Wilson's Map Operations on Regular Dessins and Cyclotomic Fields of Definition

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Abstract

Dessins d'enfants can be seen as bipartite graphs embedded in compact orientable surfaces. According to Grothendieck and others, a dessin uniquely determines a complex structure on the surface, even an algebraic structure as a projective algebraic curve defined over a number field. Combinatorial properties of the dessin should therefore determine the equations and also structural properties of the curve, such as the field of moduli or the field of definition. However, apart from a few series of examples, very few general results concerning such correspondences are known. As a step in this direction, we present here a graph theoretic characterisation of certain quasiplatonic curves defined over cyclotomic fields based on Wilson's operations on maps.

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1 The mathematical framework

For the last twenty years embeddings of graphs into surfaces have become important for the theory of algebraic curves as an essential tool in the theory of *dessins d'enfants*. In this first section we will briefly recall the necessary definitions and give an outline of the problems and results presented in this paper.

One way to define dessins d'enfants on compact oriented 2-manifolds is to use the *Walsh representation* of *hypermaps*. The Walsh map $W(\mathcal{D})$ of a hypermap $\mathcal{D}$ is a connected bipartite graph $\mathcal{B}$ embedded in a compact orientable surface, dividing it into simply connected cells; in the language of hypermaps, the white and black vertices represent the hypervertices and hyperedges of $\mathcal{D}$, the edges represent incidences between them, and the cells represent the hyperfaces.

Algebraic hypermaps are an alternative and equivalent way to describe dessins. These are triples (G, x, y) consisting of a permutation group $G \subset S_n$ (the *monodromy* or *hypercartographic group* of the dessin), acting transitively on the n edges of the Walsh map $W(\mathcal{D})$ and generated by two elements x and y. The generators x and y describe the anticlockwise cyclic permutations of the edges around their incident white and black vertices. We will call $\mathcal{D}$ a dessin of *type* (p, q, r) if

$$p = \operatorname{ord} x \ , \quad q = \operatorname{ord} y \ , \quad \operatorname{ord} xy = r \ .$$

Geometrically, p and q are the least common multiples of the valencies of the white and black vertices of $\mathcal{D}$; the faces all have even valencies, and their lcm is $2r$. We call $\mathcal{D}$ a *map* of type (p, r) in the special case where all black vertices have valency $q = 2$. Then we can omit the black vertices, and transform $\mathcal{B}$ into an ordinary graph $\mathcal{G}$: the vertices of $\mathcal{G}$ are the white vertices of $\mathcal{B}$, and its edges are formed from the pairs of edges of $\mathcal{B}$ incident with a black vertex. Now y is a permutation reversing the direction of each dart (directed edge) of $\mathcal{G}$. In this case we call G the *cartographic group* of the *algebraic map* (G, x, y). From every Walsh hypermap $\mathcal{D}$ we can pass to a map $\mathcal{M}_{\mathcal{D}}$ by forgetting vertex colours and introducing edge directions — the cartographic group will then be a subgroup of S_{2n} — and conversely we may consider every map as a hypermap, introducing black midpoints of valency 2 on every edge and replacing the two directions with the two new parts of the edge.

Dessins are linked to Riemann surfaces and algebraic curves in the following way. Riemann surfaces X uniformised by subgroups Γ of finite index in triangle groups Δ play a special role as *Belyĭ surfaces*, that is, surfaces having a (non–constant, meromorphic) *Belyĭ function*

$$\beta : X \to \mathbf{P}^1(\mathbf{C})$$

ramified over at most three points in the Riemann sphere $\mathbf{P}^1(\mathbf{C})$, corresponding to the covering map

$$\beta \ : \ \Gamma \backslash \mathbf{H} \ \to \ \Delta \backslash \mathbf{H}$$

where $\mathbf{H}$ is the hyperbolic plane if Δ is a Fuchsian triangle group. In the few cases where Δ is a spherical or euclidean triangle group, $\mathbf{H}$ has to be replaced with the Riemann sphere $\mathbf{P}^1(\mathbf{C})$ or the Gauss plane $\mathbf{C}$. As first observed by Belyĭ [**2**], the existence of such a function is equivalent to the property that X — as a smooth projective algebraic curve — can be defined over a number field. Starting with Grothendieck's theory of *dessins d'enfants* [**9**], many interesting reformulations of Belyĭ's theorem have been found, see for instance [**23**], [**4**], [**14**], the recent survey in [**26**], or the introduction in [**16**].

Every Belyĭ function β induces a bipartite map on a Riemann surface X: if we normalise its critical values to be $0, 1$ and ∞ then $V_0 = \beta^{-1}(0)$ and $V_1 = \beta^{-1}(1)$ are the sets of white and black vertices, and the connected components of the preimage of the real interval $]0, 1[$ are the edges of the graph; the elements of $V_\infty = \beta^{-1}(\infty)$ are called the *face centres* since there is one in each face of the map. Conversely, every bipartite map (equivalently, every hypermap) on a compact orientable surface arises in this way from a unique holomorphic structure and a unique Belyĭ function on the surface. The entries of the triple (p, q, r) giving the type of the resulting dessin divide the corresponding entries of the signature $\langle \bar{p}, \bar{q}, \bar{r} \rangle$ of the triangle group mentioned above; for simplicity one may take $\Delta = \langle p, q, r \rangle$, and we will always assume Δ to be given in this way. In the special case where all zeros of $1 - \beta$ are of order 2 (a *clean* Belyĭ function) we get a hypermap of type $(p, 2, r)$, hence a map of type (p, r). In the language of Belyĭ functions the passage explained above from Walsh hypermaps to maps is induced by replacing β with $4\beta(1 - \beta)$.

A fundamental problem is that of relating the combinatorial properties of the hypermap $\mathcal{D}$ to algebraic properties of the curve X, such as its moduli field, Galois orbit, defining equations, etc. In general, this problem is very difficult, but it is a little easier if the dessin has a large automorphism group; for the definition of this in terms of the monodromy group see the proof of Lemma 2 below. More precisely, we will assume that the Belyĭ function is a regular covering, that is, β is the quotient map $X \to A\backslash X$ by a group A of holomorphic automorphisms of the Riemann surface X; this is equivalent to Γ being a normal subgroup of the triangle group Δ, with $A \cong \Delta/\Gamma$, and also to the hypermap $\mathcal{D}$ being regular, that is, having an automorphism group, isomorphic to A, acting transitively on the edges of the Walsh map $W(\mathcal{D})$. It then follows that X has genus

$$g = 1 + \frac{|A|}{2}\Big(1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}\Big)\,.$$

Such surfaces X, known as *quasiplatonic surfaces*, have many interesting properties, see for instance [**26**, Thm. 4]. In particular A can be identified with the Galois group of the extension of function fields corresponding to β, and the Galois correspondence allows information about A and its action on X to be translated into information about this extension (see [**16**], [**20**], [**21**], [**22**], [**16**] and [**5**] for examples of this). All known examples follow a common pattern, and in the present paper we will work out the mathematical structure behind it: a correspondence between Galois actions on cyclotomic fields on the one hand and *Wilson's map operations* H_j on the other hand. These act on maps by raising the generating monodromy rotation of edges around each vertex to its j-th power where j is coprime to the valencies of the vertices; we will generalise this to an operation $H_{i,j}$ which acts on hypermaps by raising the permutations x and y around the white and black vertices to their i-th and j-th powers. After explaining these operations in the next section, we will establish this correspondence in Section 3, Theorems 1 to 3. With slightly simplified hypotheses and statements, their content can be summarised in the

Main Theorem *A Galois invariant family $\{\mathcal{D}_j\}$ of regular dessins forming an orbit under Wilson's operations is defined over a cyclotomic field, and the Wilson operations are equivalent to the Galois conjugations.*
Conversely, let $\mathcal{D}_j$ run over a family of Galois conjugate regular dessins defined over a cyclotomic field, and suppose that the Galois conjugations preserve adjacency between the vertices of the dessins. Then the algebraic conjugations act as Wilson operations on the dessins.

We apply these results to examples in Sections 4 and 5. Section 4 sheds new light on some series of known Galois orbits of regular dessins, and in Section 5 we will consider regular maps and hypermaps whose underlying graphs are complete graphs K_n (for their regular embeddings see [**1**], [**12**] and [**11**]). In all these examples, Galois conjugation in some sense imitates the effect of Wilson's operations. However this is not a universal phenomenon, and Section 6 contains a lemma about automorphism groups of graphs and maps which allows us to show that Galois conjugation and Wilson's operations act very differently on the family of Macbeath–Hurwitz curves. At the end of Section 3, Proposition 3 gives an indication how to treat those cases for which the correspondence between Wilson's operations and Galois conjugations no longer applies.

2 Wilson's operations

Wilson's operations H_j [**24**] can be defined as follows. Let $\mathcal{M}$ be a compact oriented map of type (p, r) and let j be an integer; we will restrict our considerations to the case where j is coprime to p. Each face of $\mathcal{M}$ is bounded by a closed path which, at each vertex, proceeds from one edge to the next incident edge in the anticlockwise direction, so that the path goes clockwise around the face. Thus $\mathcal{M}$ is uniquely determined by its underlying graph $\mathcal{G}$ and by the anticlockwise cyclic rotation ρ of edges around each vertex. Instead, consider paths in $\mathcal{G}$ which, at each vertex, proceed from one edge to the j-th incident edge in the anticlockwise direction. Of course these paths are all closed, and they form the boundaries of the faces of a new map $H_j\mathcal{M}$ which is also compact and oriented: it has the same underlying graph $\mathcal{G}$ as $\mathcal{M}$ but ρ has been replaced with ρ^j. Each such operation depends only on the congruence class of j modulo p; since $H_j \circ H_k = H_{jk}$ for all j and k coprime to p, these operations represent an action of the multiplicative group $(\mathbf{Z}/p\mathbf{Z})^*$ of units mod p. In particular, H_{-1} transforms a map into its mirror image, by reversing the orientation. For further properties of these operations see [**24**] and [**18**].

We will apply this procedure to dessins $\mathcal{D}$ via the maps $\mathcal{M}_{\mathcal{D}}$ introduced above, forgetting the colours of the vertices. The generators x and y of the monodromy group G of $\mathcal{D}$ represent the rotations of edges around the white and black vertices of $W(\mathcal{D})$, so they must be replaced with x^j and y^j where j is coprime to pq. Thus we have the following lemma (see [**18**, Section 7]):

Lemma 1 *Let $\mathcal{D}$ be a dessin of type (p, q, r), presented as an algebraic hypermap (G, x, y) with a permutation group $G \subset S_n$ acting transitively on the n edges of $\mathcal{D}$, generated by the elements x and y of order p and q respectively such that xy has order r. Let j be an integer coprime to pq. Then Wilson's operation H_j transforms $\mathcal{D}$ into the dessin $H_j\mathcal{D}$ with algebraic hypermap (G, x^j, y^j).*

In the case where $q = 2$, so that the dessin $\mathcal{D}$ is in fact already a map, there may be some confusion as to whether H_j should be applied directly to the map $\mathcal{D}$ as in the first paragraph, raising x but not y to its jth power, or to the map $\mathcal{M}_{\mathcal{D}}$ as in the second paragraph, raising both x and y to their jth powers. In fact there is no real problem: if j is odd then $y^j = y$, and if j is even (so that p is odd) then we can replace H_j with H_{j+p}, again fixing y.

Note that in Lemma 1, although x^j and y^j have order p and q respectively, x^jy^j need not have order r, so $H_j\mathcal{D}$ will have type (p, q, r') for some r', possibly distinct from r. Thus type and hence genus are not always preserved by H_j. This is illustrated by the following examples (see also Examples 3 and 8).

