in X\, \exists v \in S(u)\, \forall w \in S(v)\, (\psi(v) = \psi(w))$.

(b) *Ist* ψ *streng monoton wachsend, so folgt:* $\forall u \in X\, \exists v \in S(u)\, \forall w \in X\, (w \geq v \implies w = v)$.

Proof:
Zu (a).
Sei $u \in X$. Betrachte $(S(u), \leq), \varphi := \psi|_{S(u)}$. Dann ist φ monoton wachsend und nach oben beschränkt.
Annahme: $\forall v \in S(u)\, \exists w \geq v\, (\varphi(v) < \varphi(w))$
Nun sind die Voraussetzungen des Satzes 10.26 für $(S(u), \leq), \varphi$ erfüllt. Es folgt:

$$\forall w \in S(u)\, (\sup\{\varphi(y) | y \in S(w) \cap S(u)\} = \infty)\,.$$

Dies ist ein Widerspruch zur Beschränktheit von φ.
Zu (b).
Sei $u \in X$. Nach (a) gibt es $v \in S(u)$ mit $\varphi(w) = \varphi(v)$ für alle $w \in X, w \geq v$. Da φ streng monoton wachsend ist, gilt also:

$$w \in X, w \geq v \implies \varphi(w) = \varphi(v) \implies w = v\,.$$

■

Corollary 1.35. *Sei* (X, d) *ein metrischer Raum und sei* $(X, \leq)$ *halbgeordnet. Sei* $\eta : X \longrightarrow \mathbb{R}$ *streng monoton fallend und nach unten beschränkt. Es gelte:*

(a) *Für alle* $x \in X$ *ist* $S(x)$ *abgeschlossen.*

(b) *Für jede monoton wachsende Folge* $(x_n)_{n\in\mathbb{N}}$ *in* X *ist* $\overline{\{x_n | n \in \mathbb{N}\}}$ *kompakt.*

Dann gilt:

$$\forall u \in X\, \exists v \in S(u)\, \forall w \in X\, (w \geq v \implies w = v)\,.$$

Proof:
Sei $\psi := -\eta$. Wir weisen (10.19) nach. Sei dazu $(x_n)_{n\in\mathbb{N}}$ eine monoton wachsende Folge in X. Dann existiert wegen (b) ein $y \in X$ und eine Teilfolge $(x_{n_k})_{k\in\mathbb{N}}$ mit $y = \lim_k x_{n_k}$. Da $x_{n_k} \in S(x_{n_j})$ für alle $k \geq j$ gilt, folgt wegen (a) $y \in S(x_{n_j}) \subset S(x_n), n_j \geq n, n \in \mathbb{N}$. Also $y \leq x_n$ für alle $n \in \mathbb{N}$. Nun können wir Folgerung 10.27 (b) anwenden. ■

Theorem 1.36. *Sei* (X, d) *ein vollständiger metrischer Raum und sei* $f : X \longrightarrow (-\infty, \infty]$ *unterhalbstetig, nach unten beschränkt und eigentlich, d. h. es gibt* $z \in X$ *mit* $f(z) < \infty$. *Sei* $\varepsilon > 0, x^* \in X$ *mit*

$$f(x^*) \leq \inf_{v\in X} f(v) + \varepsilon\,. \tag{1.44}$$

Dann gibt es $\overline{x} \in X$ *mit*

1. $f(\overline{x}) \le f(x^*)$;

2. $d(\overline{x}, x^*) \le 1$;

3. $f(y) + \varepsilon d(y, \overline{x}) > f(\overline{x})$ *für alle* $y \in X \backslash \{\overline{x}\}$.

Proof:
$f(x^*)$ ist endlich wegen (10.20). $\widetilde{X} := \{z \in X | f(z) \le f(x^*)\}$ ist abgeschlossen, da f unterhalbstetig ist. Also ist $(\widetilde{X}, d)$ ein vollständiger metrischer Raum mit $x^* \in \widetilde{X}$. Haben wir das Resultat in $(\widetilde{X}, d)$ für $\widetilde{f} := f_{|\widetilde{X}}$ bewiesen, dann ist das Resultat in (X, d) bewiesen, denn die Bedingung 3. folgt dann mit 1. aus der Zeile

$$f(y) + \varepsilon d(y, \overline{x}) > f(x^*) + \varepsilon d(y, \overline{y}) \ge f(x^*) \ge f(\overline{x}), \ y \in X \backslash \widetilde{X}.$$

