Arithmetic properties of Schwarz maps

Hironori Shiga, Yoshio Suzuki and Jürgen Wolfart

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The subject of this article belongs to the general question *Under which condition(s) suitably normalized transcendental functions take algebraic values at algebraic arguments?* Already the classical examples of Weierstrass' result concerning the exponential function and Theodor Schneider's result about the elliptic modular function show that arguments and values in these cases are of particular arithmetical interest. Here we try to answer this question for the case of Schwarz maps belonging to Appell–Lauricella hypergeometric functions F_D in two and more variables, generalizing our results in [SW2] about Schwarz triangle functions, i.e. for the classical Gauss hypergeometric functions in one variable.

The first section contains the necessary basic notations, conventions and the known machineries from hypergeometric functions, transcendence and abelian varieties. The second presents the main result about necessary and sufficient conditions for algebraic and non–algebraic values of Schwarz maps. These are valid only in some Zariski open subset of the domain of definition, and the last section shows by giving some examples that this restriction is quite natural.

1 Basics

1.1 Appell-Lauricella functions

With the Pochhammer symbol $(a, 0) := 1$, $(a, n) := a(a+1)\cdot\ldots\cdot(a+n-1)$ for complex a and positive integers n the Appell–Lauricella functions F_1 in the N complex variables $x_2, x_3, \ldots, x_{N+1}$ with parameters $a, b_2, \ldots, b_{N+1}$ and $c \in \mathbf{C}$, $c \neq 0, -1, -2, \ldots$ — for $N > 2$ often denoted F_D in the literature — can be defined by the series

$$F_1(a, b_2, \ldots, b_{N+1}, c; x_2, \ldots, x_{N+1}) := \sum_{n_2} \ldots \sum_{n_{N+1}} \frac{(a, \sum_j n_j) \prod_j (b_j, n_j)}{(c, \sum_j n_j) \prod_j (1, n_j)} \prod_{j=2}^{N+1} x_j^{n_j} \ , \qquad (1)$$

each n_j running from 0 to ∞. The series converges if all $|x_j| < 1$. We will use almost everywhere its integral representation

$$\frac{1}{B(1-\mu_1, 1-\mu_{N+2})} \int u^{-\mu_0}(u-1)^{-\mu_1} \prod_j (u-x_j)^{-\mu_j} du \ , \qquad (2)$$

where the *exponential parameters* $\mu_0, \mu_1, \ldots, \mu_{N+2}$ are related to the parameters in the series representation by

$$\begin{aligned} \mu_j &= b_j \quad \text{for all} \quad j = 2, \ldots, N+1 && (3) \\ \mu_0 &= c - \sum_{j=2}^{N+1} b_j && (4) \\ \mu_1 &= 1 + a - c && (5) \\ \sum_{j=0}^{N+2} \mu_j &= 2\,, \quad \text{i.e.} \quad \mu_{N+2} := 1 - a\,. && (6) \end{aligned}$$

If μ_1 and μ_{N+2} are real < 1, the integration path can be choosen between $u = 1$ and $u = \infty$ avoiding the other singularities and choosing an appropriate branch of the differential

$$\eta := u^{-\mu_0}(u-1)^{-\mu_1} \prod_{j=2}^{N+1} (u - x_j)^{-\mu_j}\, du\,. \tag{7}$$

Throughout the present paper, we will concentrate on the following

Restricted Assumptions. *We suppose that all parameters a, b_j, c are rational numbers, hence also all exponential parameters μ_j, $j = 0, \ldots, N+2$, moreover we assume to be all $\mu_j \notin \mathbf{Z}$. To avoid the singularities of the corresponding hypergeometric differential equations, we consider only those arguments in which all variables x_j are pairwise different and distinct from* 0 *and* 1.

These assumptions are not too restrictive: on the hyperplanes avoided by the last condition the hypergeometric functions restrict to hypergeometric functions in less variables, and for an integer exponential parameter μ_j also some obvious reduction of the integral is possible to a function rational in x_j and hypergeometric in the other variables, see e.g. [CW1, §4]. If in the integral representation μ_1 or μ_{N+1} are > 1, an integral between 0 and ∞ is no longer convergent. We have to replace it by a *Pochhammer cycle* around 0 and ∞ avoiding all other singularities of η, and to modify the integral by an algebraic factor. For its precise definition, see e.g. [Kl], [Y], [Ar] and for the method of factor determination in particular [STW, Sec. 5]. Since algebraic factors do not count for our considerations, we will always assume our integrals to be integrals over Pochhammer cycles. Under our assumptions we have the additional advantage that the integrals become periods $\int \eta$ of the second kind on nonsingular projective algebraic curves $X = X(k; x_2, \ldots, x_{N+1})$ with affine (in general singular) model

$$y^k \;=\; u^{k\mu_0}(u-1)^{k\mu_1} \prod_{j=2}^{N+1} (u - x_j)^{k\mu_j}\,, \tag{8}$$

taken for the differential $\eta = \eta(x_2, \ldots, x_{N+1}) = du/y$. A basis for the solution space of the corresponding system of hypergeometric differential equations $E_1(a, b_2, \ldots, b_{N+1}, c)$ (see [AK], [Y]) is given by a system of period integrals $\int_{\gamma_0} \eta(x), \int_{\gamma_1} \eta(x), \ldots, \int_{\gamma_N} \eta(x)$ where x

denotes the n–tuple $(x_2, \ldots, x_{N+1})$ of variables and the integration paths γ_i are suitably chosen Pochhammer cycles in the u–plane, each of them going around a pair of singularities $u = 0, 1, x_2, \ldots, x_{N+1}, \infty$.

1.2 Jacobians and Prym varieties

Following the method of [CW2, Sec. 3] for $N = 2$ and [SW1, §5] we keep the right hand side of equation (8) and replace the left hand side with y^d where d denotes a proper divisor of k , obtaining a smooth complex projective algebraic curve $X(d, x)$ and an obvious epimorphism $X(k, x) \to X(d, x)$. It induces an epimorphism of Jacobians

$$m_d \ : \ \mathrm{Jac}\, X(k, x) \ \to \ \mathrm{Jac}\, X(d, x) \ .$$

Let the *Prym variety* $T(k, x)$ be the connected component of 0 in the intersection $\bigcap \mathrm{Ker}\, m_d$, d running over all proper divisors of k . Then $T(k, x)$ is an abelian variety of complex dimension $\frac{N+1}{2}\, \varphi(k)$ where φ denotes Euler's function. As in the case $N = 1$ (see [Wo] or [SW2, Sec. 1.2]) it has *generalized complex multiplication* by a cyclotomic field, more precisely we have

$$\mathbf{Q}(\zeta_k) \subseteq \mathrm{End}_0 T(k, x) := \mathbf{Q} \otimes_{\mathbf{Z}} \mathrm{End}\, T(k, x)$$

induced by the automorphism of the curve $X(k, x)$ described on its singular model by

$$\sigma \ : \ (u, y) \ \mapsto \ (u, \zeta_k^{-1} y) \ , \ \zeta_k = e^{\frac{2\pi i}{k}} \ .$$

For a more precise description of the complex analytic family of abelian varieties containing $T(k, x)$ we need also its *type*. It is determined as follows. On the vector space $H^0(T(k, x), \Omega)$ of first kind differentials on $T = T(k, x)$ we have an induced action of $\mathbf{Q}(\zeta_k)$ splitting the vector space in eigenspaces $V_\sigma = V_n, \ n \in (\mathbf{Z}/k\mathbf{Z})^*$, of differentials ω with the property

$$\omega \circ \sigma \ = \ \zeta_k^n \cdot \omega \ .$$

According to Chevalley and Weil [ChWe] the dimensions of the eigenspaces can be calculated as

$$r_n := \dim V_n = -1 + \sum_{j=0}^{N+2} \langle n\mu_j \rangle \tag{9}$$

where $\langle \alpha \rangle := \alpha - [\alpha]$ denotes the fractional part of α . It is easy to see that

$$r_n + r_{-n} \ = \ N + 1$$

for all n coprime to k . We will always identify the differentials of the first kind with certain holomorphic differentials on the curve $X(k, x)$ where we can study the action of σ in an obvious way. If e.g. all exponential parameters satisfy $\mu_j < 1$, the differential $\eta = \frac{du}{y}$

in equation (7) is in this identification an element of V_1 . Now the *type* of $T(k,x)$ can be introduced as the formal sum

$$\sum_{\sigma \in \mathrm{Gal}\,\mathbf{Q}(\zeta_k)/\mathbf{Q}} (\dim V_\sigma) \cdot \sigma$$

or in simplified version as the $\varphi(k)$–tuple $(r_n \mid n \in (\mathbf{Z}/k\mathbf{Z})^*))$.

We should remark by the way that we use also two further and similar identifications of (co)homology groups. First, by the natural action of the endomorphism algebra we can consider the homology group $H^1(T(k,x),\mathbf{Z})$ of rank $(N+1)\varphi(k)$ as being a rank $(N+1)$–module over $\mathbf{Z}[\zeta_k]$ whose cycles all come from cycles on $X(k,x)$. In particular, we can consider the Pochhammer cycles $\gamma_0, \gamma_1, \ldots, \gamma_N$ as generators of the $\mathbf{Q}(\zeta_k)$–vector space $H^1(T(k,x),\mathbf{Q}) := \mathbf{Q} \otimes_{\mathbf{Z}} H^1(T(k,x),\mathbf{Z})$.
Second, we identify the space $H^1_{DR}(T(k,x))$ of second kind differentials on our Prym variety with a subspace of the second kind differentials on the curve. As $H^0(T(k,x),\Omega)$, it splits in $\mathbf{Q}(\zeta_k)$–eigenspaces $W_n \supseteq V_n$, all of dimension $\dim W_n = N+1$. This can be shown either by complex algebraic geometry ([GH] or [Be, §4, Remarque 1]) or as another version of a well known principle concerning *associate functions* to be explained now.

1.3 Associate hypergeometric functions

Hypergeometric functions $F_1(a, b_2, \ldots, b_{N+1}, c; x_2, \ldots, x_{N+1})$ are called *associate* if their parameter $(N+2)$–tuples $(a, b_2, \ldots, b_{N+1}, c)$ are congruent mod $\mathbf{Z}^{N+2}$, in other words if all their exponential parameters μ_j differ by integers only (always under the condition that $\sum \mu_j = 2$ is preserved, of course). If we write their differential in a slightly more general way than (7) as $\eta = r(u)du/y^n \in W_n$ with a rational function $r(u) = u^{m_0}(u-1)^{m_1}\prod(u-x_j)^{m_j}$, all $m_j \in \mathbf{Z}$, then a differential for an associate hypergeometric function differs from that one by another choice of the rational function r only, but obviously remaining in W_n . So if we keep the Pochhammer cycle γ_i on the curve fixed — identified as above with a generator of $H_1(T(k,x),\mathbf{Q})$ — the periods $\int_{\gamma_i} \eta(x)$, $\eta \in W_n$, generate as functions of x a complex vector space of associate hypergeometric functions. Now it is known that this vector space has dimension $N+1$ over the space of rational functions in x [Y]. Recall that all parameters are rational, so all normalizing Beta values for associate hypergeometric functions differ by rational factors only such that this dimension result may be formulated for the differentials as follows.