Example 1 The modular group $\mathrm{PSL}_2(\mathbf{Z})$ is generated by its elements $\pm\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $\pm\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$, so for each integer $p \geq 2$ its quotient $G = \mathrm{PSL}_2(\mathbf{Z}/p\mathbf{Z})$ is generated by their images x and y in G. Since x, y and xy have orders $p, 2$ and 3 this gives a regular algebraic hypermap $\mathcal{D} = (G, x, y)$ of type $(p, 2, 3)$, that is, a regular map of type $(p, 3)$. Now $x^jy = \pm\left(\begin{smallmatrix}j&-1\\1&0\end{smallmatrix}\right)$ has trace $\pm j$, so by choosing suitable values of j we can change the conjugacy class of this element; in many cases this also changes its order and hence changes the type and genus of the map $H_j\mathcal{D}$. For instance if $p = 5$, so that $\mathcal{D}$ is the icosahedron with $G = \mathrm{PSL}_2(\mathbf{Z}/5\mathbf{Z}) \cong A_5$, then x^2y has order 5, so $H_2\mathcal{D}$ has type $(5, 5)$; it is, in fact, the great dodecahedron, a regular map of genus 4 denoted by $\{5, 5/2\}$ in [**6**]. Similarly, if $p = 8$ with $G = \mathrm{PSL}_2(\mathbf{Z}/8\mathbf{Z})$ of order 192 then x^3y has order 6, so H_3 converts a regular map of type $(8, 3)$ and genus 5 into one of type $(8, 6)$ and genus 21. These last two maps are 4-sheeted unbranched coverings of a similar pair of maps of type $(8, 3)$ and $(8, 6)$: the first is the map $\{3, 4 + 4\}$ of genus 2, a double covering of the octahedron branched over its vertices, with automorphism group

$$G = \langle x, y, z \mid x^8 = y^2 = z^3 = xyz = [x^4, z] = 1\rangle \cong \mathrm{GL}_2(\mathbf{F}_3)$$

(see [**17**]), while the second, its image under H_3, has type $(8, 6)$ and genus 6. (The authors are grateful to Steve Wilson for this example.)

Example 2 The symmetric group S_n is generated by the n-cycle $x = (1, 2, \ldots, n)$ and the transposition $y = (1, 2)$, with xy an $(n-1)$-cycle, so S_n is the monodromy group of a regular map $\mathcal{M}$ of type $(n, n-1)$. If j is coprime to n then $x^j y$ is the product of two cycles of lengths k and $n-k$ where $jk \equiv 1 \bmod n$ and $1 \leq k \leq n$; these lengths are mutually coprime, so $x^j y$ has order $k(n-k)$ and hence $H_j\mathcal{M}$ has type $(n, k(n-k))$. Since $k(n-k) = k'(n-k')$ if and only if $k' = k$ or $n-k$ we thus obtain maps of $\phi(n)/2$ different types. This is the maximum number possible for any map $\mathcal{M}$ of valency n since $H_j\mathcal{M}$ and $H_{n-j}\mathcal{M}$ always have the same type.

If n is odd, so that $x \in A_n$, then the underlying graph of the map $\mathcal{M}$ is bipartite, with white and black vertices corresponding to the cosets of $\langle x\rangle$ consisting of even and odd permutations respectively. In this case $\mathcal{M}$ is the Walsh map of a regular hypermap $\mathcal{D} = (A_n, x, x^y)$ of type $(n, n, (n-1)/2)$ such that $H_j\mathcal{D}$ has type $(n, n, k(n-k)/2)$, so again we obtain $\phi(n)/2$ different types.

In view of examples like this, and to determine whether the hypotheses of Theorem 2 in the next section are satisfied, we need a sufficient condition for $\mathcal{D}$ and $H_j\mathcal{D}$ to have the same type. A *Frobenius group* is a semidirect product G of a normal subgroup K (the *Frobenius kernel*) by a subgroup H (the *Frobenius complement*) which acts fixed-point freely by conjugation on K, i.e. such that no pair of non-identity elements of H and K commute with each other. Equivalently, G is a transitive permutation group in which only the identity element fixes more than one point: here H is the stabiliser of a point and K consists of the identity and the elements without fixed points (see [**10**, §V.8] for details). A useful example is the group $\mathrm{AGL}_1(\mathbf{F}_n)$ of affine transformations $t \mapsto at+b$ $(a, b \in \mathbf{F}_n, a \neq 0)$ of a finite field $\mathbf{F}_n$ of order n, with K (isomorphic to the additive group of $\mathbf{F}_n$) consisting of the translations $t \mapsto t+b$ and H (the stabiliser of 0, isomorphic to the multiplicative group $\mathbf{F}_n^*$) consisting of the transformations $t \mapsto at, a \neq 0$.

Lemma 2 *Let G be a finite Frobenius group with a kernel K and a complement H. Then two elements of $G \setminus K$ are conjugate in G if and only if their images in H are conjugate in H. In particular, all elements of a given coset $Kg \neq K$ of K in G have the same order.*

Proof. Each element $g \in G$ has the unique form kh where $k \in K$ and $h \in H$, so that h is the image of g under the natural epimorphism $G \to G/K \cong H$. By applying this epimorphism we see that conjugate elements of G have conjugate images in H, so each conjugacy class of G is contained in a set KC where C is a conjugacy class of H. Since G is a Frobenius group we have $C_G(h) = C_H(h)$ for each non-identity $h \in H$, so if C is the conjugacy class of h in H then the number of conjugates of h in G is

$$|G : C_G(h)| = |G : C_H(h)| = |G : H| \cdot |H : C_H(h)| = |K| \cdot |C| = |KC|\,.$$

Thus KC is a single conjugacy class of G, so the first assertion is proved. The second assertion follows immediately. □

Corollary 1 *Let $\mathcal{D} = (G, x, y)$ be a regular hypermap of type (p, q, r) for which the monodromy group G is a Frobenius group with an abelian complement H and a kernel K not containing xy. If j is coprime to pqr then $H_j\mathcal{D}$ also has type (p, q, r).*

Proof. Since j is coprime to both p and q it is sufficient to show that x^jy^j has the same order as xy, namely r. Since G/K is abelian, x^jy^j is in the same coset of K as $(xy)^j$. Now j is coprime to r, so $(xy)^j$ has order r and $(xy)^j \notin K$ since $xy \notin K$. The second part of Lemma 2 therefore implies that x^jy^j has order r. □

The following example shows that this result does not apply to all Frobenius groups.

Example 3 For each prime $p \equiv \pm 1 \bmod 5$ the group $H := \mathrm{SL}_2(\mathbf{F}_5)$ is isomorphic to a subgroup of $\mathrm{SL}_2(\mathbf{F}_p)$. The resulting action of H on $K := C_p \times C_p$ (regarded as a 2-dimensional vector space over $\mathbf{F}_p$) determines a semidirect product G of K by H, and this is a Frobenius group with a nonabelian complement H [**10**, V.8.8(b)]. The generators $a = \left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$ and $b = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ of H have order 5, with ab and a^2b^2 of order 10 and 6 respectively. Let S be the subgroup of G generated by $x := a$ and $y := kb$ for some non-identity element $k \in K$. Since S maps onto H we have $G = KS$. If $K \cap S = 1$ then S is a complement for K in G; however, all such complements are conjugate (since K and H have coprime orders, see [**10**, I.18.3]), and are therefore point-stabilisers, whereas x and y each fix a single distinct fixed point, so $K \cap S > 1$. Now $K \cap S$ is a normal subgroup of S, and S acts irreducibly on K since H does, so $K \cap S = K$. Thus $S \geq K$, so $S = KS = G$ and hence x and y generate G. By Lemma 2 the resulting hypermap $\mathcal{D} = (G, x, y)$ has type $(5, 5, 10)$ whereas $H_2\mathcal{D}$ has type $(5, 5, 6)$.

The case where $xy \in K$ is dealt with by the following slightly more general result:

Proposition 1 *Let $\mathcal{D} = (G, x, y)$ be a regular hypermap of type (p, q, r) for which xy lies in an abelian normal subgroup K of G. If j is coprime to pqr then $H_j\mathcal{D}$ also has type (p, q, r).*

Proof. As in the proof of Corollary 1 it is sufficient to show that the order r' of x^jy^j is equal to the order r of the element $w = xy$. Now

$$x^jy^j \; = \; x^j \cdot w^x \cdot w^{x^2} \cdot \ldots \cdot w^{x^j} \cdot x^{-j} \;,$$

which is conjugate (by x^{j-1}) to the element $v = w \cdot w^x \cdot w^{x^2} \cdot \ldots \cdot w^{x^{j-1}}$, so v has order r'. Since w and all its conjugates lie in K they commute with each other, so $v^r = 1$ and hence r' divides r. If $r' < r$ then let L be the characteristic subgroup of K consisting of its elements of order dividing r', so $w, w^x, \ldots, w^{x^{j-1}} \in K \setminus L$ whereas $v \in L$. Now $w^x \cdot w^{x^2} \cdot \ldots \cdot w^{x^{j-1}} \cdot w^{x^j} = v^x \in L$, so $w^{x^j} \equiv w \bmod L$ and hence the elements $w, w^x, \ldots, w^{x^{j-1}}$ represent an orbit of length l dividing j in the action of $\langle x \rangle$ by conjugation on K/L. Since j is coprime to the order p of x it follows that $l = 1$, so $w \equiv w^x \equiv \cdots \equiv w^{x^{j-1}} \bmod L$. Thus $w^j \equiv v \bmod L$, so $w^j \in L$, and since j is coprime to r it follows that $w \in L$, a contradiction. Thus $r' = r$, as required. □

Sometimes the following more general version of Wilson's operations is useful, as we shall see later in Lemma 4, Theorem 5 and Examples 5, 7 and 10.

Definition. Let $\mathcal{D}$ be a dessin of type (p, q, r) and suppose that i is coprime to p, and j is coprime to q. We define $H_{i,j}\mathcal{D}$ to be the dessin resulting from $\mathcal{D}$ by an application of the H_i procedure to the white vertices and of H_j to the black vertices. In other words, the algebraic hypermap (G, x, y) is replaced with (G, x^i, y^j).

(As in Wilson's original definition, one could even define $H_{i,j}$ for arbitrary integers i, j.) The above definition induces an action of $(\mathbf{Z}/p\mathbf{Z})^* \times (\mathbf{Z}/q\mathbf{Z})^*$ on dessins with the same monodromy group and embedded graph, commuting with the action of the usual Wilson operations defined earlier. Namely, if k is coprime to pq, we have $H_k H_{i,j} = H_{i,j} H_k = H_{ki,kj}$ and $H_k = H_{k,k}$. Note that $H_{1.-1}$ and $H_{-1,1}$ act as Petrie operations, transposing faces and Petrie polygons (closed zig-zag paths); they differ by a reversal of the orientation.

3 The main results

The next two Lemmas are certainly known to experts, but since they do not seem to be formulated explicitly in the literature, we state them here and give a short proof.

Lemma 3 *Let $\mathcal{D}$ be a regular dessin of type (p, q, r), presented as an algebraic hypermap (G, x, y) with a transitive permutation group $G \subset S_n$ acting on the n edges of $\mathcal{D}$, generated by the elements x and y of order p and q respectively. Its monodromy group G and its automorphism group $A \subset S_n$ determine each other uniquely by*

$$A = C(G) \qquad \text{and} \qquad G = C(A)$$

where C denotes the centraliser in S_n.

Proof. It is well known that, as a permutation group on the set $E(\mathcal{D})$ of edges, $A = C(G)$, see e.g. [**14**]. This immediately implies that $G \subseteq C(A)$. Since the dessin $\mathcal{D}$ is regular, A acts transitively on $E(\mathcal{D})$. It follows that $C(A)$, as a permutation group with a transitive centraliser, must act without fixed points on the set of edges. It therefore contains at most one element sending any given edge to another. However, its subgroup G always contains such an element, so $C(A) = G$. □

Remarks. 1) A regular dessin $\mathcal{D}$ does not necessarily lead to a regular map $\mathcal{M}_{\mathcal{D}}$ since the cartographic group does not necessarily act transitively on the *directed* edges. In terms of Belyĭ functions the regularity of β as a covering map does not imply the regularity of $4\beta(1-\beta)$.
2) Since $A = C(G)$ without the regularity assumption, and since G is not changed by Wilson's operation H_j (see Lemma 1), $H_j\mathcal{D}$ has the same automorphism group as $\mathcal{D}$ for any dessin $\mathcal{D}$.