O. E. können wir daher nun $\widetilde{X} = X$ annehmen.
Wir definieren

$$u \le v :\iff f(v) - f(u) \le -\varepsilon d(u, v), \ u, v \in X,$$

und zeigen, dass dadurch auf X eine Halbordnung erklärt ist.
Die Reflexivität ist klar. Seien $u \le v, v \le u$, d. h.

$$f(v) - f(u) \le -\varepsilon d(u, v), \ f(u) - f(v) \le -\varepsilon d(u, v).$$

Daraus folgt $d(u, v) = 0$ und daher ist die Symmetrie gezeigt. Seien $u, v, w \in X$ mit $u \le v, v \le w$, d. h.

$$f(v) - f(u) \le -\varepsilon d(u, v), \ f(w) - f(v) \le -\varepsilon d(v, w).$$

Mit der Dreiecksungleichung folgt

$$f(w) - f(u) \le -\varepsilon d(u, w) \ , \text{ d. h. } u \le w.$$

Damit ist auch die Transitivität klar.
Wir zeigen nun, dass f streng monoton fallend ist ist bezüglich der Halbordnung $\le$. Seien dazu $u, v \in X, u \le v, u \ne v$; also

$$f(v) - f(u) \le -\varepsilon d(u, v) < 0 \ , \text{ d. h. } f(v) < f(u).$$

Wir zeigen (a) in Folgerung 10.28. Sei dazu $u \in X$.

$$S(u) = \{y \in X | y \ge u\} = \{y \in X | f(y) + \varepsilon d(y, u) \le f(u)\}$$

ist abgeschlossen, da f und $d(u, \cdot)$ unterhalbstetig sind.
Wir zeigen (b) in Folgerung 10.28. Sei dazu $(x_n)_{n \in \mathbb{N}}$ eine monoton wachsende Folge in X. Dann konvergiert die Folge $(f(x_n))_{n \in \mathbb{N}}$, da sie monoton fallend und nach unten beschränkt ist. Wir lesen aus

$$\varepsilon d(x_n, x_m) \le \sum_{i=m}^{n-1} \varepsilon d(x_i, x_{i+1}) \le \sum_{i=m}^{n-1} (f(x_{i+1}) - f(x_i)) = f(x_n) - f(x_m), \ n, m \in \mathbb{N}, n > m,$$

ab, dass $(x_n)_{n\in\mathbb{N}}$ eine Cauchyfolge und daher konvergent ist.
Wir können nun Folgerung 10.28 anwenden: Es gibt $\overline{x} \in S(x^*)$ mit der Eigenschaft

$$u \geq \overline{x} \implies u = \overline{x}\,. \tag{1.45}$$

Wegen $\overline{x} \in S(x^*)$ folgt

$$0 \leq \varepsilon d(x^*, \overline{x}) \leq f(x^*) - f(\overline{x}) \leq \inf_{\nu \in X} f(\nu) + \varepsilon - f(\overline{x}) \leq \varepsilon\,.$$

Damit ist 1., 2. gezeigt. Sei $y \in X\backslash\{\overline{x}\}$. Wegen (10.21) gilt nicht $y \geq \overline{x}$, d. h. $f(y) - f(\overline{x}) > -\varepsilon d(y, \overline{x})$. ■

Corollary 1.37. *Sei* (X, d) *ein vollständiger metrischer Raum und sei* $f : X \longrightarrow (-\infty, \infty]$ *unterhalbstetig, nach unten beschränkt und eigentlich. Sei* $\gamma > 0, x^* \in X$*. Dann gibt es* $\overline{x} \in X$ *mit*

i) $f(\overline{x}) < f(x) + \gamma d(x, \overline{x})$ *für alle* $x \in X\backslash\{\overline{x}\}$;

ii) $f(\overline{x}) \leq f(x^*) - \gamma d(x^*, \overline{x}))$;

Proof:
Sei $Y := \{x \in X | f(x) + \gamma d(x, x^*) \leq f(x^*)\}$, $g := \frac{1}{\gamma} f|_Y$. Y ist abgeschlossen, da f unterhalbstetig ist. Also ist (Y, d) ein vollständiger metrischer Raum. g ist unterhalbstetig, nach unten beschränkt und eigentlich. Sei $\alpha := \inf_{\nu\in Y} g(\nu)$. Es gibt $x^{**} \in Y$ mit $g(x^{**}) \leq \alpha + 1$. Also ist Satz 10.29 anwendbar mit $\varepsilon = 1$:

$$\exists\, \overline{x} \in Y \forall x \in Y\backslash\{\overline{x}\}\, (\frac{1}{\gamma} f(\overline{x}) < \frac{1}{\gamma} f(x) + d(x, \overline{x}))\,.$$