Lemma 1.1. *Under our assumptions, for all fixed n coprime to k , any $N+2$ different differentials of type*

$$\eta(x) \;=\; \frac{u^{m_0}(u-1)^{m_1}\prod(u-x_j)^{m_j}}{y^n}\, du \in W_n \;, \quad \textit{all} \quad m_j \in \mathbf{Z} \;, \tag{10}$$

satisfy a nontrivial linear relation modulo exact differentials with coefficients in the polynomial ring $\mathbf{Q}[x_2, \ldots, x_{N+1}]$.

We will use this fact also in another way, concentrating on the behaviour in special fixed arguments x . Henceforth we will call these differentials of type (10) in a common eigenspace W_n *associate differentials.*

Lemma 1.2. *For all n coprime to k and any $N+1$ different associate differentials $\eta_\nu(x) \in W_n \subset H^1_{DR}(T(k,x))$ there is a Zariski dense subset $Z \subset \mathbf{C}^N$ of arguments x in which the $\eta_\nu(x)$ form a basis of W_n .*

To the complementary set $\mathbf{C}^N \backslash Z$ we always add the hyperplanes $x_j = 0\,,\ 1$ and $x_i = x_j$ forbidden by our restrictive assumptions.

1.4 Schwarz maps

For any $\eta(x)$ of type (10) and any such Zariski dense subset Z as in Lemma 1.2 we define the *Schwarz map* as the function

$$D_\eta \ : \ Z \ \rightarrow \ \mathbf{P}^N(\mathbf{C}) \ : \ x \ \mapsto \ \left(\int_{\gamma_0} \eta(x) \ : \ \ldots \ : \ \int_{\gamma_N} \eta(x) \right) \ .$$

(Since the components form a basis of a system of linear differential equations and since we consider regular points only, these components cannot all vanish.) As always with Schwarz maps, this is a priori only locally well defined since analytic continuation in Z is not globally possible without deforming the Pochhammer cycles. Since Z is not simply connected, D_η is multivalued, and this multivaluedness can be described by the (linear) action of the homotopy group of Z either on the solution space of the system of hypergeometric differential equations or on the homology $H_1(T(k,x),\mathbf{Z})$, see [Y]. The components of the image could have given also in terms of normalized basis solutions of a hypergeometric differential equation system of Appell–Lauricella type D since the normalizing Beta factors are all the same up to rational factors, hence do not count if we work with projective coordinates, at least up to projective linear transformations defined over $\mathbf{Q}$.

The central question is now: *suppose the argument $x = \tau$ is an algebraic point of Z , i.e. has all its coordinates $x_j = \tau_j \in \overline{\mathbf{Q}}$. Under which conditions $D_\eta(\tau)$ is an algebraic point, i.e. is in $\mathbf{P}^N(\overline{\mathbf{Q}})$, in other words has coordinates which are $\overline{\mathbf{Q}}$–multiples of each other?* Note first that the curve $X(k,\tau)$ is defined over $\overline{\mathbf{Q}}$ as well as its Jacobian and the Prym variety $T(k,\tau)$. Moreover, all differentials in (7) or (10) are defined over $\overline{\mathbf{Q}}$, and therefore we will consider all cohomology groups $H^0(T(k,\tau),\Omega)$, $H^1_{DR}(T(k,\tau))$ as vector spaces over $\overline{\mathbf{Q}}$. The question is well posed since under the monodromy action the base $\gamma_0,\ldots,\gamma_N$ changes only under a matrix in $\mathrm{GL}_{N+1}(\mathbf{Z}[\zeta_k])$ and the differential $\eta(\tau)$ remains unchanged, so the algebraicity of the value $D_\eta(\tau)$ remains unchanged.

1.5 Monodromy groups and modular groups

Finally we should mention that all abelian varieties T with common dimension and polarization, with generalized CM by $\mathbf{Q}(\zeta_k)$ and of the same CM type can be parametrized by a complex symmetric domain $\mathbf{D}$. According to Siegel [Si] and Shimura [Sh] $\dim \mathbf{D} = \sum_R r_n r_{-n}$ where the summation runs over a system R of representatives of $(\mathbf{Z}/k\mathbf{Z})^*$ mod $\{\pm 1\}$, in other words over a system of representations of the CM field modulo complex conjugation. The symmetric domain is a product of spaces $\mathbf{H}_{r_n, r_{-n}}$ of $r_n \times r_{-n}$–matrices z with the property that $1 - z^t\overline{z}$ is positive hermitian. Two points on $\mathbf{D}$ correspond to isomorphic abelian varieties if and only if they lie in the same orbit of the *modular group* for this family. Since the monodromy group of our hypergeometric function does not change the curve $X(k, x)$ nor its Jacobian or the Prym variety, we can consider it as a subgroup of the modular group. Several special cases have to be mentioned (recall $r_n + r_{-n} = N + 1$).

- In the case that r_n or $r_{-n} = 0$ the matrix space degenerates to one point, so the factor $\mathbf{H}_{r_n, r_{-n}}$ of $\mathbf{D}$ can be omitted. This may even occur for *all* n. In that case, there is only one isogeny class of abelian varieties of this CM type, and this is necessarily one of *complex multiplication type* in the narrow sense ([ShT] or [La, Ch 1]), i.e. isogenous to a product of simple abelian varieties A whose endomorphism algebra $\mathrm{End}_0 A$ is a (CM) field of degree $2 \dim A$. In our construction, this occurs precisely if the hypergeometric functions are algebraic functions, and these cases occur if and only if there is an x such that $T(k, x)$ is isogenous to a power of a simple abelian variety with CM. For the classical Schwarz case $N = 1$ see [SW2, Prop. 2.8], and in the Appell–Lauricalla cases $N > 1$ these possibilities are discussed in [Sa] and [CW1]. For $N > 3$ there are no such cases.

- In the one variable case ($N = 1$) $\mathbf{D}$ is isomorphic to a product of upper half planes. This case is treated in [SW2].

- In the case $N > 1$ all nontrivial factors of $\mathbf{D}$ are isomorphic to complex N–balls if r_n or $r_{-n} = 1$. This is necessarily the case for $N = 2$.

- If in these cases $\mathbf{D}$ consists of only one factor, i.e. if $r_n = 1$ for precisely one n coprime to k, the Prym varieties $T(k, x)$ form a Zariski dense subset of all abelian varieties of its CM type, and the monodromy group is commensurable to the modular group. In other words, it is arithmetically defined.

- If in these cases moreover $\eta = \omega$ is a generator of the one–dimensional eigenspace $V_n \subset H^0(T(k, x), \Omega)$, the Schwarz map D_ω has — up to linear transformations — its images in $\mathbf{D}$ and is a converse to a mapping composed by suitably normalized automorphic functions for an arithmetic group commensurable to the modular group mentioned above.

- Therefore, in these cases our central question is answered by the Main Theorem and its Corollary in [SW1]: for an algebraic τ we have an algebraic value $D_\omega(\tau)$ if and only if the Prym variety $T(k,\tau)$ is of CM type.

For the last statement, [SW1, Cor. 6] gives a slight generalization to a more complicated situation involving $N+1$ differentials of the first kind, but Thms. 3.4 and 3.5 of [SW2] show that even in the one variable case, such a neat result cannot be expected for Schwarz maps coming from differentials of the second kind. In Section 2 we will try to extend this observation to $N>1$.

1.6 Period relations and transcendence

The main instrument to get transcendence results in this context is Wüstholz' analytic subgroup theorem [Wü]. The proof for its consequence to period relations is worked out by Paula Cohen in the appendix of [STW]. We state its content as

Lemma 1.3. *Let A be an abelian variety isogenous over $\overline{\mathbf{Q}}$ to the direct product $A_1^{k_1} \times \ldots \times A_N^{k_N}$ of simple, pairwise non-isogenous abelian varieties A_ν defined over $\overline{\mathbf{Q}}$, with A_ν of dimension n_ν, $\nu = 1,\ldots,N$. Then the $\overline{\mathbf{Q}}$–vector space $\widehat{V}_A$ generated by $1, 2\pi i$ together with all periods of differentials, defined over $\overline{\mathbf{Q}}$, of the first and the second kind on A, has dimension*

$$\dim_{\overline{\mathbf{Q}}} \widehat{V}_A \;=\; 2 \,+\, 4\sum_{\nu=1}^{N} \frac{n_\nu^2}{\dim_{\mathbf{Q}} \mathrm{End}_0 A_\nu}\;.$$

This Lemma governs all $\overline{\mathbf{Q}}$–linear relations beween periods, but it does not say anything about possible nonlinear relations. Riemann's period relations give examples of those. Since we will need them in the last section, we state them here as

Lemma 1.4. *(Riemann bilinear relations)*
Let X be a compact Riemann surface of genus g, and let $\{A_1,...,A_g,B_1,...,B_g\}$ be a canonical homology basis of X with $A_iB_j = \delta_{ij}, A_iA_j = B_iB_j = 0 \quad (1 \le i,j \le g)$. For a holomorphic differential ω and a meromorphic differential η with poles s_λ we have

$$\sum_{i=1}^{g}(\int_{A_i}\omega\int_{B_i}\eta - \int_{B_i}\omega\int_{A_i}\eta) = 2\pi\sqrt{-1}\sum_{\lambda} Res_{s_\lambda}(\eta)\int_{s_0}^{s_\lambda}\omega.$$

2 Algebraic values of Schwarz maps

2.1 A necessary condition

Theorem 2.1. *Let τ be an algebraic point of $\mathbf{C}^N$, all components pairwise different and $\neq 0,1$. Under the restricted assumptions and for the differentials $\eta \in W_n$ of type (10)* *suppose that the Schwarz map D_η takes an algebraic value $D_\eta(\tau) \in \mathbf{P}^N(\overline{\mathbf{Q}})$. Then the Prym variety $T(k,\tau)$ has a simple CM factor S with complex multiplication by a CM field K such that η is induced by a K–eigendifferential on S.*

Proof. For a first kind differential $\eta = \omega$ this is a consequence of [SW1, Cor. 1]: since $\gamma_0, \ldots, \gamma_N$ form a basis of the homology of $T(k,\tau)$ as $\mathbf{Q}(\zeta_k)$–vector space and η is an eigendifferential for the action of this field, all periods $\int_\gamma \eta$ are algebraic multiples of each other, so the hypotheses of [SW1, Cor. 1] are satisfied.
For the second kind differentials η not belonging to $H_0(T(k,\tau),\Omega)$, we can deduce the existence of a corresponding first kind differential ω with the same property from this one using the fact that representations of endomorphisms on the period lattice and on the quasiperiod lattice are complex conjugate to each other, or by passing to the dual abelian variety, see [Be, §4, Remarques 1 et 2]. Therefore we can apply the argument of the first part again.