The automorphism group A of a regular dessin $\mathcal{D}$ of type (p, q, r) is generated by two elements a_0 and b_0 of order p and q, called *canonical generators*, which respectively fix a white vertex P_0 and a neighbouring black vertex Q_0, each permuting their incident edges by a cyclic shift to the anticlockwise next edge; by the regularity of $\mathcal{D}$, all such pairs P_0, Q_0

are equivalent under A, so a_0 and b_0 are unique up to conjugacy. Their product a_0b_0, which has order r, describes a clockwise rotation of the face of $\mathcal{D}$ on the left-hand side of the edge from P_0 to Q_0 (or of all such faces if there are multiple edges).

Lemma 4 *Let* $\Delta = \langle p, q, r\rangle = \langle \gamma_0, \gamma_1, \gamma_\infty \mid \gamma_0^p = \gamma_1^q = \gamma_\infty^r = 1 = \gamma_0\gamma_1\gamma_\infty\rangle$ *be a triangle group in its usual representation, and* $\mathcal{D}$ *the regular dessin defined by the Belyĭ function*

$$\beta \;:\; \Gamma\backslash\mathbf{H} \;\rightarrow\; \Delta\backslash\mathbf{H}$$

where Γ *denotes the kernel of the canonical epimorphism*

$$h : \Delta \rightarrow A = \mathrm{Aut}\,\mathcal{D} \cong \Delta/\Gamma\ , \qquad h(\gamma_0) = a_0\ ,\ h(\gamma_1) = b_0\ .$$

Suppose that i, j *are coprime to* pq*. Let* $a_0^i b_0^j$ *have order* r' *in* A*, let* $\Delta' = \langle p, q, r'\rangle = \langle \delta_0, \delta_1, \delta_\infty \mid \delta_0^p = \delta_1^q = \delta_\infty^{r'} = 1 = \delta_0\delta_1\delta_\infty\rangle$ *and let* $h' : \Delta' \rightarrow A$ *be the epimorphism defined by*

$$h'(\delta_0) := a_0^i \quad \textit{and} \quad h'(\delta_1) := b_0^j$$

with kernel Γ'*. Then* $H_{i,j}\mathcal{D}$ *is the dessin corresponding to the Belyĭ function*

$$\beta' \;: \Gamma'\backslash\mathbf{H} \rightarrow \Delta'\backslash\mathbf{H}\,.$$

(For the few signatures not defining Fuchsian triangle groups, **H** *has to be replaced with the Riemann sphere or the Gauss plane.)*

(To apply Lemma 4 to the original Wilson operation, recall that $H_j = H_{j,j}$.

Proof. Let $\mathcal{D}'$ denote the dessin corresponding to β'. Since $\mathcal{D}$ and $\mathcal{D}'$ are both regular, with automorphism group A, their edge-sets $E(\mathcal{D})$ and $E(\mathcal{D}')$ are both permuted regularly by A. We can therefore identify $E(\mathcal{D})$ with $E(\mathcal{D}')$ so that A induces the same permutation group on both edge-sets. By Lemma 2, the monodromy groups of $\mathcal{D}$ and $\mathcal{D}'$ coincide, so these dessins differ only in the local cyclic ordering of edges around the white and black vertices. The edges of each dessin can be identified with the elements of A, so that in $\mathcal{D}$ these cyclic orderings are given by right multiplication by the images a_0 and b_0 of γ_0 and γ_1, while in $\mathcal{D}'$ they are given by the images a_0^j and b_0^j of δ_0 and δ_1. It immediately follows from Lemma 1 that $\mathcal{D}' \cong H_j\mathcal{D}$. □

Lemma 5 *Let* $\mathcal{D}$ *be a dessin with an automorphism group* A*, and let* $\sigma \in \mathrm{Gal}\,\overline{\mathbf{Q}}/\mathbf{Q}$ *. Then the conjugate dessin* $\mathcal{D}^\sigma$ *has an automorphism group* $A^\sigma \cong A$*, and the mapping* $P \mapsto P^\sigma$ *induces isomorphisms between the actions of* A *on the sets* V_0, V_1 *and* V_∞ *of white and black vertices and face centres* P *of* $\mathcal{D}$ *and the actions of* A^σ *on the corresponding sets of points* P^σ *of* $\mathcal{D}^\sigma$*.*

Proof. The automorphisms of $\mathcal{D}$ are also automorphisms of the corresponding curve X, and are thus birational transformations of X. Like X they are defined over $\overline{\mathbf{Q}}$, so σ may be applied to them through their coefficients. They are in fact the automorphisms of X serving as covering transformations of the Belyĭ function β corresponding to $\mathcal{D}$, so σ induces a bijection between the automorphism groups A and A^σ of $\mathcal{D}$ and $\mathcal{D}^\sigma$. If $a, b \in A$ then the coefficients of ab are rational functions of those of a and b, so $(ab)^\sigma = a^\sigma b^\sigma$ and this bijection is an isomorphism.

The white vertices, black vertices and face centres of $\mathcal{D}$ are the points $P \in X$ such that $\beta(P) = 0, 1$ or ∞. Since β is defined over $\overline{\mathbf{Q}}$ these are all $\overline{\mathbf{Q}}$-rational points of X, so they are sent bijectively by σ to the white vertices, black vertices and face centres P^σ of $\mathcal{D}^\sigma$. The second assertion now follows from the fact that $a(P) = Q$ if and only if $a^\sigma(P^\sigma) = Q^\sigma$. □

Remark. Although Galois conjugation acts in this natural way on the vertices and face centres of a dessin, it does not do so on the edges. The first difficulty is that edges are defined topologically, not algebraically, as connected sets covering the open unit interval $]0, 1[$, and even if one restricts attention to $\overline{\mathbf{Q}}$-rational points of X, a point P on an edge of $\mathcal{D}$ may be sent to a point P^σ not on an edge of $\mathcal{D}^\sigma$ since an algebraic number which is in $]0, 1[$ may be conjugate to one which is not. Of course, edge-points which cover rational elements of $]0, 1[$ will be sent to edge-points, but two such points on the same edge may be sent to points on different edges. A second related difficulty is that if P and Q are adjacent vertices of $\mathcal{D}$ then the vertices P^σ and Q^σ of $\mathcal{D}^\sigma$ need not be adjacent: indeed it is this possibility which allows Galois conjugate dessins to have mutually non-isomorphic embedded graphs (see Example 8 and Corollary 2, for instance).

In order to avoid these difficulties we will work under additional hypotheses, e.g. that the Galois conjugation σ is *adjacency preserving* on $\mathcal{D}$, i.e. that there is a white vertex P_0 and an adjacent black vertex Q_0 of $\mathcal{D}$ such that P_0^σ and Q_0^σ are neighbours in the Galois conjugate dessin $\mathcal{D}^\sigma$. By Lemma 5 and the transitivity of the action of the automorphism group A on V_0 and V_1 in the case of regular dessins we have

Lemma 6 *Let $\mathcal{D}$ be a regular dessin with automorphism group A and σ an adjacency preserving Galois conjugation. Then for all adjacent pairs P, Q of white and black vertices in $\mathcal{D}$, their σ–images P^σ, Q^σ in $\mathcal{D}^\sigma$ are also adjacent.* □

Under the hypothesis that σ is adjacency preserving we can extend its action to the edges of $\mathcal{D}$ in a way which is compatible with the action of A .

Lemma 7 *Let $\mathcal{C}$ and $\mathcal{C}'$ denote the underlying graphs of a regular dessin $\mathcal{D}$ and its Galois conjugate $\mathcal{D}^\sigma$. If σ preserves adjacency on $\mathcal{D}$ then there is a graph isomorphism $f : \mathcal{C} \to \mathcal{C}'$ such that*

1. *f coincides with σ on the vertices, and*
2. *$f \circ a = a^\sigma \circ f$ for all automorphisms $a \in A$.*

Proof. If there is only one edge joining the neighbouring vertices P_0, Q_0 in $\mathcal{D}$, then the same is true for their σ-images in $\mathcal{D}^\sigma$, because the existence of multiple edges can be characterised by the existence of a nontrivial subgroup of A fixing both points, and because this property is σ-invariant (see the last line of the proof of Lemma 5, with $P = Q$ first a white and then a black vertex). If we exclude this possibility, edges correspond bijectively to adjacent pairs of vertices, so there is an obvious and canonical choice for f. If on the other hand there are several edges joining adjacent vertices, the definition of f is still determined canonically on the vertices by property 1, but no longer on the edge set. In this case take an arbitrary edge e_0 between P_0 and Q_0, say, and define $f(e_0)$ to be an edge between P_0^σ and Q_0^σ. Since each other edge in $\mathcal{D}$ can be written as $a(e_0)$ for a unique $a \in A$, property 2 determines f in a unique way. It is easy to see that incidence is preserved, so we have a graph isomorphism. □

This graph isomorphism f is in general not a dessin isomorphism since it need not preserve the local ordering of the edges around the vertices. To study its behaviour around the vertices, we consider a canonical pair of generators a_0 and b_0, fixing a white vertex P_0 and a neighbouring black vertex Q_0. Since A can also be regarded as a group of automorphisms of X, the action of a_0 (resp. b_0) locally around P_0 (resp. Q_0) can be written in suitable local coordinates as a rotation

$$z \;\mapsto\; \zeta_p z \quad \text{around} \quad z(P_0) = 0 \quad \text{with} \quad \zeta_p := e^{2\pi i/p}$$

(resp. $z \mapsto \zeta_q z$ around $z(Q_0) = 0$). More generally, if P is any white vertex of $\mathcal{D}$ fixed by $a \in A$ (the stabiliser in A is always cyclic of order p), then a acts locally around P as multiplication, the only difference being that the *multiplier* ζ_p must now be replaced with a power $\zeta_p^{k(a,P)}$ for some residue class $k(a,P) \bmod p$. Similarly, any $b \in A$ fixing a black vertex Q acts locally like $z \mapsto \zeta_q^{l(b,Q)} z$ for some residue class $l(b,Q) \bmod q$. Observe that conjugation does not change multipliers: for all $g \in A$ we have

$$k(gag^{-1}, gP) = k(a,P) \quad \text{and} \quad l(gbg^{-1}, gQ) = l(b,Q)\,. \tag{1}$$

As shown in Lemma 5, the automorphisms of X are birational morphisms defined over $\overline{\mathbf{Q}}$, and their fixed points are $\overline{\mathbf{Q}}$-rational points of X. In the following we will in fact identify A with A^σ by using the isomorphism $A \to A^\sigma$, $a \mapsto a^\sigma$ described in Lemma 5. The multipliers at these points satisfy

$$\zeta_p^{k(a,P^\sigma)} \;=\; \sigma(\zeta_p^{k(a,P)})\,, \tag{2}$$

see [**22**, Lemma 4] and [**5**, Lemma 1], and the multipliers $\zeta_q^{l(b,Q)}$ at the black vertex fixed points behave in the same way under Galois conjugation. Since $\sigma(\zeta_m) = \zeta_m^s$ for some $s \in (\mathbf{Z}/m\mathbf{Z})^*$, we can describe the effect of Galois conjugation on the multipliers by taking s-th powers, i.e. by writing $k(a,P^\sigma) \equiv sk(a,P) \bmod m$ and $l(b,Q^\sigma) \equiv sl(b,Q) \bmod m$. Recall that under the hypotheses of Lemmas 6 and 7 we have $P^\sigma = f(P)$ and $Q^\sigma = f(Q)$.