Also

$$f(\overline{x}) < f(x) + \gamma d(x, \overline{x})) \text{ für alle } x \in Y\backslash\{\overline{x}\})\,,\; f(\overline{x}) \leq f(x^*) - \gamma d(x^*, \overline{x})\,.$$

$ii)$ ist damit schon bewiesen.
Ist $x \in X\backslash Y$, so gilt $f(x) + \gamma d(x, x^*) > f(x^*)$ und mit $ii)$ folgt

$$f(\overline{x}) \leq f(x^*) - \gamma d(x^*, \overline{x}) < f(x) + \gamma(d(x, x^*) - d(x^*, \overline{x})) \leq f(x) + \gamma d(x, \overline{x})\,.$$

Damit ist auch $i)$ bewiesen. ■

Theorem 1.38 (Variationslemma von Ekeland, 1972)**.** *Sei* (X, d) *ein vollständiger metrischer Raum und sei* $f : X \longrightarrow (-\infty, \infty]$ *unterhalbstetig, nach unten beschränkt und eigentlich. Sei* $\varepsilon > 0, x^* \in X$ *mit*

$$f(x^*) \;\leq\; \inf_{\nu\in X} f(\nu) + \varepsilon\,. \tag{1.46}$$

Dann gibt es zu jedem $\gamma > 0$ *ein* $\overline{x} \in X$ *mit*

(a) $f(\overline{x}) \leq f(x^*)$;

(b) $d(x^*, \overline{x}) \leq \gamma$;

(c) $f(x) + \frac{\varepsilon}{\gamma} d(x, \overline{x}) > f(\overline{x})$ *für alle* $x \in X\backslash\{\overline{x}\}$.

Proof:
Sei $\gamma > 0$. Wende Satz 10.29 an unter Verwendung der zu d äquivalenten Metrik $\tilde{d} := \frac{1}{\gamma} d$.
■

Beachte: Es liegt folgender Kompromiss für die Wahl von γ nahe: $\gamma := \sqrt{\varepsilon}$; siehe Folgerung 10.17.

Das folgende Resultat stellt eine Existenzaussage im Kontext der Voraussetzungen von Satz 10.29 bereit. Ob es als Existenzsatz Verwendung finden kann, ist zweifelhaft, wir werden ihn aber verwenden können.

Theorem 1.39 (Takahashi, 1989)**.** *Sei* (X, d) *ein vollständiger metrischer Raum und sei* $f : X \longrightarrow (-\infty, \infty]$ *unterhalbstetig, eigentlich und nach unten beschränkt. Es gelte:*

$$\text{Für alle } u \in X \text{ mit } f(u) > \inf_{x \in X} f(x) \text{ gibt es } v \in X \text{ mit } v \neq u, f(v) + d(u, v) \leq f(u) .$$

Dann gibt es $\overline{x} \in X$ *mit* $f(\overline{x}) = \inf_{x \in X} f(x)$.

Proof:
Annahme: $f(u) > \mu := \inf_{x \in X} f(x)$ für alle $u \in X$.
Nach Satz 10.31 gibt es $\overline{x} \in X$ mit

$$f(y) + d(y, \overline{x}) > f(\overline{x}) \text{ für alle } y \in X \backslash \{\overline{x}\} .$$

Wegen $f(\overline{x}) > \mu$, gibt es $v \in X, v \neq \overline{x}$, mit $f(v) + d(v, \overline{x}) \leq f(\overline{x})$. Dies ist ein Widerspruch.
■

Die Vollständigkeit der metrischen Räume in den obigen Resultaten war wesentlich für die Beweise. Sie ist auch wesentlich für die Gültigkeit der Resultate, denn man kann zeigen, dass die Vollständigkeit in einem metrischen Raum schon durch die Tatsache charakterisiert wird, dass die Aussage von Satz 10.31 für jedes f mit den dortigen Eigenschaften gilt.

Theorem 1.40. *Sei* (X, d) *ein metrischer Raum. Dann sind äquivalent:*

(a) (X, d) *ist vollständig.*

(b) *Für jede unterhalbstetige Abbildung* $f : X \longrightarrow \mathbb{R}$ *mit* $\inf_{v \in X} f(v) \geq 0$ *gilt: Ist* $\varepsilon > 0, x^* \in X$ *mit* $f(x^*) \leq \inf_{v \in X} f(v) + \varepsilon$, *so gibt es* $\overline{x} \in X$ *mit*

   1. $f(\overline{x}) \leq f(x^*)$;

   2. $d(\overline{x}, x^*) \leq 1$;

   3. $f(y) + \varepsilon d(y, \overline{x}) \geq f(\overline{x})$ *für alle* $y \in X$.