In the case $N = 1$ there is a much more precise statement due to the fact that the complement of a CM factor of $T(k,\tau)$ is necessarily of CM type as well [SW2, Prop. 2.4], hence $T(k,\tau)$ is itself of CM type. For $N > 1$, any factor A of $T(k,\tau)$ with CM by $\mathbf{Q}(\zeta_k)$ has — up to isogeny — still a complement B with generalized CM by $\mathbf{Q}(\zeta_k)$ [Be, Thm 1], but since $\dim B > \frac{1}{2}\varphi(k)$, it is not necessarily of CM type in the narrow sense.

2.2 Periods on Pryms with CM factors

A closer look to the proof of Theorem 2.1 and its background in transcendence theory of periods — see [CW1] — shows that D_η can only be algebraic if (up to isogeny) there is a decomposition $T(k,\tau) = A \oplus B$ in two abelian varieties such that A has complex multiplication by $\mathbf{Q}(\zeta_k)$ and η belongs to the first factor in the corresponding decomposition

$$H^1_{DR}(T(k,\tau) \;=\; H^1_{DR}(A) \oplus H^1_{DR}(B)\;.$$

This is necessarily the case if the hypergeometric functions are algebraic (see the preceeding section) since then $T(k,x)$ is isogenous to A^{N+1}, and this decomposition can be choosen in many ways such that we can suppose that η belongs to the factor A. Moreover we can then suppose $\eta \in H^1_{DR}(S)$, S a simple factor of A with complex multiplication by a CM field $K \subseteq \mathbf{Q}(\zeta_k)$ such that $2\dim S = [K:\mathbf{Q}]$ and η is a K–eigendifferential. In that case the vector space $\Pi_\eta \subset \mathbf{C}$ generated over $\overline{\mathbf{Q}}$ by all periods of η has dimension 1, see the arguments of [SW2, Prop. 2.8] which easily extend to the present case. Another situation where $\eta \in H^1_{DR}(A)$ with $\dim \Pi_\eta = 1$ occurs if $\eta = \omega$ is a first kind differential and $r_n = 1$, $V_n \cap H_0(A,\Omega) \neq \{0\}$ such that necessarily $V_n \cap H_0(B,\Omega) = \{0\}$ and ω belongs to A. In general, we cannot expect these hypotheses to be satisfied, as the next result shows.

Theorem 2.2. *Let P be a finite set of associate differentials $\eta_\nu(x) \in W_n$ of type (10), and suppose their monodromy group to be infinite. Then there is a Zariski open subset $Z \subset \mathbf{C}^N$ depending on P with the following property. For all algebraic $\tau \in Z$ at most $N+1$ among the Schwarz maps D_{η_ν} take algebraic values $D_{\eta_\nu}(\tau)$.*

Here we tacitely include the fact that associate hypergeometric functions have the same monodromy group: as above, we may read the monodromy group as an automorphism group of the homology of $T(k,x)$ keeping unchanged the differentials.
Proof. As we have seen in Section 1.5, $T(k,\tau)$ is never isogenous to a pure power of a simple abelian variety with complex multiplication. This will enable us to show that $\eta \in H^1_{DR}(T(k,\tau))$ with a one–dimensional $\overline{\mathbf{Q}}$–vector space Π_η generated by its periods are quite rare. In fact, by Wüstholz' analytic subgroup theorem and its consequences described in Lemma 1.3, period spaces Π_{η_j} of dimension 1 for all differentials η_j in a basis of H^1_{DR} can occur only if the abelian variety is of CM type, and even then the periods of the η_j are linearly independent over $\overline{\mathbf{Q}}$ if they belong to non–isogenous simple factors of $T(k,\tau)$ or to different K–eigenspaces of the same factor S, $K := \mathrm{End}_0 S$, since nontrivial linear combinations of these basis differentials have period spaces of higher dimension, see the arguments in the first part of the proof of [STW, Prop. 4.4]. Since S may occur with higher multiplicity in the decomposition of $T(k,\tau)$, these η with $\dim \Pi_\eta = 1$ may however form subspaces of $H^1_{DR}(T(k,\tau))$ of dimension m_i, more precisely they are K_i–eigenspaces of $H^1_{DR}(S_i^{m_i})$ if $T(k,\tau)$ is isogenous to the product $S_1^{m_1} \times \ldots \times S_s^{m_s}$ of simple non–isogenous factors S_j and if the factor S_i has CM by the field K_i. These K_i–eigenspaces are proper subspaces of $H^1_{DR}(T(k,\tau))$, and their intersection with W_n are proper subspaces $W^{(i)}$ of W_n as well, since the decomposition of $H^1_{DR}(T(k,\tau))$ in the subspaces W_n is compatible with the decomposition in the K_i–eigenspaces, and since $s > 1$ because $T(k,\tau)$ is not a pure power of a simple CM abelian variety.
A priori it is possible that all $\eta \in P \subset W_n$ fall in these $W^{(i)} \subset W_n$, $i = 1, \ldots, s$, common eigenspaces for the action of K_i and $\mathbf{Q}(\zeta_k)$. By construction, the different $W^{(i)}$ belong to factors $S_i^{m_i}$ of $T(k,\tau)$, all S_i simple with CM and pairwise non–isogenous. Then they form a direct sum of dimension $\sum_i \dim W^{(i)} \leq N+1$ inside W_n.
If P contains $\leq N+1$ elements, the statement of the theorem is trivial, so we may suppose that we have to consider at least $N+2$ differentials. If all of them lie in that finite union $\bigcup_i W^{(i)}$ of proper subspaces, there is — by the pidgeonhole principle — at least one i such that $W^{(i)}$ contains more than $\dim W^{(i)}$ elements of P. Since $\dim W^{(i)} < N+1$, there are in particular $N+1$ elements of P which are linearly dependent.
Now, Lemma 1.2 gives us the means to exclude this possibility: we can choose the Zariski dense subset $Z \subset \mathbf{C}^N$ in such a way that for all $\tau \in Z$ any $N+1$ among all (finitely many) subsystems of $N+2$ differentials in P are linearly independent. $\square$

3 Some application

In this section we show some illustrating examples of the theory stated in the preceeding section.

Recall that we considered the integral of the differential form

$$\eta = \eta(x,y) = u^{-\mu_0}(u-1)^{-\mu_1}(u-x)^{-\mu_2}(u-y)^{-\mu_3}du.$$

By the correspondence (3) to (6)

$$\begin{cases} b_j = \mu_j (j = 2, 3) \\ a = \mu_0 + \mu_1 + \mu_2 + \mu_3 - 1 \\ c = \mu_0 + \mu_2 + \mu_3. \end{cases}$$

the integral $\int \eta$ is a solution of the Appell hypergeometric differential equation $E_1(a, b_2, b_3, c)$.

We study here the Appell hypergeometric curves

$$P(x, y) : w^3 = u(u-1)(u-x)(u-y),$$

$$Q(x, y) : w^5 = u(u-1)(u-x)(u-y)$$

together with their corresponding differential equations $E_1(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1)$ and $E_1(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 1)$.

Note that in these cases the Prym variety $T(P(x, y))$ (resp. $T(Q(x, y))$) coincides with the Jacobi varity $\mathrm{Jac}(P(x, y))$ (resp. $\mathrm{Jac}(Q(x, y))$).

For the differential η, we define the Schwarz map by

$$D(\eta, x, y) = (\int_0^1 \eta(x, y) : \int_x^y \eta(x, y) : \int_1^\infty \eta(x, y)).$$

We investigate the Schwarz images of the differentials of second kind at some CM points (x, y).

For a hypergeometric curve $C : w^k = u^{k\mu_0}(u-1)^{k\mu_1}(u-x)^{k\mu_x}$ or an Appell curve $C : w^k = u^{k\mu_0}(u-1)^{k\mu_1}(u-x)^{k\mu_2}(u-y)^{k\mu_3}$, we have an action of $\zeta_k = e^{2\pi i/k}$, so the cyclotomic field $K = \boldsymbol{Q}(\zeta_k)$ can be considered as a subfield of $\mathrm{End}_0(\mathrm{Jac}(C))$.

As we studied in the preceeding section, the Schwarz image $D(\varphi, x, y)$ is algebraic only if $\mathrm{Jac}(C(x, y))$ has a simple component of CM type. But it is not a sufficient condition. We are going to look at this situation in detail by several examples.

3.1 General results

3.1.1 Decomposition of the Jacobian variety

Here we state how our Jacobi variety $\mathrm{Jac}(C(x, y))$ is decomposed in simple components. For it, we refer to the following theorems in Lang's text book [L].

Lemma 3.1. ([L], Theorem 3.1, p.8-9) *Let A be a g-dimensional abelian variety. Suppose F to be a subfield of* $\mathrm{End}_0(A)$. *Then*

(1) $[F : \boldsymbol{Q}] \leq g$

(2) $[F : \boldsymbol{Q}] = 2g$ *implies* $A \sim_{isog} B \times \cdots \times B$, *$B$: simple. If A is defined over $\overline{\boldsymbol{Q}}$ then the isogeny is defined over $\overline{\boldsymbol{Q}}$.*

Let X be a g -dimensional complex torus, and let $F = \boldsymbol{Q}(\xi)$ be a CM field of degree $2g$. We assume that $F \subseteq \mathrm{End}_0(X)$, and F to be a Galois extension of $\boldsymbol{Q}$.

In this case we have a basis system $\{\omega_1, \cdots, \omega_g\}$ of holomorphic differentials which are eigendifferentials for the action of F. Set

$$\xi(\omega_1) = \xi_1\omega_1, \ldots, \xi(\omega_g) = \xi_g\omega_g$$

We define the type $\Phi(F)$ of the action of F by

$$\Phi(F) = (\xi_1, \ldots, \xi_g).$$

$(\xi_1, \cdots, \xi_g)$ are conjugates of ξ. Together with their complex conjugates $\xi_1, \cdots, \xi_g, \overline{\xi_1}, \cdots, \overline{\xi_g}$ becomes a full set of conjugates of ξ. They correspond to $2g$ different embeddings $\sigma_1, \cdots, \sigma_g, \overline{\sigma}_1, \cdots, \overline{\sigma}_g$ of F into $\boldsymbol{C}$ with $\sigma_j(\xi) = \xi_j$. So $\Phi(F)$ is a "type" of F in the sense of subsection 1.2 with all $\dim V_\sigma \leq 1$. In case X is an abelian variety, we say X is an abelian variety of type (F, Φ).