For some applications — see the remarks after Theorem 4 and Example 9, for instance — it is useful to generalise the condition of *adjacency preserving* as follows.

Definition. Let A be the automorphism group of a regular dessin $\mathcal{D}$. For each σ in the absolute Galois group $\mathrm{Gal}\,\overline{\mathbf{Q}}/\mathbf{Q}$ we can identify A and A^σ in the obvious way, so that A is regarded as the automorphism group of $\mathcal{D}^\sigma$. We call $\mathcal{D}$ and $\mathcal{D}^\sigma$ *equivariant for* A if there is a colour-preserving isomorphism $f : \mathcal{C} \to \mathcal{C}'$ of the underlying graphs of $\mathcal{D}$ and $\mathcal{D}^\sigma$ commuting with the actions of A and changing the multipliers in a uniform way. More precisely we require that

1. $f \circ a = a \circ f$ for all automorphisms $a \in A$, and
2. there is an integer s coprime to the least common multiple m of p and q such that $k(a, f(P)) \equiv sk(a, P) \bmod m$ for all $a \in A$ and all fixed points P of a.

We then call f an A-*equivariant* graph isomorphism. We say that $\mathcal{D}$ is *Galois compatible* if $\mathcal{D}$ and $\mathcal{D}^\sigma$ are equivariant for A for each $\sigma \in \mathrm{Gal}\,\overline{\mathbf{Q}}/\mathbf{Q}$.

Lemmas 6 and 7 and equation (2) immediately imply the following:

Proposition 2 *If $\mathcal{D}$ is a regular dessin with automorphism group A, and the Galois conjugation $\sigma : \mathcal{D} \to \mathcal{D}^\sigma$ is adjacency preserving, then $\mathcal{D}$ and $\mathcal{D}^\sigma$ are equivariant for A.* □

Theorem 1 *Let X be a quasiplatonic surface with a regular dessin $\mathcal{D}$ of type (p, q, r). As a smooth projective algebraic curve, let X be defined over a subfield of the cyclotomic field $\mathbf{Q}(\zeta_m)$, $\zeta_m := \exp(2\pi i/m)$, where m is the least common multiple of p and q. Let $\sigma \in \mathrm{Gal}\,\mathbf{Q}(\zeta_m)/\mathbf{Q}$, so that $\sigma(\zeta_m) = \zeta_m^s$ for some s coprime to m. If there is an A-equivariant graph isomorphism f between the graphs underlying $\mathcal{D}$ and $\mathcal{D}^\sigma$, where $A = \mathrm{Aut}\,\mathcal{D}$, then the Galois conjugate dessin $\mathcal{D}^\sigma$ results from $\mathcal{D}$ by an application of Wilson's operation H_j where $js \equiv 1 \bmod m$, so $\mathcal{D}^\sigma \cong H_j\mathcal{D}$.*

Proof. We will use the terminology and the results of Lemmas 4 to 7, in particular the isomorphism $f : \mathcal{C} \to \mathcal{C}'$ between the graphs underlying $\mathcal{D}$ and $\mathcal{D}^\sigma$, and the generation of the automorphism group $A = \mathrm{Aut}\,\mathcal{D}$ by a_0 and b_0 as described above.

The dessin $\mathcal{D}$ is uniquely determined by $\mathcal{C}$ and the anticlockwise cyclic rotations of edges around its vertices. The rotation around P_0 is given by $(e_0, a_0e_0, a_0^2e_0, \ldots, a_0^{p-1}e_0)$ where e_0 is an edge incident with P_0, and since $\mathcal{D}$ is regular, the rotation around any white vertex P is given by applying successive powers of an automorphism a fixing P and conjugate in A to a_0. Similarly the rotation around a black vertex Q is given by successive powers of an automorphism b fixing Q and conjugate to b_0. Identifying via f we can regard the dessin $\mathcal{D}^\sigma$ as having the same graph $\mathcal{C}$ as $\mathcal{D}$, but possibly with a different local ordering of the edges around the vertices, corresponding to different generating rotations a and b of A fixing them. Now a_0 acts on $\mathcal{D}^\sigma$ by fixing $f(P_0)$ (which we have identified with P_0), and by the definition of A–equivariance it acts locally around this vertex as multiplication by ζ_p^s. Thus if we were to choose a conjugate to a_0, the resulting cyclic order of edges of $\mathcal{D}^\sigma$ around each white vertex P would be given locally by multiplication by ζ_p^s. Instead, in order to obtain the correct anticlockwise cyclic order of edges around P we need to multiply by ζ_p, so we must choose a conjugate to a_0^j where $js \equiv 1 \bmod m$. Of course, the same idea applies to the black vertices, with b conjugate to b_0^j. By Lemma 4, this is equivalent to replacing $\mathcal{D}$ with $H_j\mathcal{D}$. □

Remark. If $\mathcal{C}' \not\cong \mathcal{C}$, we cannot have $\mathcal{D}^\sigma \cong H_j\mathcal{D}$ for any j since $H_j\mathcal{D}$ has the same embedded graph $\mathcal{C}$ as $\mathcal{D}$, whereas $\mathcal{D}^\sigma$ has $\mathcal{C}'$ as its embedded graph. See Example 8 and Corollary 2 below.

Theorem 2 *Let X_j, $j \in (\mathbf{Z}/m\mathbf{Z})^*$, denote a family of quasiplatonic surfaces of type (p, q, r) where m denotes the least common multiple of p and q, with regular dessins $\mathcal{D}_j := H_j\mathcal{D}_1$ forming an orbit under Wilson's map operations H_j and corresponding to Belyĭ functions β_j on X_j. Suppose that the family of pairs $\{(X_j, \beta_j),\ j \in (\mathbf{Z}/m\mathbf{Z})^*\}$ is invariant under the action of the absolute Galois group $\mathrm{Gal}\,\overline{\mathbf{Q}}/\mathbf{Q}$. Then, as smooth projective algebraic curves, the curves X_j and their Belyĭ functions β_j can be defined over a subfield K of the cyclotomic field $\mathbf{Q}(\zeta_m)$, and they form a single orbit under the action of the absolute Galois group. Here, K is the fixed field of the subgroup*

$$H \ := \ \{j \in (\mathbf{Z}/m\mathbf{Z})^* \mid H_j\mathcal{D}_1 \cong \mathcal{D}_1\},$$

where we identify $(\mathbf{Z}/m\mathbf{Z})^*$ with the Galois group $\mathrm{Gal}\,\mathbf{Q}(\zeta_m)/\mathbf{Q}$.

Proof. Suppose that $\sigma \in \mathrm{Gal}\,\overline{\mathbf{Q}}/\mathbf{Q}$ is a Galois conjugation with $X_1^\sigma \cong X_j$, $\mathcal{D}_1^\sigma \cong \mathcal{D}_j$. Equation (1) also applies in this case to all multipliers. Therefore, if $a \in A$ generates the stabiliser of the white vertex P of $\mathcal{D}_1$, with multiplier ζ_p, it fixes P^σ with multiplier $\sigma(\zeta_p)$. By definition of H_j — see Lemma 1 — a^j has the correct multiplier ζ_p as a generator of the stabiliser P^σ, so $\sigma(\zeta_p^j) = \zeta_p$. The situation is completely analogous for the black vertices, so for all σ fixing the cyclotomic field $\mathbf{Q}(\zeta_m)$ elementwise we have $\mathcal{D}_1^\sigma \cong \mathcal{D}_1$ and hence $X_1^\sigma \cong X_1$. This is moreover the case for each σ whose restriction to $\mathbf{Q}(\zeta_m)$ belongs to H. This means that the moduli field of (X_1, β_1) is contained in the cyclotomic field $K \subseteq \mathbf{Q}(\zeta_m)$ corresponding to H, and by [**3**] (see also [**7**], [**25**, Remark 4] or [**26**, Theorem 5]), X_1 and β_1 can be defined over this moduli field. On the other hand, any $\sigma \in \mathrm{Gal}\,\mathbf{Q}(\zeta_m)/\mathbf{Q}$ with $\sigma(\zeta_m) = \zeta_m^s$ sends $\mathcal{D}_1$ to a Galois conjugate dessin $\mathcal{D}_1^\sigma$ which, by hypothesis, belongs to the family $\{\mathcal{D}_j\}$. The behaviour at the fixed vertices again shows that j must satisfy $sj \equiv 1 \bmod m$. □

Remarks. 1. Since $H_iH_j = H_jH_i$ for all i and j, we can use any $\mathcal{D}_i$ instead of $\mathcal{D}_1$ in the definition of H.
2. Since we are considering isomorphisms of dessins, and not just curves, K is in fact the minimal field of definition of a *Belyĭ pair* (X_j, β_j) where β_j is the Belyĭ function corresponding to $\mathcal{D}_j$. It may happen that the moduli field (minimal field of definition) of X_j is smaller, see [**21**, Remark 1], if there are isomorphisms $X_j \cong X_k$ not compatible with the corresponding Belyĭ functions.

If we consider the white and the black vertices separately in the proof of Theorem 2, we get an obvious extension to Wilson's generalised operations $H_{i,j}$ introduced at the end of Section 2.

Theorem 3 *Let $X_{i,j}$, with (i,j) running over a subset $S \subseteq (\mathbf{Z}/p\mathbf{Z})^* \times (\mathbf{Z}/q\mathbf{Z})^*$, denote a family of quasiplatonic surfaces of type (p,q,r). Let m denote the least common multiple of p and q, and suppose that S admits an action of $(\mathbf{Z}/m\mathbf{Z})^*$, i.e. whenever S contains (i,j) it also contains (ki,kj) for all $k \in (\mathbf{Z}/m\mathbf{Z})^*$. Suppose that these curves $X_{i,j}$ are equipped with regular dessins $\mathcal{D}_{i,j} := H_{i,j}\mathcal{D}_{1,1}$ forming an orbit under Wilson's generalised operations $H_{i,j}$ and corresponding to Belyĭ functions $\beta_{i,j}$ on $X_{i,j}$. Suppose that the family of pairs $\{(X_{i,j},\beta_{i,j}),\ (i,j) \in S\}$ is invariant under the action of the absolute Galois group $\mathrm{Gal}\,\overline{\mathbf{Q}}/\mathbf{Q}$. Then, as smooth projective algebraic curves, the curves $X_{i,j}$ and their Belyĭ functions $\beta_{i,j}$ can be defined over a subfield $K(i,j)$ of the cyclotomic field $\mathbf{Q}(\zeta_m)$. The orbits of the absolute Galois group consist of the dessins*

$$H_{ki,kj}\mathcal{D}_{1,1} \;=\; H_k \circ H_{i,j}\mathcal{D}_{1,1}\ , \quad k \in (\mathbf{Z}/m\mathbf{Z})^*\ ,$$

and the minimal field of definition of the Belyĭ pair $(X_{i,j},\beta_{i,j})$ is the fixed field $K(i,j) \subseteq \mathbf{Q}(\zeta_m)$ of all $k \in (\mathbf{Z}/m\mathbf{Z})^*$ with the property that $\mathcal{D}_{i,j} \cong \mathcal{D}_{ki,kj}$.

Proof. By the same multiplier arguments as in the proof of Theorem 2, all the dessins $\mathcal{D}_{i,j}$ in question are invariant under $\mathrm{Gal}\,\overline{\mathbf{Q}}/\mathbf{Q}(\zeta_m)$, and algebraic conjugations within $\mathbf{Q}(\zeta_m)$ act like Wilson's operations H_k. □

4 Known Galois orbits

We will examine the next example in some detail, in order to illustrate the general principles underlying the preceding theorems.