Proof:
$(a) \Longrightarrow (b)$. Satz 10.29.
$(b) \Longrightarrow (a)$.
Sei $(x_n)_{n \in \mathbb{N}}$ eine Cauchyfolge in X. Dann existiert offenbar $\lim_n d(x_n, y)$ für jedes $y \in X$.
Definiere $f : X \longrightarrow \mathbb{R}$ durch

$$f(y) := \lim_n d(x_n, y), y \in X .$$

f ist unterhalbstetig, da $d(y,\cdot)$ stetig ist für alle $y \in X$, und es gilt $\inf_{\nu\in X} f(\nu) = 0$. Sei $\varepsilon \in (0,1), x^* \in X$ mit $f(x^*) \leq \varepsilon$. Dann gibt es nach Satz 10.29 $\overline{x} \in X$ mit

$$f(\overline{x}) \leq f(x^*) \leq \varepsilon\,,\; f(y) + \varepsilon d(y,\overline{x}) \geq f(\overline{x}) \text{ für alle } y \in X\,.$$

Wir zeigen: $f(\overline{x}) \leq \varepsilon^l$ für alle $l \in \mathbb{N}$.
Der Induktionsanfang $l = 1$ ist klar.
Sei $\eta > 0$ beliebig. Da $f(\overline{x}) \leq \varepsilon^l$ ist nach Induktionsvoraussetzung, gibt es x_m mit $d(x_m,\overline{x}) < \varepsilon^l + \eta, f(x_m) < \eta$. Mit 3. folgt

$$\eta + \varepsilon(\varepsilon^l + \eta) > f(x_m) + \varepsilon d(x_m,\overline{x}) \geq f(\overline{x})\,.$$

Durch Grenzübergang $\eta \to 0$ folgt $f(\overline{x}) \leq \varepsilon^{l+1}$.
Da $\varepsilon \in (0,1)$ ist, gilt also $f(\overline{x}) = 0$, d. h. $\overline{x} = \lim_n x_n$. ■

Theorem 1.41. *Sei* $(X, \|\cdot\|)$ *ein Banachraum und sei* $f : X \longrightarrow (-\infty,\infty]$ *eigentlich, unterhalbstetig, Gateaux–differenzierbar in* $dom(f) := \{x \in X | f(x) < \infty\}$ *und nach unten beschränkt Dann gibt es zu jedem* $\varepsilon > 0, \gamma > 0$ *und* $x^* \in X$ *mit*

$$f(x^*) \leq \inf_{\nu\in X} f(\nu) + \varepsilon$$

ein $\overline{x} \in X$ *mit*

1. $f(\overline{x}) \leq f(x^*)$.
2. $\|\overline{x} - x^*\| \leq \gamma$.
3. $\|D_G f(\overline{x})\| \leq \frac{\varepsilon}{\gamma}$.

Proof:
Wähle $\overline{x} \in X$ gemä"s Satz 10.31. Dann gilt

$$f(y) \geq f(\overline{x}) - \frac{\varepsilon}{\gamma}\|y - \overline{x}\|, y \in X\,. \tag{1.47}$$

Also

$$f(\overline{x} + tw) \geq f(\overline{x}) - \frac{\varepsilon}{\gamma}\|w\|t \text{ für alle } t > 0, \text{ d. h. } D_G f(\overline{x})(w) \geq -\frac{\varepsilon}{\gamma}\|w\|\,.$$

Dies zeigt

$$\|D_G f(\overline{x})\| \leq \frac{\varepsilon}{\gamma}\,.$$

■

Remark 1.42. *Satz 10.34 zeigt, dass für ein Optimierungsproblem*

$$\textit{Minimiere } f(x), x \in X,$$

unter den dortigen Voraussetzungen Punkte $\overline{x} \in X$ *existieren, die* f *„fast minimieren" und die die notwendige Bedingung* $D_G f(\overline{x}) = \theta$ *„fast erfüllen":*

$$f(\overline{x}) \leq \inf_{\nu\in X} f(\nu) + \varepsilon\,,\; \|D_G f(\overline{x})\| \leq \frac{\varepsilon}{\gamma}\,.$$