Lemma 3.2. ([L], Theorem 3.5 (p.13))

Let A be an abelian variety of type (F, Φ). Set

$$H = \{\sigma \in \mathrm{Gal}(F/\boldsymbol{Q}) : \sigma\Phi = \Phi\}$$

and suppose B to be a simple factor of A with $K = \mathrm{End}_0(B)$. Then we have $H = \mathrm{Gal}(F/K)$. Especially $H = \{1\} \Longleftrightarrow A$ is simple.

3.2 CM Picard curves

We consider the Picard curve

$$P(x,y) : w^3 = u(u-1)(u-x)(u-y) \quad \text{for} \quad xy(x-1)(y-1)(x-y) \neq 0$$

and differentials of second kind of the form

$$\varphi = \frac{u^\ell du}{w^n}.$$

We have

$$\text{genus of } P(x,y) = 3$$

We assume the variables x, y to be algebraic numbers. Set

$$\begin{cases} \varphi_1 = \frac{du}{w}, \varphi_2 = \frac{du}{w^2}, \varphi_3 = \frac{udu}{w^2}, \\ \\ \varphi_4 = \frac{udu}{w}, \varphi_5 = \frac{u^2du}{w}, \varphi_6 = \frac{u^2du}{w^2}. \end{cases} \tag{11}$$

They form a basis of the deRham cohomology group $H^1_{DR}(P(x,y),\overline{\boldsymbol{Q}})$ and the system $\{\varphi_1,\varphi_2,\varphi_3\}$ gives a basis of $H^0(P(x,y),\Omega)$, the space of holomorphic differentials.

We note the following fact. If we know the algebraicity of $D(\varphi_i,x,y)$ for a fixed point $(x,y)\in\overline{\boldsymbol{Q}}^2$ for every index i, it does not mean that we can see the algebraicity of $D(\varphi,x,y)$ for a generic differential of $H^1_{DR}(P(x,y),\overline{\boldsymbol{Q}})$.

Set $\varphi=\frac{u^\ell du}{w^n}$. Now we study the Schwarz values $D(\varphi,\frac{1+i}{2},\frac{1-i}{2})$ for the special Picard curve $P(\frac{1+i}{2},\frac{1-i}{2}): w^3=u(u-1)(u^2-u+\frac{1}{2})$.

We have the following results

Theorem 3.1. *For the Picard curve $P(\frac{1+i}{2},\frac{1-i}{2}): y^3=x(x-1)(x^2-x+\frac{1}{2})$, we have*

$$\mathrm{Jac}(P(\frac{1+i}{2},\frac{1-i}{2}))\sim E(\zeta_3)\oplus E(i)^2\ ,$$

and

$$D(\varphi_1;\frac{1+i}{2},\frac{1-i}{2}),D(\varphi_2;\frac{1+i}{2},\frac{1-i}{2})\in\overline{\boldsymbol{Q}},$$

$$D(\varphi_3;\frac{1+i}{2},\frac{1-i}{2}),D(\varphi_4;\frac{1+i}{2},\frac{1-i}{2}),D(\varphi_5;\frac{1+i}{2},\frac{1-i}{2}),D(\varphi_6;\frac{1+i}{2},\frac{1-i}{2})\notin\overline{\boldsymbol{Q}}.$$

Remark 3.1. *Suppose $n=1$ or $2\pmod 3$ and $0\le\ell\le 30$. We have $D(\varphi,\frac{1+i}{2},\frac{1-i}{2})\in\boldsymbol{P}^2(\overline{\boldsymbol{Q}})$ if and only if $\ell=0$. This result illustrates our general Theorem 2.2 that inside some Zariski open set Z we have at most 3 algebraic Schwarz values $D(\varphi,\tau)$ for varying differentials $\varphi\in V_n$.*

For the Schwarz values $D(\varphi,\zeta_3,\zeta_3^2)$ corresponding to the special Picard curve $P(\zeta_3,\zeta_3^2)$: $w^3=u(u^3-1)$, we have:

Theorem 3.2. *$D(\varphi,\zeta_3,\zeta_3^2)\in\boldsymbol{P}^2(\overline{\boldsymbol{Q}})$ for any $\ell,n\in\boldsymbol{Z}$.*

This fact illustrates that (ζ_3,ζ_3^2) belongs to the exceptional set $\mathbf{C}^2\setminus Z$ in Theorem 2.2.

Proof of Theorem 3.1

Set $P=P(\frac{1+i}{2},\frac{1-i}{2}): w^3=u(u-1)(u^2-u+\frac{1}{2})$, and set $\Sigma: t^3=s^4-1$. We have a biholomorphic isomorphism $T:P\to\Sigma$ over $\overline{\boldsymbol{Q}}$ by

$$(s,t)=T(u,w)=(2u-1,2^{\frac{4}{3}}w).$$

The inverse is given by

$$(u,w)=T^{-1}(s,t)=(\frac{1}{2}(s+1),2^{-\frac{4}{3}}t).$$

We use this isomorphism in our argument. Set

$$\begin{cases}\psi_1=\frac{ds}{t},\psi_2=\frac{ds}{t^2},\psi_3=\frac{sds}{t^2},\\ \\ \psi_4=\frac{sds}{t},\psi_5=\frac{s^2ds}{t},\psi_6=\frac{s^2ds}{t^2}.\end{cases}\tag{12}$$

$\{\psi_1, \psi_2, \psi_3\}$ forms a basis of $H^0(\Sigma, \Omega)$, and $\{\psi_1, \psi_2, \cdots, \psi_6\}$ forms a basis of $H^1_{DR}(\Sigma, \overline{\boldsymbol{Q}})$. We have the relation between 2 systems:

$$\begin{cases} (T^{-1})^*\varphi_1 = \frac{16^{(1/3)}}{2}\psi_1 \\ (T^{-1})^*\varphi_2 = \frac{16^{(2/3)}}{2}\psi_2 \\ (T^{-1})^*\varphi_3 = \frac{16^{(2/3)}}{4}\psi_2 + \frac{16^{(2/3)}}{4}\psi_3 \\ (T^{-1})^*\varphi_4 = \frac{16^{(1/3)}}{2}\psi_1 + \frac{16^{(1/3)}}{2}\psi_4 \\ (T^{-1})^*\varphi_5 = \frac{16^{(1/3)}}{4}\psi_1 + \frac{16^{(1/3)}}{2}\psi_4 + \frac{16^{(1/3)}}{4}\psi_5 \\ (T^{-1})^*\varphi_6 = \frac{16^{(2/3)}}{4}\psi_2 + \frac{16^{(2/3)}}{2}\psi_3 + \frac{16^{(2/3)}}{4}\psi_6. \end{cases} \tag{13}$$

We define paths $\alpha_{01}, \alpha_{12}, \alpha_{23}$ in the s space as indicated in Figure 1. Let $\alpha^{(k)}_{j,j+1}$ be the path on Σ with $\alpha^{(k+1)}_{j,j+1} = \rho\alpha^{(k)}_{j,j+1} \quad (j = 0, 1, 2)$, where $\rho(s, t) = (s, \zeta_3 t)$.

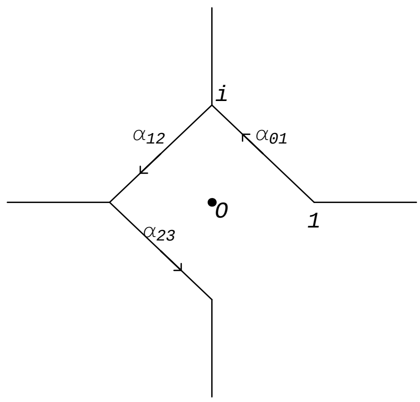

Figure 1: paths for integrals on Σ

Let $\mathcal{I}_P$ be the $\overline{\boldsymbol{Q}}$ vector space generated by the periods

$$\int_\gamma \varphi_i \quad (i = 1, \cdots, 6,\ \gamma \in H_1(P, \boldsymbol{Z})),$$

and let $\mathcal{I}_\Sigma$ be the $\overline{\boldsymbol{Q}}$ vector space generated by the integrals

$$\int_{\alpha^{(k)}_{j,j+1}} \psi_i \quad (j = 0, 1, 2,\ k = 0, 1,\ i = 1, \cdots, 6).$$

We can make up a basis of $H_1(\Sigma, \boldsymbol{Z})$ in terms of $\alpha^{(k)}_{j,j+1}$. So we have $\mathcal{I}_P = \mathcal{I}_\Sigma$.
Here we note that

$$D(\varphi, \frac{1+i}{2}, \frac{1-i}{2}) \in \boldsymbol{P}^2(\overline{\boldsymbol{Q}}) \Longleftrightarrow \left(\int_{\alpha^{(0)}_{01}} (T^{-1})^*(\varphi) : \int_{\alpha^{(0)}_{12}} (T^{-1})^*(\varphi) : \int_{\alpha^{(0)}_{23}} (T^{-1})^*(\varphi) \right) \in \boldsymbol{P}^2(\overline{\boldsymbol{Q}}).$$

(a) Decomposition of $\mathrm{Jac}(\Sigma)$.

For the basis

$$\psi_1 = \frac{ds}{t}, \psi_2 = \frac{ds}{t^2}, \psi_3 = \frac{sds}{t^2},$$

$F = \boldsymbol{Q}(\zeta_{12})$ acts by

$$\sigma : (s, w) \mapsto (\zeta^3 s, \zeta^4 w) \quad (\zeta = \zeta_{12}). \tag{14}$$

Set $P_0 : w'^3 = s'^2 - 1$. The correspondence $(s, w) \mapsto (s', w') = (s^2, w)$ defines a map $\pi : P \to P_0$. The differential ψ_3 is a lifting of the holomorphic differential on P_0. So it induces a projection $H^0(C, \Omega) \to H^0(C_0, \Omega)$. Hence we have a decomposition

$$\mathrm{Jac}(P) \sim_{isog} \mathrm{Jac}(P_0) \oplus B,$$

where B is the kernel of this projection. There is an action of $F = \boldsymbol{Q}(\zeta_{12})$ on the space $\langle \psi_1, \psi_2 \rangle$ induced from (14).