Example 4 Suppose that p is a prime and q an integer > 2 dividing $p-1$, and let A be the semidirect product of a cyclic normal subgroup $\langle a\rangle$ of order p by a cyclic subgroup $\langle b\rangle$ of order q acting on $\langle a\rangle$ by $b^{-1}ab = a^u$, where the integer u represents a fixed residue class of order q in $(\mathbf{Z}/p\mathbf{Z})^*$ (the isomorphism class of A is independent of the choice of this class). Along the lines of [**21**, Sections 2 and 3] one can prove that there are precisely $\phi(q)$ non-isomorphic regular dessins $\mathcal{D}(s)$, $s \in (\mathbf{Z}/q\mathbf{Z})^*$, of type (p,q,q) with automorphism group A, determined by the generators $h(\gamma_0) = a_0 = a$, $h(\gamma_1) = b_0 = b^s$. (In [**21**] more restrictive assumptions on p and q were made which are not necessary for the present paper, see e.g. [**21**, Remark 2].) These dessins $\mathcal{D}(s)$ are all cyclic coverings of a common quotient $\mathcal{M} = \mathcal{D}(s)/\langle a\rangle$, a genus 0 dessin with one white vertex of valency q at $x = 0$ and q black vertices of valency 1 at $x = \zeta_q^j$, $j \bmod q$, corresponding to the Belyĭ function x^q. Each $\mathcal{D}(s)$ is ramified of order p at all the black vertices of $\mathcal{M}$, so that the embedded graph of $\mathcal{D}(s)$ is the complete bipartite graph $K_{p,q}$.

We use this example first for an application of Theorem 2. Since type and automorphism group are Galois invariants, these dessins form a Galois invariant family. Then we can modify the homomorphism h without changing the kernel by defining

$$h(\gamma_0) = a_0 = a^s \ , \quad h(\gamma_1) = b_0 = b^s$$

for any s prime to pq (the dessin still depends only on $s \bmod q$ and one may assume that $s \equiv 1 \bmod p$), so by Lemma 4 we have $\mathcal{D}(s) = H_s\mathcal{D}(1)$. Hence Theorem 2 implies that the resulting Belyĭ pairs are all defined over $\mathbf{Q}(\zeta_q)$ and form a single Galois orbit, see [**21**, Thm 1]. Specifically, an affine model of the curve X_s corresponding to $\mathcal{D}(s)$ is given by

$$y^p = \prod_{j=1}^{q} (x - \zeta_q^{\bar{s}j})^{u^j}$$

where $\zeta_q = \exp(2\pi i/q)$ and $\bar{s}s \equiv 1 \bmod q$.

Conversely we can use this field of definition and the common graph to see that these Galois conjugate dessins result from each other by Wilson's operations H_s: since the graphs are complete, all Galois conjugations are clearly adjacency preserving, so Theorem 1 applies.

For completeness we consider the behaviour of the generating automorphisms around the fixed points. We may take y as local variable near the q white vertices $P_i = (\zeta_q^i, 0)$, and x near the p black vertices $Q_j = (0, \zeta_p^j)$. As rational maps, the generating automorphisms are given by

$$a \, : \, (x, y) \mapsto (x, \zeta_p y) \, , \quad b \, : \, (x, y) \mapsto (\zeta_q^{\bar{s}} x, \zeta_p^{\bar{u}} y^u (x-1)^{(1-u^q)/p})$$

where we define $\bar{u}$ by $\bar{u}u \equiv 1 \bmod p$. One can check the relation $ab = ba^u$ and verify that a^s and b^s have the correct multipliers (recall that $s \equiv 1 \bmod p$).

Example 5 Now we consider the same group A as in Example 4 under the more restrictive assumptions that p and q are both primes > 3 with $p \equiv 1 \bmod q$ and $q \neq 7$. The group A also occurs in [**21**, Cor. 1] as the automorphism group of $(q-1)(q-2)$ regular dessins $\mathcal{D}(n, t)$ of type (q, q, q) for all $n, t \in (\mathbf{Z}/q\mathbf{Z})^*$ with $n + t \not\equiv 0 \bmod q$, in this case with $h(\gamma_0) := b^n$, $h(\gamma_1) := ab^t$. We can replace these generators with

$$a_0 = b^n \quad , \quad b_0 = (ab)^t$$

by composing h with an automorphism of A, see [**21**, Lemma 3]. With these generators, Lemma 4 applies and shows that for Wilson's generalised operations $H_{n,t}\mathcal{D}(1, 1) = \mathcal{D}(n, t)$ for all n, t coprime to q with $n+t \not\equiv 0 \bmod q$. It is shown in [**21**, Thm 3] that these dessins are all defined over $\mathbf{Q}(\zeta_q)$, and that for fixed n and t the dessins $\mathcal{D}(kn, kt)$, $k \in (\mathbf{Z}/q\mathbf{Z})^*$, form a single Galois orbit. With $S := \{(n, t) \in ((\mathbf{Z}/q\mathbf{Z})^*)^2 \mid n + t \not\equiv 0 \bmod q\}$ this is now a direct application of Theorem 3 since Galois conjugation leaves invariant the automorphism group and ramification properties. Thus these dessins $\mathcal{D}(n, t)$ in fact form a Galois invariant family.

On the other hand, knowing that the corresponding Belyĭ pairs are defined over $\mathbf{Q}(\zeta_q)$ we can also apply Theorem 1 to them — provided, for instance, we can show that Galois conjugation is adjacency preserving on the dessins $\mathcal{D}(n,t)$. To do so, we remark that these dessins all have the same graph: if we label the white and black vertices with $\mathbf{F}_p$ so that A acts as a subgroup of $\mathrm{AGL}_1(\mathbf{F}_p)$, then P_i is joined to Q_j if and only if $j-i$ is in a specific coset of $\langle u\rangle$ depending on u. (For a given p and q, all such graphs are isomorphic, so the choice of a generator u is unimportant.) In particular, a generates translations on the sets of white and black vertices (without fixed points), and the elements b and ab have precisely one fixed point each on the respective sets of white and black vertices. If we think of Galois conjugations as identifying the automorphism group and preserving the labelling of the vertices, then neighbouring vertices P_i and Q_j (as the unique white and black fixed points of b and ab) must remain neighbouring fixed points under the Galois action because $j-i$ remains invariant. Thus Proposition 2 and Theorem 1 apply, showing that Galois conjugacy implies Wilson conjugacy.

More generally, if we apply $H_{i,j}$, replacing a_0 and b_0 with a_0^i and b_0^j for arbitrary i and j coprime to q, then we obtain all $(q-1)(q-2)$ dessins of this type (q,q,q), together with $q-1$ dessins of type (q,q,p) where $n+t \equiv 0 \bmod q$; these are the duals of the dessins in Example 4, interchanging white vertices and face centres.

Example 6 Let $n = p^e$ be an odd prime power, and $\mathcal{M}$ a regular dessin based on an embedding of the complete bipartite graph $\mathcal{B} = K_{n,n}$ such that the map, and not just the hypermap, is regular. These dessins have been classified in [**13**]: they are of type (n,n,n) with automorphism group (of the hypermap)

$$G_f := \langle g, h \mid g^n = h^n = 1\,,\, h^g = h^{1+p^f}\rangle \qquad (\, f \in \{1,\dots,e\}\,)\,. \tag{3}$$

Their isomorphism classes are $\mathcal{M}(f;u)$ where $u \in (\mathbf{Z}/p^{e-f}\mathbf{Z})^*$, the different dessins being distinguished by the choice of a pair of generators $a_0 = g^u$, $b_0 = (gh)^u$, see [**13**, Thm 1] and [**16**, Lemma 1]: here one has to take a representative $u \bmod n$, but the isomorphism class of the dessin depends only on the residue class $u \bmod p^{e-f}$. It follows from Lemma 4 (see also [**13**]) that for each fixed f the maps $\mathcal{M}(f;u)$ form a single orbit under Wilson's operations, with $H_j\mathcal{M}(f;u) \cong \mathcal{M}(f;ju)$ for all j coprime to p; Theorems 1 and 2 of [**16**] show that they correspond to Belyĭ pairs defined over $\mathbf{Q}(\zeta_{p^{e-f}})$, and that they form a single Galois orbit. Since all graphs are complete bipartite, adjacency is preserved under Galois conjugation, so these results again illustrate Theorems 1 and 2.

Example 7 Once again consider a regular embedding of $K_{n,n}$, $n = p^e$ an odd prime power, but now regular only as a hypermap (i.e. with an edge-transitive automorphism group which is not necessarily transitive on directed edges). We again get hypermaps of type (n,n,n) with automorphism group G_f as in (3), but now many more pairs of generators are allowed. Up to a permutation of the three generators of $\Delta = \langle n,n,n\rangle$ (i.e. exchange of black vertices, white vertices and face centres) we can take them to be

$$a_0 = g^u\,, \quad b_0 = (gh)^v\,, \qquad u,v \in (\mathbf{Z}/n\mathbf{Z})^*\,.$$

As before, the isomorphism classes of dessins $\mathcal{M}(f;u,v)$ depend only on $u,v \bmod p^{e-f}$, and by Lemma 4 we again have

$$\mathcal{M}(f;iu,jv) \cong H_{i,j}\mathcal{M}(f;u,v) \quad \text{for all} \quad i,j \quad \text{coprime to} \quad p\,.$$

This series of dessins is also Galois invariant because they are — up to the permutations of the generators mentioned above — the only ones sharing regularity, type and automorphism group G_f. Therefore Theorem 3 applies and shows that the family $\mathcal{M}(f;u,v)$ splits into Galois orbits $\mathcal{M}(f;ku,kv)$, $k \in (\mathbf{Z}/n\mathbf{Z})^*$. Theorem 2 applies to these orbits. For more detailed information, in particular the fields of definition of the underlying curves and their equations, see [**5**]. Since all graphs are again complete bipartite, adjacency is also preserved in these cases.

Example 8 The preceding examples may give the impression that dessins which are Galois conjugate are always equivalent under Wilson's operations. The following example shows that this is not always the case.

A curve X of genus $g > 1$ is a *Hurwitz curve* if $|\mathrm{Aut}\,X|$ attains Hurwitz's upper bound of $84(g-1)$. This is equivalent to X being uniformised by a normal subgroup Γ of finite index $|\Delta : \Gamma| > 1$ in the triangle group Δ of signature $(7,2,3)$, so such surfaces are quasiplatonic and correspond to regular dessins of type $(7,2,3)$, that is, regular maps of type $(7,3)$.

Macbeath [Macb] showed that for each prime $p \equiv \pm 1 \bmod 7$ there are three Hurwitz curves, known as *Macbeath–Hurwitz curves*, with an automorphism group $\mathrm{PSL}_2(\mathbf{F}_p)$. As shown in [**20**] they form a family of three Galois conjugate curves, defined over the cubic field $\mathbf{Q}(\cos\frac{2\pi}{7}) \subset \mathbf{Q}(\zeta_7)$. In the generating triples (x,y,z) for the corresponding dessins the elements x lie in the three conjugacy classes of elements of order 7 in $\mathrm{PSL}_2(\mathbf{F}_p)$, distinguished from each other by their traces. In fact, not only are these three curves and their corresponding dessins mutually non-isomorphic, but so are their underlying graphs, as shown in the Appendix.