□

Theorem 1.43 (Kontraktionssatz/Fixpunktsatz von Banach). *Sei* (X, d) *ein vollständiger metrischer Raum und sei* $F : X \longrightarrow X$ *eine Kontraktion, d. h.*

$$d(F(x), F(y)) \leq L\, d(x, y) \text{ für alle } x, y \in X$$

mit $L \in [0, 1)$. *Dann besitzt* F *einen eindeutigen Fixpunkt.*

Proof:
Betrachte

$$f : X \ni x \longmapsto d(x, F(x)) \in \mathbb{R} .$$

Offenbar ist f stetig, nach unten beschränkt und eigentlich. Sei $x^* \in X, \varepsilon := d(x^*, F(x^*)), \gamma := 1 - L$. Wende damit Satz 10.29 an: Es gibt $\overline{x} \in X$ mit

$$d(\overline{x}, F(\overline{x})) < d(x, F(x)) + (1 - L)d(x, \overline{x}) \text{ für alle } x \neq \overline{x} .$$

Dann muss $\overline{x}$ ein Fixpunkt sein, denn anderenfalls würde $x := F(\overline{x})$ einen Widerspruch ergeben gemä"s

$$d(\overline{x}, F(\overline{x})) < d(F(\overline{x}), F(F(\overline{x}))) + (1-L)d(\overline{x}, F(\overline{x})) \leq Ld(\overline{x}, F(\overline{x})) - Ld(\overline{x}, F(\overline{x})) + d(\overline{x}, F(\overline{x})) .$$

Die Eindeutigkeit liest man mit mit zwei Fixpunkten $u, v \in X$ aus

$$d(u, v) = d(F(u), F(v)) \leq L\, d(u, v)$$

sofort ab. ■

Theorem 1.44 (Fixpunktsatz von Caristi–Kirk, 1976). *Sei* (X, d) *vollständiger metrischer Raum, sei* $f : X \longrightarrow \mathbb{R}$ *nach unten beschränkt und unterhalbstetig. Sei* $F : X \longrightarrow X$; *es gelte:*

$$d(x, F(x)) \leq f(x) - f(F(x)) \text{ für alle } x \in X.$$

Dann besitzt F *einen Fixpunkt, d. h. es gibt* $z \in X$ *mit* $F(z) = z$.

Proof:
Annahme: Es gilt $F(z) \neq z$ für alle $z \in X$.
Nach Satz 10.29 gibt es $\overline{x} \in X$ mit $f(z) - f(\overline{x}) > -d(z, \overline{x})$ für alle $z \in X \backslash \{\overline{x}\})$. Also

$$f(\overline{x}) - f(F(\overline{x})) < d(\overline{x}, F(\overline{x})) \leq f(\overline{x}) - f(F(\overline{x})) ,$$

was ein Widerspruch ist. ■

Remark 1.45. *Bemerkenswert an Satz 10.37 ist, dass von der Abbildung* F *nicht einmal Stetigkeitseigenschaften verlangt werden.*
Unter den Voraussetzungen von Satz 10.37 liegt i. a. keine Eindeutigkeit vor, wie folgendes Beispiel zeigt:

$$X := [0, 1], d(x, y) := |x - y|, f(x) := |x|, F(x) := x$$

Beachte: Man kann den Satz 10.29 auch aus dem Fixpunktsatz von Caristi–Kirk (Satz 10.37) herleiten. □

Betrachte folgende Sätze:

Theorem 1.46. *Sei* (X, d) *ein vollständiger metrischer Raum und sei* $f : X \longrightarrow (-\infty, \infty]$ *unterhalbstetig, eigentlich und nach unten beschränkt. Sei* $T : X \longrightarrow POT(X)\backslash\{\emptyset\}$*. Es gelte:*

$$\forall x \in X, x \notin T(x) \, \exists y \neq x \, (f(y) + d(y, x) \leq f(x)) \, .$$

Dann gibt es $z \in X$ *mit* $z \in T(z)$.

Theorem 1.47. *Sei* (X, d) *ein vollständiger metrischer Raum und sei* $f : X \longrightarrow (-\infty, \infty]$ *unterhalbstetig, eigentlich und nach unten beschränkt. Sei* $T : X \longrightarrow POT(X)\backslash\{\emptyset\}$*. Es gelte für alle* $x \in X$ *:*

$$f(y) + d(y, x) \leq f(x) \text{ für alle } y \in T(x) \, .$$

Dann gibt es $z \in X$ *mit* $\{z\} = T(z)$.

Theorem 1.48. *Sei* (X, d) *ein vollständiger metrischer Raum und sei* $f : X \longrightarrow (-\infty, \infty]$ *unterhalbstetig, eigentlich und nach unten beschränkt. Sei* $T : X \longrightarrow POT(X)\backslash\{\emptyset\}$ *. Es gelte für alle* $x \in X$

$$f(y) + d(y, x) \leq f(x) \text{ für alle } y \in T(x) \, .$$

Dann gibt es $z \in X$ *mit* $z \in T(z)$.