Namely

$$\sigma\psi_1 = \zeta^{-1}\psi_1, \sigma\psi_2 = \zeta^{-5}\psi_2.$$

So we have the type $\Phi = (\zeta^{-1}, \zeta^{-5}) = (-1, -5)$ on B. By considering Lemma 3.2 and shifting Φ by the residue classes in $(\boldsymbol{Z}/12\boldsymbol{Z})^* = \{1, 5, 7, 11\}$ we obtain

$$5(-1, -5) = (-1, -5) \ , \quad 7(-1, -5) = (1, 5) \ , \quad 11(-1, -5) = (1, 5) \ .$$

So we have

$$H = \{\sigma \in (\boldsymbol{Z}/12\boldsymbol{Z})^* : \sigma\Phi = \Phi\} = \{1, 5\}.$$

Hence B is non-simple, and by Lemma 3.1 it is a product of an elliptic curve with itself. By Lemma 3.2 again we know that $B \sim_{isog} E(i)^2$. Finally we obtain

Lemma 3.3.

$$\mathrm{Jac}(P) = \mathrm{Jac}(\Sigma) \sim_{isog} E(\zeta_3) \oplus (E(i))^2.$$

(b) By referring to Lemma 1.3 and the fact that $E(i)$ and $E(\zeta_3)$ are non–isogenous, the above lemma induces

Lemma 3.4.

$$\dim_{\overline{\boldsymbol{Q}}} \mathcal{I}_P = \dim_{\overline{\boldsymbol{Q}}} \mathcal{I}_\Sigma = 4.$$

(c) Set $p_k = \int_{\alpha_{01}^{(0)}} \psi_k$. We can describe the period matrix of Σ with respect to the basis $\{\psi_1, \psi_2, \psi_3\}$ of $\mathrm{H}^0(\Sigma, \Omega)$ and the basis $\{\alpha_{01}^{(0)}, \alpha_{12}^{(0)}, \alpha_{23}^{(0)}, \alpha_{01}^{(1)}, \alpha_{12}^{(1)}, \alpha_{23}^{(1)}\}$ of $\mathrm{H}_1(\Sigma, \boldsymbol{Q})$ in symbolic notation by

$$\Lambda = \begin{pmatrix} p_1 & ip_1 & -p_1 & \zeta_3^2 p_1 & i\zeta_3^2 p_1 & -\zeta_3^2 p_1 \\ p_2 & ip_2 & -p_2 & \zeta_3 p_2 & i\zeta_3 p_2 & -\zeta_3 p_2 \\ p_3 & -p_3 & p_3 & \zeta_3 p_3 & -\zeta_3 p_3 & \zeta_3 p_3 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \left(\alpha_{01}^{(0)}, \alpha_{12}^{(0)}, \alpha_{23}^{(0)}, \alpha_{01}^{(1)}, \alpha_{12}^{(1)}, \alpha_{23}^{(1)}\right) \ .$$

Set

$$S = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

By the change of basis by S we get

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \left(\alpha_{01}^{(0)}, \alpha_{12}^{(0)}, \alpha_{23}^{(0)}, \alpha_{01}^{(1)}, \alpha_{12}^{(1)}, \alpha_{23}^{(1)}\right) S = \begin{pmatrix} p_1 & ip_1 & 0 & \zeta_3^2 p_1 & i\zeta_3^2 p_1 & 0 \\ p_2 & ip_2 & 0 & \zeta_3 p_2 & i\zeta_3 p_2 & 0 \\ 0 & 0 & p_3 & 0 & 0 & \zeta_3 p_3 \end{pmatrix}.$$

As we mentioned in Lemma 3.1, the cofactor B of $E(\zeta_3)$, is isogenous to $E(i)^2$ over $\overline{\boldsymbol{Q}}$. We have

$$\begin{pmatrix} \frac{p_2}{p_1} & -1 \\ \frac{p_2}{p_1} & -\zeta_3 \end{pmatrix} \begin{pmatrix} p_1 & ip_1 & \zeta_3^2 p_1 & i\zeta_3^2 p_1 \\ p_2 & ip_2 & \zeta_3^2 p_2 & i\zeta_3^2 p_2 \end{pmatrix} = p_2 \begin{pmatrix} 0 & 0 & (-\sqrt{3}i) & i(-\sqrt{3}i) \\ (-\sqrt{3}\zeta_3) & i(-\sqrt{3}\zeta_3) & 0 & 0 \end{pmatrix}.$$

So we see that $\frac{p_2}{p_1} \in \overline{\boldsymbol{Q}}$. Set $\gamma_{j,j+1}^k = \alpha_{j,j+1}^k - \alpha_{j,j+1}^{k+1}$ and set

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 1 \\ 1 & -1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then

$$(A_1, A_2, A_3, B_1, B_2, B_3) = (\gamma_{01}^{(0)}, \gamma_{01}^{(0)}, \gamma_{12}^{(0)}, \gamma_{12}^{(1)}, \gamma_{23}^{(1)}, \gamma_{23}^{(1)})M_1$$

is a canonical homology basis of $H_1(\Sigma, \boldsymbol{Z})$. We can describe the extended period matrix as the following diagram:

	A_1	A_2	A_3
ψ_1	$\frac{1}{2}(3+(2+i)\sqrt{3})p_1$	$(-1-i)\sqrt{3}p_1$	$\frac{1}{2}(-1+i)(3+\sqrt{3})p_1$
ψ_2	$\frac{1}{2}(3-(2+i)\sqrt{3})p_2$	$(1+i)\sqrt{3}p_2$	$\frac{1}{2}(1-i)(-3+\sqrt{3})p_2$
ψ_3	$\frac{1}{2}(3+i\sqrt{3})p_3$	0	$-i\sqrt{3}p_3$
ψ_4	$\frac{1}{2}(3-i\sqrt{3})p_4$	0	$i\sqrt{3}p_3p_4$
ψ_5	$\frac{1}{2}(3-(2-i)\sqrt{3})p_5$	$(1-i)\sqrt{3}p_5$	$\frac{1}{2}(1+i)(-3+\sqrt{3})p_5$
ψ_6	$\frac{1}{2}(3+(2-i)\sqrt{3})p_6$	$(-1+i)\sqrt{3}p_6$	$\frac{1}{2}(-1-i)(3+\sqrt{3})p_6$

	B_1	B_2	B_3
ψ_1	$\frac{1}{2}(-1+i)(3+\sqrt{3})p_1$	$\frac{1}{2}(-1+i)\sqrt{3}(i+\sqrt{3})p_1$	$\frac{1}{2}(-3-(2+i)\sqrt{3})p_1$
ψ_2	$\frac{1}{2}(1-i)(-3+\sqrt{3})p_2$	$\frac{1}{2}(1+i)(3i+\sqrt{3})p_2$	$\frac{1}{2}(-3+(2+i)\sqrt{3})p_2$
ψ_3	$i\sqrt{3}p_3$	0	$\frac{1}{2}(3+i\sqrt{3})p_3$
ψ_4	$-i\sqrt{3}p_3p_4$	0	$\frac{1}{2}(3-i\sqrt{3})p_4$
ψ_5	$\frac{1}{2}(1+i)(-3+\sqrt{3})p_5$	$\frac{1}{2}(-1-i)\sqrt{3}(i+\sqrt{3})p_5$	$\frac{1}{2}\{-3+(2-i)\sqrt{3}\}p_5$
ψ_6	$\frac{1}{2}(-1-i)(3+\sqrt{3})p_6$	$\frac{1}{2}(-1+i)(3i+\sqrt{3})p_6$	$\frac{1}{2}\{-3-(2-i)\sqrt{3}\}p_6$

. According to Lemma 1.4 we obtain three nontrivial relations

$$p_1p_6 = -\frac{3}{3+\sqrt{3}}\pi,\ p_2p_5 = \frac{3}{5(-3+\sqrt{3})}\pi,\ p_3p_4 = -\frac{\sqrt{3}}{2}\pi. \tag{15}$$

Combining it with $p_2/p_1 \in \overline{\boldsymbol{Q}}$ we obtain $p_6/p_5 \in \overline{\boldsymbol{Q}}$. By Lemma 3.4 we obtain

Lemma 3.5. *The $\overline{\boldsymbol{Q}}$ vector space P_Σ is generated by*

$$p_1, p_2, \cdots, p_6,$$

and we have only two nontrivial linear relations $p_2/p_1 \in \overline{\boldsymbol{Q}}$ and $p_6/p_5 \in \overline{\boldsymbol{Q}}$.

(d) By putting $w^3 = A(u) = u(u-1)(u^2-u+\frac{1}{2})$ we have $dw^3 = A'(u)du$, hence

$$dw = \frac{1}{3}\frac{A'(u)du}{w^2}.$$

By observing $d(u^\ell w^m) = \ell u^{\ell-1}w^m + mu^\ell w^{m-1}dw$ we obtain

$$\ell u^{\ell-1}w^m du \equiv -\frac{1}{3}mu^\ell w^{m-3}A'(u)du.$$

Therefore we have

Lemma 3.6. *(1) If $n \equiv 0 \pmod 3$, then φ is an exact differential.*

(2) $$\frac{x^{\ell+3}dx}{y^n} \equiv \frac{6\ell-6n+18}{3\ell-4n+12}\frac{x^{\ell+2}dx}{y^n} - \frac{9\ell-6n+18}{6\ell-8n+24}\frac{x^{\ell+1}dx}{y^n} + \frac{3\ell-n+3}{6\ell-8n+24}\frac{x^\ell dx}{y^n}$$

(3) $$\frac{x^\ell dx}{y^n} \equiv \frac{12\ell-16n+60}{n-3}\frac{x^\ell dx}{y^{n-3}} - \frac{6\ell}{n-3}\frac{x^{\ell-1}dx}{y^{n-3}}$$

The equality (2) in the above lemma is essentially a contiguity relation between Appell's hypergeometric functions.

(e) Because ψ_k is an eigendifferential for the action of $\boldsymbol{Q}(\zeta_{12})$, we have always

$$\left(\int_{\alpha_{01}^{(0)}}\psi_k : \int_{\alpha_{12}^{(0)}}\psi_k : \int_{\alpha_{23}^{(0)}}\psi_k\right) \in \boldsymbol{P}^2(\overline{\boldsymbol{Q}}) \quad \text{for all} \quad k = 1, 2, \ldots, 6\ .$$

Looking at (13), we know that $D(\varphi_1)$ and $D(\varphi_2)$ are algebraic.