There is another reason as well showing that Galois conjugations and Wilson's operations do not coincide on Macbeath–Hurwitz curves: Wilson's operations may change the valencies of the faces of the maps. The case $p = 13$ is simple and instructive. The elements of order 7 in $\mathrm{PSL}_2(\mathbf{F}_{13})$ consist of the three classes $C(t)$ of elements with traces $t = \pm 3$, ± 6 and ± 5, permuted in that cyclic order by squaring. The corresponding dessins $\mathcal{D}(t)$ are distinguished from each other by the order of the commutator $c = [x,y] = x^{-1}y^{-1}xy$, which is respectively $7, 6$ or 13, so that the Petrie polygons in these maps have lengths $14, 12$ or 26. (This example shows that Petrie length, unlike various other combinatorial parameters such as valencies of vertices and faces, is not a Galois invariant.) We can apply Wilson's operation H_2 to each map $\mathcal{D}(t)$, replacing x with $x' = x^2$ but using the same y, or equivalently we can apply H_9 to the Walsh map, sending x to $x^9 = x^2$ and y to $y^9 = y$, so that $z = (xy)^{-1}$ is replaced with $z' = (x^2y)^{-1} = yx^5$. In all three cases z' has order 7 rather than 3, so we get three non-isomorphic regular maps $H_2\mathcal{D}(t)$ of type $(7,7)$, with x' and z' in classes $C(6)$ and $C(3)$, $C(5)$ and $C(6)$, and $C(3)$ and $C(5)$ respectively. However, if we apply the inverse operation H_4 to each $D(t)$ we get maps $H_4\mathcal{D}(t)$ of three different types, namely type $(7,7)$ with $x', z' \in C(5)$, type $(7,6)$ with $x' \in C(3)$, and type $(7,13)$ with $x' \in C(6)$.

The situation is similar for the next possible prime $p = 29$. In this case the elements of order 7 have traces ± 3, ± 7 and ± 11, and are permuted by squaring in this cyclic order. The commutator $c = [x,y]$ has trace ± 8, ± 10 and ± 4 respectively, so that its order is $15, 14$ and 15 giving Petrie lengths $30, 28$ and 30. This distinguishes the second map from the first and third, and for a combinatorial distinction between these two we can consider the element x^4yx^2y, representing a generalised Petrie path turning alternately fourth and second right: in these two cases it has order 29 and 14 respectively, giving path-lengths 58 and 28. Applying H_2 to any of these three maps gives a map of type $(7,7)$, with z' conjugate to x, whereas applying H_4 gives maps of types $(7,15)$, $(7,14)$ and $(7,15)$, with z' conjugate to c in all cases. Once again we have a Galois orbit which is not an orbit under Wilson's operations.

To conclude this section, we recall the common ideas behind the earlier papers which considered the Galois actions on the families of dessins in Examples 4 to 8. It is unclear whether this method has a common graph–theoretic interpretation, but its advantage is the possible application to additional examples such as the Macbeath–Hurwitz curves, where no graph isomorphism is available.

Definition. Let $\mathcal{D}$ be a regular dessin of type (p,q,r), and let its automorphism group A have canonical generators a_0 and b_0. Suppose that for each $i \in (\mathbf{Z}/p\mathbf{Z})^*$, a_0 fixes l_i white vertices of $\mathcal{D}$ with multiplier ζ_p^i, that for each $j \in (\mathbf{Z}/q\mathbf{Z})^*$, b_0 fixes m_j black vertices with multiplier ζ_q^j, and that for each $k \in (\mathbf{Z}/r\mathbf{Z})^*$, a_0b_0 fixes n_k face centres with multiplier ζ_r^k. The sets of all such pairs

$$(i, l_i) \in (\mathbf{Z}/p\mathbf{Z})^* \times \mathbf{N}_0\ , \quad (j, m_j) \in (\mathbf{Z}/q\mathbf{Z})^* \times \mathbf{N}_0\ , \quad (k, n_k) \in (\mathbf{Z}/r\mathbf{Z})^* \times \mathbf{N}_0\ ,$$

which are independent of the choice of a_0, b_0 by the conjugacy of canonical generating pairs, are called the *multiplier data* for $\mathcal{D}$.

Using formula (2) again one can easily see how the absolute Galois group induces an action on the multiplier data: each σ acts on roots of unity ζ of orders p, q and r as $\sigma(\zeta) = \zeta^s$ for some s coprime to pqr, so that the multiplier data for $\mathcal{D}^\sigma$ are obtained from those for $\mathcal{D}$ by replacing the above sets of pairs with the sets of pairs (si, l_i), (sj, m_j) and (sk, n_k).

Proposition 3 *Let A be a finite group, and let $\mathcal{F}$ be the family of all regular dessins $\mathcal{D}$ of type (p,q,r) with automorphism group A. Suppose that there is a bijection between $\mathcal{F}$ and the multiplier data for its dessins $\mathcal{D}$. Then these dessins are all defined over the cyclotomic field $\mathbf{Q}(\zeta_p, \zeta_q, \zeta_r)$, and the orbits of the absolute Galois group on $\mathcal{F}$ are in one-to-one correspondence with its orbits on the multiplier data.*

Proof. As in the proof of Theorem 2, the type and automorphism group of a dessin are Galois invariants, so $\mathcal{F}$ is a Galois invariant family. If σ fixes $\mathbf{Q}(\zeta_p, \zeta_q, \zeta_r)$ elementwise, it fixes all the multiplier data. Since the dessins in $\mathcal{F}$ are uniquely determined by their multiplier data, they are invariant under those σ and are therefore defined over a subfield of $\mathbf{Q}(\zeta_p, \zeta_q, \zeta_r)$. The precise field of moduli of a dessin, equal to its minimal field of definition, is determined by its Galois orbit, and hence by the corresponding Galois orbit on the multiplier data. □

5 Regular dessins based on complete graphs

By work of Lynne James and the first author of the present paper [**12**] it is known that there are regular maps whose underlying graph is the complete graph K_n if and only if n is a prime power p^e. The regular embeddings of K_n were previously constructed by Biggs [**1**] in the following way. Label the vertices with the elements of the finite field $\mathbf{F}_n$, choose a generator u of the cyclic multiplicative group $\mathbf{F}_n^*$, and join each vertex $v \in \mathbf{F}_n$ by an edge to each of the other vertices in the anticlockwise cyclic order

$$v+1 \,,\; v+u \,,\; v+u^2 \,,\; \ldots \,,\; v+u^{n-2}$$

around v. This ordering defines a regular map $\mathcal{M}(n,u)$. Considered as a hypermap it is of type $(n-1, 2, r)$ with

$$r = \frac{1}{2}(n-1) \quad \text{for} \quad 3 < n \equiv 3 \bmod 4 \quad \text{and} \quad r = n-1 \quad \text{otherwise}$$

on a Riemann surface $X(n,u)$ of genus $(n^2-7n+4)/4$ if $n \equiv 3 \bmod 4$ and $(n-1)(n-4)/4$ for all other $n > 3$ [**12**, p. 362]. The automorphism group of $\mathcal{M}(n,u)$ is the affine group $\mathrm{AGL}_1(\mathbf{F}_n)$, acting naturally on the vertices. Two such dessins $\mathcal{M}(n,u)$ and $\mathcal{M}(n,u')$ are isomorphic if and only if u and u' belong to the same orbit under the action of the Galois group $\mathrm{Gal}\,\mathbf{F}_n/\mathbf{F}_p$ [**12**, Thm B]. Since this group is cyclic of order e, generated by the Frobenius automorphism $u \mapsto u^p$, there are $\phi(n-1)/e$ maps $\mathcal{M}(n,u)$ up to isomorphism. It is clear from the construction of these maps that $H_j\mathcal{M}(n,u) = \mathcal{M}(n,u^j)$ for all j coprime to $n-1$ [**12**, p. 362], thus illustrating Corollary 1 and also showing that for fixed n they are all equivalent under Wilson's operations. Theorem 2 implies the following

Theorem 4 *Let n be a prime power p^e and let $\{\mathcal{M}(n,u) \mid u \in (\mathbf{Z}/(n-1)\mathbf{Z})^*\}$ be the family of regular maps resulting from all regular embeddings of the complete graph K_n into orientable surfaces $X(n,u)$, equipped with the structure of smooth complex projective algebraic curves over number fields and with the regular Belyĭ functions β_u corresponding to the dessins $\mathcal{M}(n,u)$. Then these Belyĭ pairs $(X(n,u), \beta_u)$ form a single Galois orbit. Their minimal field of definition is the splitting field K of the prime p in the cyclotomic field extension $\mathbf{Q}(\zeta_{n-1})/\mathbf{Q}$.*

Proof. Since regularity is a Galois invariant [**15**] and since by [**12**] we have a complete list of regular maps based on K_n, our family is Galois invariant, so Theorem 2 applies:

Wilson's operation H_j replaces u with u^j in the above definition of the cyclic order around the vertices.

The only remaining point is to determine the field of definition K. By Theorem 2 and [**12**] again, K is the fixed field of the subgroup

$$H \;=\; \{\, p^k \bmod n-1 \mid k=1,\ldots,e\,\} \;\subseteq\; (\mathbf{Z}/(n-1)\mathbf{Z})^* \;\cong\; \mathrm{Gal}\,\mathbf{Q}(\zeta_{n-1})/\mathbf{Q}$$

inducing all residue class Galois groups for all prime ideals extending p (unramified in $\mathbf{Q}(\zeta_{n-1})$ since p is coprime to $n-1$). Therefore K is the splitting field of p, i.e. the maximal subextension of $\mathbf{Q}(\zeta_{n-1})/\mathbf{Q}$ in which p decomposes completely into d different prime ideals of degree 1 over p, where $d=[K:\mathbf{Q}]=\phi(n-1)/e$. □

Example 9 Let $n=8$, so $p=2$, $e=3$ and there are $d=\phi(7)/3=2$ regular maps $\mathcal{M}(8,u)$. If we take $\mathbf{F}_8=\mathbf{F}_2[x]/(x^3+x+1)$ then these two maps correspond to choosing u from the subsets $\{x,x^2,x^4=x^2+x\}$ or $\{x+1,(x+1)^2=x^2+1,(x^2+1)^2=x^2+x+1\}$ of $\mathbf{F}_8^*$. These two sets are mutually inverse in $\mathbf{F}_8$, so the two maps form a chiral (mirror-image) pair $\mathcal{M}$, $H_{-1}\mathcal{M}$, namely the Edmonds maps of genus 7. If $\zeta:=\zeta_7$ and $\eta:=\zeta+\zeta^2+\zeta^4=(-1+\sqrt{-7})/2$ then the two corresponding Belyĭ pairs are defined over the field $K=\mathbf{Q}(\eta)=\mathbf{Q}(\sqrt{-7})$ fixed by the automorphism $\zeta\mapsto\zeta^2$ of order 3 of the cyclotomic field $\mathbf{Q}(\zeta)$, and they are conjugate under its automorphism $\zeta\mapsto\zeta^{-1}$, or equivalently $\sqrt{-7}\mapsto-\sqrt{-7}$. The decomposition of $p=2$ in this field is given by $2=-\eta(\eta+1)$.

In this example, considered as a hypermap with black vertices as midpoints of all edges in K_n, it is obvious that Galois conjugation preserves adjacency between the white and black vertices of valencies $n-1$ and 2, since it acts as a reflection. In other instances of Theorem 4 it is unclear whether adjacency is always preserved. For instance, when $n=25$ there are $\phi(24)/2=4$ regular maps, forming two chiral pairs: if u is a primitive element of $\mathbf{F}_{25}$, then one pair corresponds to $u^{\pm1}$ and the other to $u^{\pm7}$. Here Wilson's operations and the absolute Galois group both act on these dessins as the Klein four-group $\{\pm1,\pm7\}$ in $(\mathbf{Z}/24\mathbf{Z})^*$, while the complementary subgroup $\{\pm5\}$ (corresponding to the Galois group of $\mathbf{F}_{25}$) acts by dessin automorphisms only. As in the case $n=8$ the Galois conjugations corresponding to $\{\pm1\}$ preserve adjacency, but it is unclear whether those corresponding to $\{\pm7\}$ also do. However, since all dessins of the family are based on the graph K_8, it is easy to see that there are always A–equivariant graph isomorphisms between them, so we could use this weaker hypothesis to apply Theorem 1 in these cases as well.