Theorem 1.49. *Sei* (X, d) *ein vollständiger metrischer Raumund sei* $f : X \longrightarrow (-\infty, \infty]$ *unterhalbstetig, eigentlich und nach unten beschränkt. Sei* $T : X \longrightarrow POT(X)\backslash\{\emptyset\}$ *abgeschlossen. Es gelte:*

$$d(x, T(x)) \leq f(x) - \sup_{y \in T(x)} f(y) \text{ für alle } x \in X \, .$$

Dann gibt es $z \in X$ *mit* $T(z) = \{z\}$.

Lemma 1.50. *Die Sätze 10.32, 10.39, 10.40, 10.41, 10.42 sind äquivalent.*

Proof:
Aus Satz 10.32 folgt Satz 10.39, denn:
Annahme: T hat keinen Fixpunkt, d. h. $x \notin T(x)$ für alle $x \in X$..
Sei $u \in X$ mit $f(u) > \inf_{x \in X} f(x)$. Da $T(u) \neq \emptyset$ und da $u \notin T(u)$ gibt es $v \in T(u), v \neq u$, mit $f(v) + d(v, u) \leq f(u)$. Damit liefert nun Satz 10.32 die Existenz von $\overline{x}$ mit $f(\overline{x}) = \inf_{x \in X} f(x)$. Sei $\overline{y} \in T(\overline{x})$. Aus

$$0 < d(\overline{x}, \overline{y}) \leq f(\overline{x}) - f(\overline{y}) \leq f(\overline{y}) - f(\overline{y}) = 0$$

lesen wir einen Widerspruch ab.
Aus Satz 10.39 folgt Satz 10.40, denn:
Annahme: Es gibt kein $u \in X$ mit $\{u\} = T(u)$.
Definiere $g : X \ni x \longmapsto T(x)\backslash\{x\} \in \mathrm{POT}(X)$. Es gilt

$$f(g(x)) + d(g(x), x) \leq f(x) \, , \, x \in X \, .$$

Aus Satz 10.39 folgt die Existenz von $z \in X$ mit $z \in T(z)$. Dies ist auch ein Fixpunkt von g nach Satz 10.37.
Aus Satz 10.40 folgt Satz 10.32, denn:

Definiere $T : X \ni x \longmapsto \{y \in X | f(y) + d(x,y) \le f(x)\} \in \mathrm{POT}(X)$. Auf Grund der Voraussetzung in Satz 10.32 gilt $T(x) \neq \emptyset$ für alle $x \in X$.
Annahme: Es gibt kein $\overline{x} \in X$ mit $f(\overline{x}) = \inf_{v \in X} f(v)$. Nun gilt

$$f(y) + d(x,y) \le f(x)\,, y \in T(x), x \in X\,.$$

Aus Satz 10.40 folgt die Existenz von $z \in X$ mit $\{z\} = T(z)$. Dies zeigt, dass kein $v \in X$ esistiert mit $f(y) + d(y,z) \le f(z)$, was ein Widerspruch ist.
Aus Satz 10.40 folgt Satz 10.41, denn:
Dies ist offensichtlich. Aus Satz 10.41 folgt Satz 10.42, denn:
Setze $g(x) := \frac{1}{2} f(x), x \in X$. Ist $d(x, T(x)) = 0$, dann ist $x \in T(x)$, da $T(x)$ abgeschlossen ist. Hat also T keinen Fixpunkt, dann ist $d(x, T(x)) > 0$ für alle $x \in X$. Sei $y \in T(x)$ so, dass $d(x,y) < \frac{1}{2} d(x, T(x))$. Dann haben wir

$$d(x,y) \le \frac{1}{2} d(x, T(x)) \le \frac{1}{2}(f(x) - \sup_{y \in T(x)} f(y)) \le g(x) - g(y)\,.$$

Da g eigentlich, unterhalbstetig und nach unten beschränkt ist, hat T einen Fixpunkt nach Satz 10.41.
Aus Satz 10.42 folgt Satz 10.40, denn Satz 10.40 ist ein Spezialfall von Satz 10.42. ■

1.9 Conclusion and comments

Nonlinear functional analysis is the study of operators lacking the property and equations which are governed by such operators. The fixed point principle is the bunch of methods to translate or reformulate (analytic) problems in the applied sciences to equations of fixed point type, mostly governed by nonlinear mappings.

The most earliest observation that a fixed point principle is helpful is the method of computing the square root of a nonnegative number known as Babylonian method which is nowadays considered as a Newton-type method. Another method which shows that the fixed point principle is very powerful is the proof that a matrix with nonnegative entries has an eigenvalue which is nonnegative (see exercises). Other examples we have seen in Section 1.1.