Let us observe $D(\varphi_3)$. If $D(\varphi_3)$ is algebraic, then

$$\left(\int_{\alpha_{12}^{(0)}} \psi_2 + \psi_3 : \int_{\alpha_{01}^{(0)}} \psi_2 + \psi_3\right) = \beta \in \overline{\boldsymbol{Q}}.$$

Because we have

$$\int_{\alpha_{12}^{(0)}} \psi_2 = i \int_{\alpha_{01}^{(0)}} \psi_2 \ , \quad \int_{\alpha_{12}^{(0)}} \psi_3 = - \int_{\alpha_{01}^{(0)}} \psi_3,$$

it induces a relation $(i - \beta)p_2 = (1 + \beta)p_3$. It contradicts Lemma 3.5. So $D(\varphi_3)$ is transcendental.

In case we have $\varphi = \frac{u^\ell du}{w^n}$ $(1 \le \ell \le 30, n = 1, 2)$, by a similar argument we obtain $D(\varphi) \notin \boldsymbol{P}^2(\overline{\boldsymbol{Q}})$.

q.e.d.

Proof of Theorem 3.2. For $P = P(\zeta_3, \zeta_3^2) : w^3 = u(u^3 - 1)$ we get an action of ζ_9 :

$$(u, w) \mapsto (\zeta_9^{-3} u, \zeta_9^{-1} w).$$

We can use again the system (11) as a deRham basis. By the similar argument as (a) in the previous example but more easily we obtain

Lemma 3.7. $\mathrm{Jac}(P)$ *is a simple CM abelian variety with complex multiplication by* $\boldsymbol{Q}(\zeta_9)$.

We see easily the following

Lemma 3.8. φ_k $(k = 1, 2, \cdots, 6)$ *is an eigendifferential with respect to the action of* $\boldsymbol{Q}(\zeta_9)$. *Moreover, every* $\varphi = \frac{u^\ell du}{w^n}$ *is an eigendifferential.*

These two lemmas induce the algebraicity $D(\varphi_k) \in \boldsymbol{P}^2(\overline{\boldsymbol{Q}})$. More explicitly we can observe this fact in a direct way.

$$\boldsymbol{Q}(\zeta_9) \subseteq \mathrm{End}_0(\mathrm{Jac}(P)).$$

The action of ζ_9 on $H^0(Q, \Omega)$ is given by

$$\sigma(\varphi_1) = \zeta_9^{-2} \varphi_1 \ , \ \sigma(\varphi_2) = \zeta_9^{-1} \varphi_2 \ , \ \sigma(\varphi_3) = \zeta_9^{-4} \varphi_3.$$

So we obtain the type of the action of $\boldsymbol{Q}(\zeta_9)$ on $\mathrm{Jac}(P)$ of type $\Phi = (2, 1, 4)$. We can show the type Φ is simple. Hence $\mathrm{Jac}(P)$ is a simple CM abelian variety.

Let $\alpha_{01}, \alpha_{12}, \alpha_{23}$ be oriented line segments on P given by $[0, 1], [0, \zeta_3], [0, \zeta_3^2]$ on a fixed sheet. Set $\rho : w \mapsto \zeta_3 w$ be the covering transformation map. Let us denote $\rho^k \alpha_{ij}$ $(k = 0, 1, 2)$ by

$\alpha_{ij}^{(k)}$. Put $\gamma_i^{(k)} = \alpha_{ij}^{(k)} - \alpha_{ij}^{(k+1)}$ $(k = 0, 1)$. Putting

$$M_2 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 & 1 & 1 \\ 0 & -1 & -1 & 1 & 0 & -1 \\ 1 & -1 & 0 & 1 & -1 & 0 \end{pmatrix},$$

$$(A_1, A_2, A_3, B_1, B_2, B_3) = (\gamma_{01}^{(0)}, \gamma_{01}^{(0)}, \gamma_{12}^{(0)}, \gamma_{12}^{(1)}, \gamma_{23}^{(1)}, \gamma_{23}^{(1)})M_2$$

becomes a symplectic basis of $H_1(P, \boldsymbol{Z})$. By putting $q = \int_{\alpha_{01}^{(0)}} \varphi$ we obtain the following table of integrals.

	$\alpha_{01}^{(0)}$	$\alpha_{12}^{(0)}$	$\alpha_{23}^{(0)}$	$\alpha_{01}^{(1)}$	$\alpha_{12}^{(1)}$	$\alpha_{23}^{(1)}$	$\alpha_{01}^{(2)}$	$\alpha_{12}^{(2)}$	$\alpha_{23}^{(2)}$
$\frac{x^m dx}{y^n}$	q	$\zeta_3^{-n} q$	$\zeta_3^{-2n} q$	$\zeta_3^{m+1} q$	$\zeta_3^{m-n+1} q$	$\zeta_3^{m-2n+1} q$	$\zeta_3^{2m+2} q$	$\zeta_3^{2m-n+2} q$	$\zeta_3^{2m-2n+2} q$.

So we can see $D(\varphi) \in \overline{\boldsymbol{Q}}$.

3.3 CM Pentagonal curves

Let us consider the following pentagonal curve

$$Q(\lambda, \mu) : y^5 = x(x-1)(x-\lambda)(x-\mu) \quad \text{for} \quad (\lambda\mu(\lambda-1)(\mu-1)(\lambda-\mu) \neq 0).$$

This is a curve of genus 6, and we have the following deRham basis:

$$\text{1st kind} : \varphi_1 = \frac{dx}{y^2}, \varphi_2 = \frac{dx}{y^3}, \varphi_3 = \frac{xdx}{y^3}, \varphi_4 = \frac{dx}{y^4}, \varphi_5 = \frac{xdx}{y^4}, \varphi_6 = \frac{x^2dx}{y^4},$$
$$\text{2nd kind} : \varphi_7 = \frac{dx}{y}, \varphi_8 = \frac{xdx}{y}, \varphi_9 = \frac{x^2dx}{y}, \varphi_{10} = \frac{xdx}{y^2}, \varphi_{11} = \frac{x^2dx}{y^2}, \varphi_{12} = \frac{x^2dx}{y^3}.$$

For the special CM point $(x, y) = (\frac{1+i}{2}, \frac{1-i}{2})$, we have the following

Theorem 3.3.

$$D(\varphi_1; \frac{1+i}{2}, \frac{1-i}{2}), D(\varphi_2; \frac{1+i}{2}, \frac{1-i}{2}), D(\varphi_4; \frac{1+i}{2}, \frac{1-i}{2}), D(\varphi_7; \frac{1+i}{2}, \frac{1-i}{2}) \in \overline{\boldsymbol{Q}},$$

$$D(\varphi_3; \frac{1+i}{2}, \frac{1-i}{2}), D(\varphi_5; \frac{1+i}{2}, \frac{1-i}{2}), D(\varphi_6; \frac{1+i}{2}, \frac{1-i}{2}), D(\varphi_8; \frac{1+i}{2} \frac{1-i}{2}),$$

$$D(\varphi_9; \frac{1+i}{2}, \frac{1-i}{2}), D(\varphi_{10}; \frac{1+i}{2}, \frac{1-i}{2}), D(\varphi_{11}; \frac{1+i}{2}, \frac{1-i}{2}), D(\varphi_{12}; \frac{1+i}{2}, \frac{1-i}{2}) \notin \overline{\boldsymbol{Q}}.$$

Remark 3.2. *We expect that we have* $D(\frac{x^\ell dx}{y^m};\frac{1+i}{2},\frac{1-i}{2})\notin\overline{\boldsymbol{Q}}$ *for all* $1\le\ell$ *and all* $1\le m\le 4$.

Proof of the theorem.

Set

$$\Sigma_1 : w^5 = s^4 - 1.$$

Then we have a deRham basis on Σ_1:

$$\begin{aligned}\text{1st kind}: \psi_1 &= \frac{ds}{w^2}, \psi_2 = \frac{ds}{w^3}, \psi_3 = \frac{sds}{w^3}, \psi_4 = \frac{ds}{w^4}, \psi_5 = \frac{sds}{w^4}, \psi_6 = \frac{s^2ds}{w^4},\\ \text{2nd kind}: \psi_7 &= \frac{ds}{w}, \psi_8 = \frac{sds}{w}, \psi_9 = \frac{s^2ds}{w}, \psi_{10} = \frac{sds}{w^2}, \psi_{11} = \frac{s^2ds}{w^2}, \psi_{12} = \frac{s^2ds}{w^3}.\end{aligned}$$

We have two cyclic actions by ζ_4 and ζ_5 on Σ_1:

$$(s,w)\mapsto(\zeta_4 s,w)\ ,\ (s,w)\mapsto(s,\zeta_5 w).$$

They are generated by a single action

$$(s,w)\mapsto(\zeta_{20}^5 s,\zeta_{20}^4 w).$$

And it induces an action of $\boldsymbol{Q}(\zeta_{20})$ on the space of the deRham cohomology group $H^1_{DR}(\Sigma_1,\overline{\boldsymbol{Q}})$. Namely

$$\boldsymbol{Q}(\zeta_{20})\subseteq \mathrm{End}_0(\mathrm{Jac}(\Sigma_1)).$$

Every ψ_i $(1\le i\le 12)$ is an eigen differential for this action. Define

$$T : s(x) = 2x-1, w(y) = 2^{\frac{4}{5}}y.$$

The CM curve Σ_1 is shifted to the pentagonal curve

$$Q(\frac{1+i}{2},\frac{1-i}{2}) : y^5 = x(x-1)(x^2-x+\frac{1}{2}).$$

Set

$$C_1 : w^5 = u^2 - 1.$$

By putting $s\mapsto u = s^2$, we have a double covering map $\Sigma\to C_1$. Hence, $\mathrm{Jac}(\Sigma_1)$ is nonsimple and $A_1 = \mathrm{Jac}(C_1)$ is a component. Here the differentials

$$\psi_3 = \frac{sds}{w^3}, \psi_5 = \frac{sds}{w^4}$$

are liftings of the differential on C_1.