In [**11**], James classified the orientable embeddings of K_n which are edge-transitive but not regular, so that the automorphism group acts transitively on edges but not on directed edges. Such maps exist if and only if n is a prime power p^e, with $n\equiv 3 \bmod 4$ and $n>3$. As before the vertices of such a map are labelled with the elements of $\mathbf{F}_n$. Choosing a generator u of $\mathbf{F}_n^*$ and an odd integer k with $1<k<n-1$, we define a map $\mathcal{M}(n,u,k)$ by joining each vertex $v\in\mathbf{F}_n$ by an edge to each of the other vertices in the anticlockwise cyclic order

$$v+1\,,\; v+u^k\,,\; v+u^2\,,\; v+u^{k+2}\,,\; v+u^4\,,\; v+u^{k+4}\,,\;\ldots\;,\; v+u^{n-3}\,,\; v+u^{k+n-3}.$$

Two such maps $\mathcal{M}(n,u,k)$ and $\mathcal{M}(n,u',k')$ are isomorphic if and only if u and u' are in the same orbit of $\mathrm{Gal}\,\mathbf{F}_n/\mathbf{F}_p$ and $k' \equiv k$ or $2-k \bmod n-1$ [**11**, Thm B′], so there are $(n-3)\phi(n-1)/4e$ isomorphism classes of such maps [**11**, Thm A].

The automorphism group A of $\mathcal{M}(n,u,k)$ is the unique subgroup of index 2 in $\mathrm{AGL}_1(\mathbf{F}_n)$, consisting of the affine transformations $t \mapsto at+b$ of $\mathbf{F}_n$ such that a is a square in $\mathbf{F}_n^*$. This group acts transitively on the vertices and on the edges of the map, but it has two orbits on the faces, each edge separating faces from different orbits. For instance, the edge joining vertices 0 and u^k separates two faces whose stabilisers in A are cyclic groups generated by the affine transformations $a_0 : t \mapsto -u^{-k}t+1$ and $b_0 : t \mapsto -u^{k-2}t+u^k$, rotating the incident vertices in the anticlockwise cyclic orders $\ldots, u^k, 0, 1, \ldots$ and $\ldots, u^2, 0, u^k, \ldots$. The valencies m_0 and m_1 of these faces are the orders of a_0 and b_0. Since $-1 = u^{(n-1)/2}$ and k is odd we find that $m_0 = (n-1)/2(n-1,k)$ unless $k \equiv (n-1)/2 \bmod (n-1)$, in which case $m_0 = p$; similarly $m_1 = (n-1)/2(n-1,k-2)$ unless $k-2 \equiv (n-1)/2 \bmod (n-1)$, in which case $m_1 = p$. This allows us to count the faces, and hence to compute the genus of the map [**11**].

The dual map $\mathcal{M}(n,u,k)^*$ is bipartite, with white and black vertices corresponding to the two orbits of A on faces of $\mathcal{M}(n,u,k)$, so it is the Walsh map of a hypermap $\mathcal{D}(n,u,k)$ of type $(m_0, m_1, (n-1)/2)$. In fact, the product $a_0b_0 : t \mapsto u^{-2}t$ has order $(n-1)/2$ and rotates the face of $\mathcal{M}(n,u,k)^*$ resulting from the vertex $v=0$ of the original map $\mathcal{M}(n,u,k)$. Since A acts transitively on the edges of $\mathcal{M}(n,u,k)^*$, this hypermap is regular, corresponding to the algebraic hypermap with automorphism group A and generators a_0, b_0. Observe that the involution $k \mapsto 2-k$ transposes the vertex colours of the hypermap.

Theorem 5 *With $n = p^e \equiv 3 \bmod 4$, >3 and u, k, m_0, m_1 as above, let*

$$S \; := \; \{ \, (i,j) \mid i \in (\mathbf{Z}/m_0\mathbf{Z})^* \,,\, j \in (\mathbf{Z}/m_1\mathbf{Z})^* \,,\, j(k-2) - ik \textit{ is coprime to } \frac{n-1}{2} \, \} \, .$$

For these $(i,j) \in S$, we have $H_{i,j}\mathcal{M}(n,u,k)^* \cong \mathcal{M}(n,u^s,k')^*$ with $s \equiv \frac{1}{2}[ik-j(k-2)] \bmod \frac{n-1}{2}$ and $k' \equiv ik/s \bmod n-1$ if we assume i,j,s to be represented by odd integers. They form a Galois invariant family of dessins. If we identify dessins resulting from each other by transposition of the vertex colours, this family splits into $(n-3)/4$ Galois orbits

$$\{ \, H_{ri,rj}\mathcal{M}(n,u,k)^* \,,\, r \in (\mathbf{Z}/(n-1)\mathbf{Z})^* \, \} \; = \; \{ \, \mathcal{M}(n,u^s,k)^* \,,\, s \in (\mathbf{Z}/(n-1)\mathbf{Z})^* \, \}$$

characterised by the odd numbers k, $1 < k < n-1$ modulo the involution $k \mapsto 2-k$. Independently of k, their minimal field of definition K is the splitting subfield of the prime p in $\mathbf{Q}(\zeta_{n-1})$.

Proof. Clearly, a_0^i and b_0^j generate the same automorphism group A as a_0 and b_0, and by Lemma 4 they correspond to a regular dessin of the same type if their product has the same order $\frac{n-1}{2}$ as a_0b_0. The additive constant is irrelevant: renumbering the elements of $\mathbf{F}_n$ by a translation makes it disappear. In the same way one may check that replacing a_0, b_0 with a_0^i, b_0^j amounts to replacing (u,k) with (u^s, k'). Then, as an element of $\mathrm{AGL}_1(\mathbf{F}_n)$, the action is given by

$$a_0^i b_0^j \;:\; t \;\mapsto\; (-1)^{i+j}\, u^{-ik+jk-2j} t \,+\, c$$

for some constant c, where we can omit the factor (-1) since i and j are assumed to be odd. Thus the choice of S guarantees the correct order, and the claim follows directly from Theorem 3 with some rather obvious calculations. The determination of K relies on the same argument as in the proof of Theorem 4. □

Example 10 For the reasons given above, we can restrict to odd k with $1 < k \leq (n-1)/2$.

a) The first case to consider is $p = n = 7$, leading with $k = 3$ to two isomorphism classes of regular hypermaps of type $(7,3,3)$ and genus 3, both on Klein's quartic, compare [**22**] or Example 4.

b) The next case is $p = n = 11$, leading with $k = 3$ and 5 to two Galois orbits, each containing four regular dessins of types $(5,5,5)$ and $(11,5,5)$ in genera 12 and 15, all defined over $\mathbf{Q}(\zeta_{10}) = \mathbf{Q}(\zeta_5)$ (where the prime 11 splits completely). These orbits are special cases of Examples 5 and 4.

c) The first really new case occurs for $n = 27$, $p = 3$, $e = 3$, giving five Galois orbits of regular dessins of type $(13,13,13)$ (for $k = 3,5,7,9,11$) and for $k = 13$ one Galois orbit of type $(3,13,13)$, all defined over the biquadratic field $K = \mathbf{Q}(\zeta+\zeta^3+\zeta^9)$, $\zeta = \zeta_{26}$.

As always, it is much more demanding to find explicit equations for the curves than to determine the fields of definition. We restrict our consideration to an accessible but already sufficiently complicated subcase of the regular hypermaps considered in Thm. 5. In the notation and under the hypothesis given there, we concentrate on the case

$$k := \frac{n-1}{2}\;, \qquad m_0 = p\;, \qquad m_1 = \frac{n-1}{2(n-1,k-2)} = \frac{n-1}{2}$$

(recall that $n \equiv 3 \bmod 4$, hence k and $k-2$ are odd) and the subfamily

$$\{\, H_r\mathcal{M}(n,u,\frac{n-1}{2})^*\,,\; r \in (\mathbf{Z}/(n-1)\mathbf{Z})^*\,\} \;=\; \{\, H_{r,r}\mathcal{M}(n,u,\frac{n-1}{2})^*\,,\; r \in (\mathbf{Z}/(n-1)\mathbf{Z})^*\,\}$$

$$=\; \{\, \mathcal{M}(n,u^s,\frac{n-1}{2})^*\,,\; s \in (\mathbf{Z}/(n-1)\mathbf{Z})^*\,\}\;.$$

According to Theorem 3 they form a Galois orbit, therefore it is sufficient to determine equations for $s = 1$. Recall that $\mathbf{Q}(\zeta_{n-1}) = \mathbf{Q}(\zeta_k)$, and that the splitting subfield of the prime p in $\mathbf{Q}(\zeta_k)$ is the fixed field of the subgroup of $\mathrm{Gal}\,\mathbf{Q}(\zeta_k)/\mathbf{Q}$ generated by $\zeta_k \mapsto \zeta_k^p$.

Theorem 6 *Let $n > 3$ be a prime power $p^e \equiv 3 \bmod 4$. An affine (singular) model of the curve determined by the regular dessin $\mathcal{M}(n,u,\frac{n-1}{2})^*$ is given by e equations in $\mathbf{C}^{e+1}$*

of type

$$y_\nu^p \;=\; \prod_j (z-\zeta^{-p^\nu j})^{\mu_j}\ , \quad \zeta := \zeta_k\ ,\ k=\frac{n-1}{2}\ , \quad \nu=0,\ldots,e-1\ .$$

The integer exponents μ_j depend on the interplay between the additive and the multiplicative structures of $\mathbf{F}_n$ and can be calculated explicitly.

Proof. We consider a regular dessin of type (p,k,k) and with an automorphism group $A\cong C_p^e \rtimes C_k$, defined as that index 2 subgroup of $\mathrm{AGL}_1(\mathbf{F}_n)=\{x\mapsto ax+b \mid a\in \mathbf{F}_n^*, b\in \mathbf{F}_n\}$ for which a runs only over the cyclic group C_k of squares in $\mathbf{F}_n^*$. The normal subgroup $(\mathbf{F}_n,+)$ of translations is isomorphic to C_p^e, and we can fix three generators g_j of A acting on $\mathbf{F}_n$ as

$$g_0: x\mapsto x+1\ , \quad g_1: x\mapsto u^{-2}x\ , \quad g_\infty : x\mapsto u^2(x-1)$$

where u denotes as before a generator of the multiplicative group of the field. The respective orders of the generators are p,k and k, and they satisfy $g_0g_1g_\infty=1$. The kernel N of the homomorphism h of $\Delta=\langle p,k,k\rangle$ onto A given by

$$\gamma_j\mapsto g_j\ , \quad j=0\,,\,1\,,\,\infty$$

is the surface group of the curve X. First we calculate the signature of the normal subgroup $\Gamma_1 := h^{-1}(C_p^e)$ of Δ. To do so, we determine the cycle structure of the action of each generator on the cosets of C_p^e. Considered as permutations, they act as

$$g_0\mapsto (1)\ , \quad g_1\mapsto (1\,2\,\ldots\,k)\ , \quad g_\infty\mapsto (k\,\ldots\,2\,1)\ .$$

Using Singerman's procedure described in [**19**] and the Riemann–Hurwitz formula we can deduce from this information that Γ_1 has genus 0 and signature $\langle 0;p,p,\ldots,p\rangle$ with k generators $\gamma_0,\,\gamma_1^{-1}\gamma_0\gamma_1,\,\ldots,\,\gamma_1^{-(k-1)}\gamma_0\gamma_1^{k-1}$ of order p, mapped by h to the elements

$$g_0: x\to x+1\,,\quad g_1^{-1}g_0g_1: x\mapsto x+u^2\,,\ \ldots\,,\quad g_1^{-(k-1)}g_0g_1^{k-1}: x\mapsto x+u^{n-3}\ .$$

Therefore we can consider the covering $\beta:\Gamma_1\backslash\mathbf{H}\to\Delta\backslash\mathbf{H}$ as a cyclic cover of the Riemann sphere over itself of order k and ramified in two points. In the usual normalisation, β ramifies over 1 and ∞. We renormalise β defining $z:=1-\beta$ so that the covering looks like $z\mapsto z^k$, ramified in and over 0 and ∞, and so that the covering group is the group of k–th roots of unity acting by multiplication. The critical values of the covering $N\backslash\mathbf{H}\to\Gamma_1\backslash\mathbf{H}$ can be identified with these roots of unity ζ^j, $j \bmod k$, and the function field of X is clearly a Kummer extension of $\mathbf{C}(z)$, i.e. a composite extension of e cyclic field extensions of degree p corresponding to e index p subgroups U of C_p^e.