In the sections of this chapter we have presented the most important fixed point theorems in the analytic study of problems in the applied science: Banach's and Schauder's fixed point theorem. In our context, Brouwer's fixed point may be considered as a result which enables us to proof the Schauder fixed point theorem. The fixed point theorem of Darbo is the central result of a unified approach for Banach's and Schauder's theorem.

A first impression for fixed point theorems for nonexpansive mappings is the proof of a version of the fixed point theorem of Browder, Göhde, Kirk. As we will see, fixed point theory for nonexpansive mappings benefit mostly of geometric properties of Banach spaces.

There is a lot of fixed point theorems mostly located in other topics of mathematics: the fixed point theorems of Borsuk, Krasnoselski, Caristi (see Section 10.6), Knaster-Tarski, Lefschetz, Kakutani; see [2, 8, 9].

1.10 Exercises

1.) Let $a < b$ let $F : [a, b] \longrightarrow [a, b]$ be continuous and let F be differentiable in (a, b). We assume: There exists $q \in [0, 1)$ with

$$|F'(\xi)| \leq q \text{ for all } \xi \in (a, b)\,. \tag{1.48}$$

Show: F posesses a uniquely determined fixed point.

2.) Let $X := \mathbb{R}$, $\|\cdot\| := |\cdot|$, $M := (0, \infty)$, $F : M \ni t \longmapsto qt \in M$ with $q \in [0, 1)$. Show that no $z \in M$ exists with $z = F(z)$.

3.) Consider in the inner product space $(\mathbb{R}^n, \|\cdot\|)$ and a continuous mapping $g : \overline{B}_r \longrightarrow \mathbb{R}^n$ with

$$\langle g(x)|x\rangle \geq 0\,,\ x \in \mathbb{R}^n \text{ with } \|x\| = r\,.$$

Show: The equation $g(x) = \theta$ has a solution in $\overline{B}_r$.

4.) Let $\mathcal{X}$ be a Banach space, let $C \subset X$ open and convex, let $\theta \in C$.
Show: For all $x \in \mathcal{X}\backslash C$ there exists a uniquely determined $\sigma(x) \in (0, 1]$ with $\sigma(x)x \in \partial C$.

5.) Let $\mathcal{X}$ be a Banach space and let $A \subset X$ be compact. Consider continuous mappings $F_i : A \longrightarrow X, i \in \mathbb{N}$, and a mapping $F : A \longrightarrow X$. Suppose that $\lim_i \sup_{x\in A} \|F_i(x) - F(x)\| = 0$. Show:

(a) F is continuous.

(b) If each F_i has a fixed point then F has a fixed point.

6.) Consider in $C[0, 1]$, endowed with the supremum norm, the (nonlinear) mapping

$$G(x)(t) := \int_0^1 \kappa(t, s)f(s, x(s))ds\,,\ t \in [0, 1], x \in C[0, 1]\,.$$

Here, $\kappa \in C([0, 1] \times [0, 1]), f \in C([0, 1] \times \mathbb{R})$. Show: If

$$|f(s, u) - f(s, v)| \leq L|u - v|\,,\ s \in [0, 1], u, v \in \mathbb{R}, L \cdot \max_{t\in[0,1]} \int_0^1 |\kappa(t, s)|ds < 1,$$

then $G : C[0, 1] \longrightarrow C[0, 1]$ is a contraction.

7.) Let $\mathcal{X}$ be a Banach space, let V be a compact subset of $\mathcal{X}$ and let $F : V \longrightarrow V$ be contractive, i. e.

$$\forall x, y \in V, x \neq y,\ (\|F(x) - F(y)\| < \|x - y\|)\,. \tag{1.49}$$

Show:

(a) F has a uniquely determined fixed point $\overline{x}$.

(b) The iteration $x^{n+1} := F(x^n), x^0 = x$, converges for each $x \in \mathcal{X}$ to $\overline{x}$.

Hint for (b): One can show that the function $x \longmapsto \|x-F(x)\|$ along the iteration is monotone decreasing.

8.) Let X be a compact metric space. Show: X is separable.

9.) Let $C[a,b]$ be the space of continuous functions on $[a,b]$ endowed with the maximum-norm. Show that $C[a,b]$ is separable but not compact.

10.) Consider the following system of nonlinear equations:

$$2x_1^2 - x_2^2 - 8x_1 = 0\,,\ x_1^2 + x_1x_2 - 4x_2 + 1 = 0$$

Find a reformulation as a fixed point equation in $\mathbb{R}^2$ and solve this equation (by Brouwer's fixed point theorem).