The action of ζ_{20} on the space of holomorphic differentials given by $\sigma : (s,w)\mapsto(\zeta_{20}^5 s,\zeta_{20}^4 w)$ is described as follows:

$$\sigma(\psi_1) = \zeta_{20}^{-3}\psi_1, \sigma(\psi_2) = \zeta_{20}^{-7}\psi_2, \sigma(\psi_3) = \zeta_{10}^{-1}\psi_3, \sigma(\psi_4) = \zeta_{20}^{-11}\psi_4, \sigma(\psi_5) = \zeta_{10}^{-3}\psi_5, \sigma(\psi_6) = \zeta_{20}^{-1}\psi_6.$$

Thus the cofactor of $\mathrm{Jac}(\Sigma_1)$ has the CM type $(1,3,7,11)$. We can see easily this is a simple CM-type . So we have the decomposition

$$\begin{aligned}&\mathrm{Jac}(\Sigma_1) \sim A_1 \oplus A_2 \ \text{ with } \dim A_1 = 2 \ , \ \dim A_2 = 4, \\ &\boldsymbol{Q}(\zeta_{10}) = \mathrm{End}_0(\mathrm{Jac}A_1), \boldsymbol{Q}(\zeta_{20}) = \mathrm{End}_0(\mathrm{Jac}A_2).\end{aligned}$$

According to Lemma 1.3 we have

$$\dim_{\overline{\boldsymbol{Q}}} \langle \int_{\gamma_i} \psi_k \rangle = 12 \qquad (i = 1, ..., 12 \ , k = 1, ..., 12). \tag{16}$$

Let $\alpha_{01}, \alpha_{12}, \alpha_{23}$ be the oriented arcs on Σ_1 with the projection $[1, i], [i, -1], [-1, -i]$ on the same sheet, respectively. We make the exchange of the sheets by $\rho : w \mapsto \zeta_5 w$, and let $\alpha_{ij}^{(k)}$ denote $\rho^k \alpha_{ij}$ $(k = 0, 1, 2, 3, 4)$. Set $\gamma_{ij}^{(k)} = \alpha_{ij}^{(k)} - \alpha_{ij}^{(k+1)}$ $(k = 0, 1, 2, 3)$. By putting

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

we have a symplectic basis

$$\begin{aligned}&(A_1, A_2, A_3, A_4, A_5, A_6, B_1, B_2, B_3, B_4, B_5, B_6) \\ &= (\gamma_{01}^{(0)}, \gamma_{01}^{(1)}, \gamma_{01}^{(2)}, \gamma_{01}^{(3)}, \gamma_{12}^{(0)}, \gamma_{12}^{(1)}, \gamma_{12}^{(2)}, \gamma_{12}^{(3)}, \gamma_{23}^{(0)}, \gamma_{23}^{(1)}, \gamma_{23}^{(2)}, \gamma_{23}^{(3)})M\end{aligned}$$

of $H_1(\Sigma_1, \boldsymbol{Z})$. The extended period matrix of Σ_1 with respect to the basis $\{\alpha_{01}^{(0)}, \alpha_{01}^{(1)}, \alpha_{01}^{(2)}, \alpha_{01}^{(3)}, \alpha_{12}^{(0)}, \alpha_{12}^{(1)}, \alpha_{12}^{(2)}, \alpha_{12}^{(3)}, \alpha_{23}^{(0)}, \alpha_{23}^{(1)}, \alpha_{23}^{(2)}, \alpha_{23}^{(3)}\}$ is given by the following table.

	$\alpha_{01}^{(0)}$	$\alpha_{01}^{(1)}$	$\alpha_{01}^{(2)}$	$\alpha_{01}^{(3)}$	$\alpha_{12}^{(0)}$	$\alpha_{12}^{(1)}$	$\alpha_{12}^{(2)}$	$\alpha_{12}^{(3)}$	$\alpha_{23}^{(0)}$	$\alpha_{23}^{(1)}$	$\alpha_{23}^{(2)}$	$\alpha_{23}^{(3)}$
ψ_1	p_1	$\zeta_5^3 p_1$	$\zeta_5^1 p_1$	$\zeta_5^4 p_1$	ip_1	$i\zeta_5^3 p_1$	$i\zeta_5^1 p_1$	$i\zeta_5^4 p_1$	$-p_1$	$-\zeta_5^3 p_1$	$-\zeta_5^1 p_1$	$-\zeta_5^4 p_1$
ψ_2	p_2	$\zeta_5^2 p_2$	$\zeta_5^4 p_2$	$\zeta_5^1 p_2$	ip_2	$i\zeta_5^2 p_2$	$i\zeta_5^4 p_2$	$i\zeta_5^1 p_2$	$-p_2$	$-\zeta_5^2 p_2$	$-\zeta_5^4 p_2$	$-\zeta_5^1 p_2$
ψ_3	p_3	$\zeta_5^2 p_3$	$\zeta_5^4 p_3$	$\zeta_5^1 p_3$	$-p_3$	$-\zeta_5^2 p_3$	$-\zeta_5^4 p_3$	$-\zeta_5^1 p_3$	p_3	$\zeta_5^2 p_3$	$\zeta_5^4 p_3$	$\zeta_5^1 p_3$
ψ_4	p_4	$\zeta_5^1 p_4$	$\zeta_5^2 p_4$	$\zeta_5^3 p_4$	ip_4	$i\zeta_5^1 p_4$	$i\zeta_5^2 p_4$	$i\zeta_5^3 p_4$	$-p_4$	$-\zeta_5^1 p_4$	$-\zeta_5^2 p_4$	$-\zeta_5^3 p_4$
ψ_5	p_5	$\zeta_5^1 p_5$	$\zeta_5^2 p_5$	$\zeta_5^3 p_5$	$-p_5$	$-\zeta_5^1 p_5$	$-\zeta_5^2 p_5$	$-\zeta_5^3 p_5$	p_5	$\zeta_5^1 p_5$	$\zeta_5^2 p_5$	$\zeta_5^3 p_5$
ψ_6	p_6	$\zeta_5^1 p_6$	$\zeta_5^2 p_6$	$\zeta_5^3 p_6$	$-ip_6$	$-i\zeta_5^1 p_6$	$-i\zeta_5^2 p_6$	$-i\zeta_5^3 p_6$	$-p_6$	$-\zeta_5^1 p_6$	$-\zeta_5^2 p_6$	$-\zeta_5^3 p_6$
ψ_7	q_1	$\zeta_5^4 q_1$	$\zeta_5^3 q_1$	$\zeta_5^2 q_1$	iq_1	$i\zeta_5^4 q_1$	$i\zeta_5^3 q_1$	$i\zeta_5^2 q_1$	$-q_1$	$-\zeta_5^4 q_1$	$-\zeta_5^3 q_1$	$-\zeta_5^2 q_1$
ψ_8	q_2	$\zeta_5^4 q_2$	$\zeta_5^3 q_2$	$\zeta_5^2 q_2$	$-q_2$	$-\zeta_5^4 q_2$	$-\zeta_5^3 q_2$	$-\zeta_5^2 q_2$	q_2	$\zeta_5^4 q_2$	$\zeta_5^3 q_2$	$\zeta_5^2 q_2$
ψ_9	q_3	$\zeta_5^4 q_3$	$\zeta_5^3 q_3$	$\zeta_5^2 q_3$	$-iq_3$	$-i\zeta_5^4 q_3$	$-i\zeta_5^3 q_3$	$-i\zeta_5^2 q_3$	$-q_3$	$-\zeta_5^4 q_3$	$-\zeta_5^3 q_3$	$-\zeta_5^2 q_3$
ψ_{10}	q_4	$\zeta_5^3 q_4$	$\zeta_5^1 q_4$	$\zeta_5^4 q_4$	$-q_4$	$-\zeta_5^3 q_4$	$-\zeta_5^1 q_4$	$-\zeta_5^4 q_4$	q_4	$\zeta_5^3 q_4$	$\zeta_5^1 q_4$	$\zeta_5^4 q_4$
ψ_{11}	q_5	$\zeta_5^3 q_5$	$\zeta_5^1 q_5$	$\zeta_5^4 q_5$	$-iq_5$	$-i\zeta_5^3 q_5$	$-i\zeta_5^1 q_5$	$-i\zeta_5^4 q_5$	$-q_5$	$-\zeta_5^3 q_5$	$-\zeta_5^1 q_5$	$-\zeta_5^4 q_5$
ψ_{12}	q_6	$\zeta_5^2 q_6$	$\zeta_5^4 q_6$	$\zeta_5^1 q_6$	$-iq_6$	$-i\zeta_5^2 q_6$	$-i\zeta_5^4 q_6$	$-i\zeta_5^1 q_6$	$-q_6$	$-\zeta_5^2 q_6$	$-\zeta_5^4 q_6$	$-\zeta_5^1 q_6$

Here we use

$$\int_{\alpha^{(k)}_{i,i+1}} \psi_m / \int_{\alpha^{(k')}_{i',i'+1}} \psi_m \in \boldsymbol{Q}(\zeta_{20}) \quad (0 \le k, k' \le 3,\ 0 \le i, i' \le 2,\ 0 \le m \le 12).$$

According to (16), the $\overline{\mathbf{Q}}$–vector space generated by all periods of $H^1_{DR}(\Sigma_1, \overline{\boldsymbol{Q}})$ is generated by the $\overline{\mathbf{Q}}$–linearly independent integrals

$$\{\int_{\alpha^{(0)}_{01}} \psi_i\} \quad (i = 1, ..., 12)\ . \tag{17}$$

We have the pull backs:

$$T^*(\varphi_1) = 2^{\frac{3}{5}}\psi_1,\ T^*(\varphi_2) = 2^{\frac{7}{5}}\psi_2,$$
$$T^*(\varphi_3) = 2^{\frac{2}{5}}(\psi_2 + \psi_3),\ T^*(\varphi_4) = 2^{\frac{11}{5}}\psi_4,$$
$$T^*(\varphi_5) = 2^{\frac{6}{5}}(\psi_4 + \psi_5),\ T^*(\varphi_6) = 2^{\frac{1}{5}}(\psi_4 + 2\psi_5 + \psi_6)$$
$$T^*(\varphi_7) = 2^{-\frac{1}{5}}\psi_7,\ T^*(\varphi_8) = 2^{-\frac{6}{5}}(\psi_7 + \psi_8),$$
$$T^*(\varphi_9) = 2^{-\frac{11}{5}}(\psi_7 + 2\psi_8 + \psi_9),\ T^*(\varphi_{10}) = 2^{-\frac{2}{5}}(\psi_1 + \psi_{10}),$$
$$T^*(\varphi_{11}) = 2^{-\frac{7}{5}}(\psi_1 + \psi_{10} + \psi_{11}),\ T^*(\varphi_{12}) = 2^{-\frac{3}{5}}(\psi_2 + 2\psi_3 + \psi_{12}).$$

Here $\varphi_1, \varphi_2, \varphi_4, \varphi_7$ are eigen differentials for the action of $\boldsymbol{Q}(\zeta_{20})$. So we have

$$D(\varphi_1; \frac{1+i}{2}, \frac{1-i}{2}), D(\varphi_2; \frac{1+i}{2}, \frac{1-i}{2}), D(\varphi_4; \frac{1+i}{2}, \frac{1-i}{2}), D(\varphi_7; \frac{1+i}{2}, \frac{1-i}{2}) \in \overline{\boldsymbol{Q}}.$$

On the other hand, if we assume

$$D(\varphi_3; \frac{1+i}{2}, \frac{1-i}{2}) \in \overline{\boldsymbol{Q}},$$

it induces a $\overline{\boldsymbol{Q}}$ linear relation between ψ_2 and ψ_3. This is a contradiction, so

$$D(\varphi_3; \frac{1+i}{2}, \frac{1-i}{2}) \notin \overline{\boldsymbol{Q}}.$$

We obtain the results for other cases by similar arguments.