11.) Consider

$$F : \mathbb{R}^2 \ni (x_1, x_2) \longmapsto \frac{1}{6}(x_1 e^{-x_2^2} + x_1x_2 + 3, \ln(1 + x_1^2 + x_2^2) - 1) \in \mathbb{R}^2\,.$$

Let $\mathbb{R}^2$ be endowed with the maximum norm $\|\cdot\|_\infty$.

(a) Show that F is Lipschitz-continuous in $[0,1]\times[0,1]$ with Lipschitz-constant $c = \frac{5}{6}$.

(b) Show that there exists a uniquely determined fixed point of F.

(c) How many steps k do we need to obtain an accuracy $\|x^k - z\|_\infty \le 10^{-3}$ using the fixed point iteration $x^{k+1} := F(x^k)$ starting with $x^0 := (0,0)$.

12.) Show that

$$f : [0,\infty) \ni x \longmapsto \frac{x + 1/2}{x+1} \in [0,\infty)$$

is a contraction. Compute the fixed point of f and an approximation x^1 using the starting point $x^0 = 1$ for the fixed point iteration.

13.) Show that the following system of nonlinear equations

$$\sin(x_1 + x_2) - x_2 = 0\,,\ \cos(x_1 + x_2) - x_1 = 0$$

is solvable in $\mathbb{R}^2$.

14.) Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be a matrix with entries $a_{ij} \ge 0$. Show that A has a nonnegative eigenvalue λ with associated eigenvector $x = (x_i)$ with entries $x_i \ge 0$.

15.) Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a continuous mapping with

$$\|F(x) - F(y)\| \ge c\|x - y\| \text{ for all } x, y \in \mathbb{R}^n$$

with $c > 1$. Show: If F is surjective then F has a uniquely determined fixed point.

16.) Let $C[0,1]$ endowed with the maximum-norm $\|\cdot\|_\infty$. Let $C := \{x \in C[0,1] : 0 = x(0) \le x(s) \le x(1) = 1\}$ and consider

$$T : C[0,1] \ni x \longmapsto T(x) \in C[0,1] \text{ with } T(x)(s) := sx(s), s \in [0,1]\,.$$

Show:

(1) T is a linear nonexpansive mapping with $\|T\| = 1$.

(2) $T(x) \in C$ if $x \in C$.

(3) C is a bounded closed convex set.

(4) T posesses a uniquely determined fixed point belonging to $C[0,1]\backslash C$.

(5) There exists $x_t \in C$ with $x_t = tu + (1-t)T(x_t)$ where $u \in C, u(s) := s$.

(6) $\inf_{x\in C} \|x - T(x)\| = 0$.

(7) $\lim_n \|x^0 - T^n(x^0)\|_\infty = \text{diam}(C)$.

17.) Let $\mathcal{X}$ be a Banach space, let $C \subset \mathcal{X}$ be compact and let $F : X \longrightarrow X$ be continuous. Then the following statements are equivalent:

(a) $\inf_{x\in X} \|x - F(x)\| = 0$.

(b) F posesses a fixed point.

18.) Let Xs be a Banach space, let $C \subset \mathcal{X}$ be compact and convex and let $F : X \longrightarrow X$ be nonexpansive. Then F posesses a fixed point.

19.) Let $\mathcal{H}$ be the euclidean space $\mathbb{R}^2$ and let $C_0 := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$, $C_1 := \{(x,y) \in \mathbb{R}^2 : x \ge 0, |y| \le x\}$, $C_2 := \{(x,y) \in \mathbb{R}^2 : x \le 0, |y| \le |x|\}$ und $C := (C_0 \cap C_1) \cup (C \cap C_2)$. Show: $C \subset \text{co}(\{P_C(x,y) : (x,y) \in \mathbb{R}^2\backslash C\})$.

20.) Let $\mathcal{X}$ be a Banach space and let U be a closed subspace of $\mathcal{X}$. Consider the quotient space

$$\mathcal{X}/U := \{[x] : x \in \mathcal{X}\} \text{ where } [x] := \{x + u : u \in U\}.$$

Show that $\|[x]\| := \text{dist}(x, U), x \in \mathcal{X}$, defines a norm in $\mathcal{X}/U$.

21.) Let $\mathcal{X}$ be a Banach space nd let U be a closed subspace of $\mathcal{X}$, different from $\mathcal{X}$. Show that for every $\varepsilon >=$ there exists $x \in \overline{B}_1$ with $\text{dist}(x, U) \ge 1 - \varepsilon$.

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