Theorem 3.4. *For the CM pentagonal curve $Q(\zeta_3, \zeta_3^2) : y^5 = x(x^3 - 1)$, every Schwarz value $D(\varphi_i; \zeta_3, \zeta_3^2)$ $(i = 1, ..., 12)$ is algebraic. Moreover, we have*

$$D(\frac{x^m dx}{y^n}; \zeta_3, \zeta_3^2) \in \overline{\boldsymbol{Q}} \quad \forall m, \forall n \in \boldsymbol{Z}.$$

proof. Let $\alpha_{01}, \alpha_{12}, \alpha_{23}$ be the oriented arcs on Σ_1 with the projection $[0,1], [0,\zeta_3], [0,\zeta_3^2]$ on the same sheet, respectively. We make the exchange of the sheets by $\rho : w \mapsto \zeta_5 w$, and let $\alpha_{ij}^{(k)}$ denote $\rho^k \alpha_{ij}$ $(k = 0,1,2,3,4)$. Set $\gamma_{ij}^{(k)} = \alpha_{ij}^{(k)} - \alpha_{ij}^{(k+1)}$ $(k = 0,1,2,3)$. By putting

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & 0 \end{pmatrix},$$

we obtain a symplectic basis

$$\begin{aligned} &(A_1, A_2, A_3, A_4, A_5, A_6, B_1, B_2, B_3, B_4, B_5, B_6) \\ &= (\gamma_{01}^{(0)}, \gamma_{01}^{(1)}, \gamma_{01}^{(2)}, \gamma_{01}^{(3)}, \gamma_{12}^{(0)}, \gamma_{12}^{(1)}, \gamma_{12}^{(2)}, \gamma_{12}^{(3)}, \gamma_{23}^{(0)}, \gamma_{23}^{(1)}, \gamma_{23}^{(2)}, \gamma_{23}^{(3)})M \end{aligned}$$

of $H_1(Q(\zeta_3, \zeta_3^2), \boldsymbol{Z})$. We have the following table of path integrals

	$\alpha_{01}^{(0)}$	$\alpha_{12}^{(0)}$	$\alpha_{23}^{(0)}$	$\alpha_{01}^{(1)}$	$\alpha_{12}^{(1)}$	$\alpha_{23}^{(1)}$
$\frac{x^m dx}{y^n}$	q	$\omega^{m+1} q$	$\omega^{2m+2} q$	$\zeta_5^{-n} q$	$\omega^{m+1} \zeta_5^{-n} q$	$\omega^{2m+2} \zeta_5^{-n} q$

$\alpha_{01}^{(2)}$	$\alpha_{12}^{(2)}$	$\alpha_{23}^{(2)}$	$\alpha_{01}^{(3)}$	$\alpha_{12}^{(3)}$	$\alpha_{23}^{(3)}$
$\zeta_5^{-2n} q$	$\omega^{m+1} \zeta_5^{-2n} q$	$\omega^{2m+2} \zeta_5^{-2n} q$	$\zeta_5^{-3n} q$	$\omega^{m+1} \zeta_5^{-3n} q$	$\omega^{2m+2} \zeta_5^{-3n} q$

namely we have

$$\int_{\alpha_{i,i+1}^{(k)}} \frac{x^m dx}{y^n} \Big/ \int_{\alpha_{i',i'+1}^{(k')}} \frac{x^m dx}{y^n} \in \boldsymbol{Q}(\zeta_{15}) \quad (0 \le k, k' \le 3,\ 0 \le i, i' \le 2).$$

So we obtain the required result.

We can make the argument by the decomposition of the Jacobian varity instead of the above direct proof. We have an action of ζ_{15} on $Q(\zeta_3, \zeta_3^2)$:

$$\sigma : (x, y) \mapsto (\zeta_{15}^5 x, \zeta_{15}^1 y).$$

Hence

$$\boldsymbol{Q}(\zeta_{15}) \subseteq \mathrm{End}_0(\mathrm{Jac}(C(\zeta_3, \zeta_3^2))).$$

Every holomorphic differential $\varphi_1, \cdots, \varphi_6$ is an eigen-differential for this action:

$$\sigma(\varphi_1) = \zeta_5^1\varphi_1, \sigma(\varphi_2) = \zeta_{15}^2\varphi_2, \sigma(\varphi_3) = \zeta_{15}^7\varphi_3,$$

$$\sigma(\varphi_4) = \zeta_{15}^1\varphi_1, \sigma(\varphi_5) = \zeta_5^2\varphi_2, \sigma(\varphi_6) = \zeta_{15}^{11}\varphi_3.$$

So we have

$$\mathrm{Jac}(C(\zeta_3, \zeta_3^2)) \sim A_1 \oplus A_2 \quad \text{with} \quad \dim A_1 = 2\ ,\ {}_2 = 4\ ,$$
$$\boldsymbol{Q}(\zeta_5) = \mathrm{End}_0(\mathrm{Jac}A_1), \boldsymbol{Q}(\zeta_{15}) = \mathrm{End}_0(\mathrm{Jac}A_2).$$

The CM type of $\mathrm{Jac}(A_1)$ is (ζ_5^1, ζ_5^2). This is a simple CM type. So $\mathrm{Jac}(A_1)$ is simple. The CM type of $\mathrm{Jac}(A_2)$ is $(\zeta_{15}^2, \zeta_{15}^7, \zeta_{15}^1, \zeta_{15}^{11})$. By $\zeta_{15} \mapsto \zeta_{15}^{11}$ we have

$$(\zeta_{15}^2, \zeta_{15}^7, \zeta_{15}^1, \zeta_{15}^{11}) \mapsto (\zeta_{15}^7, \zeta_{15}^2, \zeta_{15}^{11}, \zeta_{15}^1).$$

So it is decomposed with CM type $(\zeta_{15}^2, \zeta_{15}^7)$ and CM type $(\zeta_{15}^1, \zeta_{15}^{11})$. These two CM types are isomorphic by $\zeta_{15} \mapsto \zeta_{15}^{-2}$. Hence we have

$$\mathrm{Jac}(A_2) \sim B^2 \text{ with } \dim B = 2.$$

Consequently

$$\mathrm{Jac}(C(\zeta_3, \zeta_3^2)) \sim A_1 \oplus B^2 \quad \text{with } \dim A_1 = 2\ ,\ \dim B = 2.$$

Our differentials φ_i, $i = 1, \ldots, 6$, are eigen-differentials for the CM–actions.

3.4 CM Hypergeometric curves

Set

$$HP(x) : w^3 = u^2(u-1)(u-x) \quad (x \in \overline{\boldsymbol{Q}} - \{0, 1\}).$$

It is an algebraic curve of genus 2 defined over $\overline{\boldsymbol{Q}}$. We have a deRham basis

$$\text{1st kind} : \varphi_1 = \frac{du}{w}, \varphi_2 = \frac{udu}{w^2},$$
$$\text{2nd kind} : \varphi_3 = \frac{udu}{w}, \varphi_4 = \frac{du}{w^2}.$$

Theorem 3.5. *For $HP(-1) : w^3 = u^2(u^2-1)$ we have*

$$\mathrm{Jac}(HP(-1)) \sim_{isog} E(\zeta_3)^2$$

and

$$D(\varphi_i, -1) \in \overline{\boldsymbol{Q}} \quad \textit{for} \quad i = 1, 2, 3, 4\ .$$

Set

$$HQ(x) : w^5 = u^2(u-1)(u-x) \quad (x \in \overline{\boldsymbol{Q}} - \{0,1\}).$$

It is an algebraic curve of genus 4 defined over $\overline{\boldsymbol{Q}}$. We have a deRham basis

$$\begin{aligned}
\text{1st kind} : \varphi_1 &= \frac{du}{w^2}, \varphi_2 = \frac{udu}{w^3}, \varphi_3 = \frac{udu}{w^4}, \varphi_4 = \frac{u^2du}{w^4}, \\
\text{2nd kind} : \varphi_5 &= \frac{du}{w}, \varphi_6 = \frac{udu}{w}, \varphi_7 = \frac{udu}{w^2}, \varphi_8 = \frac{du}{w^3}.
\end{aligned}$$

Theorem 3.6. *For* $HQ(-\zeta_3) : w^5 = u^2(u-1)(u+\zeta_3)$, $\mathrm{Jac}(HQ(-\zeta_3))$ *is a simple abelian variety of CM type with* $\mathrm{End}_0(\mathrm{Jac}(HQ(-\zeta_3)) \cong \boldsymbol{Q}(\zeta_{15})$. *We have*

$$D(\varphi_i, -1) \in \overline{\boldsymbol{Q}} \quad \textit{for} \quad i = 1, 2, 4$$

and

$$D(\varphi_i, -1) \notin \overline{\boldsymbol{Q}} \quad \textit{for} \quad i = 3, 5, 6, 7, 8\ .$$

To prove this result, we use a biholomorphic isomorphism from $HQ(-\zeta_3)$ to $H\Sigma : t^5 = s^2(s^3-1)$:

$$HT : (u, w) \mapsto (s, t) = \left(\frac{u}{\zeta_3(-1+\zeta_3+u)}, \frac{(-1)^{1/10}3^{3/10}w}{-1+\zeta_3+u} \right) .$$

Theorem 3.7. *For* $HQ(-1) : w^5 = u^2(u^2-1)$, *we have a decomposition*

$$\mathrm{Jac}(HQ(-1)) \sim_{isog} A^2$$

here A *is a two dimensional simple abelian variety with* $\mathrm{End}_0(A) \cong \boldsymbol{Q}(\zeta_5)$.

$$D(\varphi_i; -1) \in \overline{\boldsymbol{Q}} \qquad \textit{for all } 1 \le i \le 8\ .$$

Remark. To get the results for $HP(-1), HQ(-1), HQ(-\zeta_3)$ we can use results of Koblitz and Ogus on classical relations among Γ values at rational points, see [KO] and [Su].

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Address of the Authors

Hironori Shiga, Yoshio Suzuki
Inst. of Math. and Physics
Chiba University
Yayoi–Cho 1–33, Inage–ku
Chiba 263–8522
Japan
e–mail: shiga@math.s.chiba-u.ac.jp

Jürgen Wolfart
Math. Seminar der Univ.
Postfach 111 932
D–60054 Frankfurt a.M.
Germany
e–mail: wolfart@math.uni-frankfurt.de