We concentrate first on the more difficult case $e>1$, identify C_p^e with an e–dimensional vector space over $\mathbf{F}_p$ and consider its subgroup U as an $e-1$–dimensional subspace. Besides 0 it contains as many $\mathbf{F}_n$–squares as non–squares because it is the union of one–dimensional $\mathbf{F}_p$–vector spaces inside $\mathbf{F}_n$ intersecting only in 0, and every such one–dimensional vector space contains as many nonzero $\mathbf{F}_n$–squares as non–squares because they are $\mathbf{F}_p^*$–multiples of one additive generator, and $\mathbf{F}_p^*$ contains as many squares as non–squares, and since e is odd, non–squares in $\mathbf{F}_p$ remain non–squares in $\mathbf{F}_n$. So the action of $g_1^{-j} g_0 g_1^j : x \mapsto x + u^{2j}$ on the cosets of U is

- either trivial if $u^{2j} \in U$ (in $(p^{e-1} - 1)/2$ cases)
- or given by a cycle of order p if $u^{2j} \notin U$.

In the trivial case the generator $\gamma_1^{-j} \gamma_0 \gamma_1^j$ of Γ_1 also belongs to the index p subgroup $\Gamma_U := h^{-1}(U)$, therefore the cyclic covering $\Gamma_U \backslash \mathbf{H} \to \Gamma_1 \backslash \mathbf{H}$ is unramified over the fixed point of this generator. If we suppose that γ_1 acts on the Riemann sphere $\Gamma_1 \backslash \mathbf{H}$ as $z \mapsto \zeta z$, and that $z = 1$ is the fixed point of γ_0 under its action, then ζ^{-j} is the fixed point of the generator $\gamma_1^{-j} \gamma_0 \gamma_1^j$. The function field of $\Gamma_U \backslash \mathbf{H}$ is therefore $\mathbf{C}(z, y)$ with

$$y^p \;=\; \prod_{u^{2j} \notin U} (z - \zeta^{-j})^{\mu_j}$$

for some integer exponents μ_j to be determined as follows. In the nontrivial case the covering is ramified of order p over this point ζ^{-j} . Since the local behaviour of $\gamma_1^{-j} \gamma_0 \gamma_1^j$ around its fixed point induces a multiplication of y with ζ_p , and since on the other hand the different branches of y correspond to the different cosets of U , we can determine the μ_j in the following way. Suppose for simplicity that $1 = u^0 \notin U$ such that $\gamma_0 \notin \Gamma_U$ and such that the covering ramifies over $z = 1$. We label the residue classes of U in $C_p^e \cong \mathbf{F}_n$ (corresponding to the branches of y) such that

$$U_\mu := U + \mu \quad \text{for} \quad \mu = 0, 1, \ldots, p-1$$

and such that γ_0 acts on the branches — here on the indices of the residue classes U_μ — as the cyclic permutation $\tau = (0\,1\,\ldots\,p-1)$. Clearly, if $\gamma_1^{-j} \gamma_0 \gamma_1^j \notin U$, it acts as the power τ^{μ_j} where μ_j is determined by

$$U + u^{2j} \;=\; U_{\mu_j} \;=\; U + \mu_j \tag{4}$$

(in particular $\mu_0 = 1$). With this normalisation for the other μ_j the first equation is determined. The explicit calculation of the μ_j depends on the choice of U and requires a nontrivial comparison between the additive and the multiplicative structures of $\mathbf{F}_n$.

We can apply this procedure for any index p subgroup of $C_p^e \cong \mathbf{F}_n$. On the other hand it is clear that e of these subgroups are sufficient provided their common intersection is 0 . This is equivalent to the property that their one–dimensional duals $\hat{U} \subset \hat{\mathbf{F}}_n \cong \mathbf{F}_n$ generate $\mathbf{F}_n$ as an additive group. By the normal basis theorem we can choose some Galois orbit as a basis for the field extension $\mathbf{F}_n / \mathbf{F}_p$. By dualisation, this means that we can choose a subspace U with the property that by taking p^ν–powers, $\nu = 0, \ldots, e-1$, we obtain e subspaces U^{p^ν} with common intersection 0 . Their cosets are obtained by applying the Frobenius automorphisms to the cosets $U_\mu = U + \mu$ of U, of course. In the resulting equations the same effect is obtained if we replace j with $p^\nu j$: observe that $u^{2j} \in (U+\mu)^{p^{e-\nu}}$ is equivalent to $u^{2jp^\nu} \in (U+\mu)^{p^e} = U + \mu$.

The case $e = 1$, $n = p$ is considerably simpler since we have only to consider $U = \{0\}$, and the u^{2j} automatically have integer representatives. We can simply replace the definition (4) with $\mu_j := u^{2j}$, then all other arguments — as far as needed — remain valid. □

Remark. The construction shows again that the Galois automorphisms of $\mathbf{Q}(\zeta)/\mathbf{Q}$ leaving invariant the splitting field of p induce permutations of the coordinates y_ν only, therefore inducing isomorphisms, as predicted by Theorem 5.

Example 11 We take up Example 10 c) with the dessins of type $(3, 13, 13)$. To determine the three equations explicitly, we have to make a good choice of the index 3 subspace U in $\mathbf{F}_{27}$. For some generator u of the multiplicative group $\mathbf{F}_{27}^*$ a [**8**] calculation gives the following exponents in $U \setminus \{0\}$.

$$E(U) \;=\; \{3, 16, 4, 17, 6, 19, 12, 25\} \;\subset\; \mathbf{Z}/26\mathbf{Z}$$

(Since $u^{13} = -1$ it would be sufficient to give the residue classes $3, 4, 6, 12 \bmod 13$ which form in fact a difference set $\bmod 13$, see the remark below.) The images under the operation of $\mathrm{Gal}\,\mathbf{F}_n/\mathbf{F}_p$ correspond to the exponent sets $E(U)$, $3E(U)$ and $9E(U)$ having an empty intersection, so we have $\bigcap U^{p^\nu} = \{0\}$, hence a good choice of U. The u–exponents for the additive residue classes of U can be calculated as

$$\begin{aligned} E(U_1) &= \{0, 1, 2, 8, 11, 18, 20, 22, 23\}\,, \\ E(U_2) &= \{5, 7, 9, 10, 13, 14, 15, 21, 24\}\,. \end{aligned}$$

Therefore, the first of the three equations can be written as

$$y_0^3 \;=\; \prod_{2j \in E(U_1)} (z - \zeta^{-j}) \prod_{2j \in E(U_2)} (z - \zeta^{-j})^2\,,$$

the other two for y_1 and y_2 following from this by replacing ζ^{-j} with ζ^{-3j} and ζ^{-9j}, respectively.

Remark. The prime number cases $n = p$ can be treated as in [**21**], see Example 4; there, take $q = k$, $s = -1$ and replace u with u^2. In the next case $n = p^3$ we may consider the 2–dimensional subspaces $U \subset \mathbf{F}_{p^3}$ as lines in the (cyclic) projective plane $\mathbf{P}^2(\mathbf{F}_p)$, therefore it is not surprising that its elements $\neq 0$, seen as exponents of a fixed generator u of $\mathbf{F}_{p^3}$, correspond to the elements of a difference set $\bmod (p^3-1)/(p-1)$. A similar rule is true for p–exponents $e > 3$; e.g. for $e = 5$ we get the elements of U via generalised difference sets corresponding to the points of a projective 3–space in $\mathbf{P}^4(\mathbf{F}_p)$.

6 Appendix

Lemma 8 *Let $\mathcal{M}$ be a reflexible embedding of a graph $\mathcal{G}$ of girth 3 and prime valency p such that $(p-1)/2$ is also prime (and thus a Sophie Germain prime). Then $\mathrm{Aut}\,\mathcal{G} = \mathrm{Aut}\,\mathcal{M}$, the full automorphism group of $\mathcal{M}$, including orientation–reversing elements.*

Proof. Clearly $\mathrm{Aut}\,\mathcal{G} \geq \mathrm{Aut}\,\mathcal{M}$. Since $\mathcal{M}$ is reflexible the subgroup $(\mathrm{Aut}\,\mathcal{M})_v$ of $\mathrm{Aut}\,\mathcal{M}$ fixing a vertex v acts on the set $N(v)$ of neighbouring vertices of v as the dihedral group D_p, so the subgroup $(\mathrm{Aut}\,\mathcal{G})_v$ of $\mathrm{Aut}\,\mathcal{G}$ fixing v induces on $N(v)$ a permutation group P of degree p containing D_p. By theorems of Galois and Burnside, any transitive group of prime degree p is either doubly transitive or is a proper subgroup of the affine group $\mathrm{AGL}_1(\mathbf{F}_p)$.

First suppose that P is doubly transitive. Since $\mathcal{G}$ has girth 3 it contains a triangle, and since $\mathrm{Aut}\,\mathcal{G}$ acts transitively on the vertices of $\mathcal{G}$ we may assume that v is a vertex of this triangle. Thus at least two of the neighbours of v are adjacent, and since $(\mathrm{Aut}\,\mathcal{G})_v$ is doubly transitive on $N(v)$ they are all adjacent. Thus v and $N(v)$ span a complete subgraph $K(v) \cong K_{p+1}$ in $\mathcal{G}$, and since $\mathcal{G}$ is connected and has valency p it follows that $\mathcal{G} = K(v)$. However, the regular embeddings of complete graphs K_n are all known [**12**], and for $n > 4$ none of them is reflexible, so this contradiction shows that P is a proper subgroup of $\mathrm{AGL}_1(\mathbf{F}_p)$. Now P contains D_p, which is a maximal subgroup of $\mathrm{AGL}_1(\mathbf{F}_p)$ since its index $(p-1)/2$ is prime, so $P = D_p$.

The stabiliser in D_p of any two points is the identity subgroup, so the kernel $K(v)$ of the action of $(\mathrm{Aut}\,\mathcal{G})_v$ on $N(v)$ consists of those $g \in \mathrm{Aut}\,\mathcal{G}_v$ fixing at least two neighbours w of v. Since $\mathcal{G}$ contains a triangle and $\mathcal{M}$ is regular, such a pair v, w have a common neighbour u in $\mathcal{G}$, so g fixes at least two elements u, v of $N(w)$ and hence $g \in K(w)$. Thus $K(v) \leq K(w)$, so the connectedness of $\mathcal{G}$ implies that $K(v) = K(v')$ for all vertices v, v' of $\mathcal{G}$. Since $\mathcal{G}$ has no multiple edges it follows that $K(v) = 1$, so $(\mathrm{Aut}\,\mathcal{G})_v \cong D_p$. Thus $(\mathrm{Aut}\,\mathcal{G})_v = (\mathrm{Aut}\,\mathcal{M})_v$ and hence $\mathrm{Aut}\,\mathcal{G} = \mathrm{Aut}\,\mathcal{M}$. □

Note that one cannot remove that the condition that $\mathcal{G}$ has girth 3: for instance the Fermat curve of degree p gives a reflexible embedding $\mathcal{M}$ of the complete bipartite graph $\mathcal{G} = K_{p,p}$, which has girth 4, with $\mathrm{Aut}\,\mathcal{M} \cong D_p \times D_p$ of order $4p^2$ and $\mathrm{Aut}\,\mathcal{G}$ isomorphic to the wreath product $S_p \wr S_2$ of order $2(p!)^2$. Similarly the reflexibility condition is essential here: for each Mersenne prime p the complete graph $\mathcal{G} = K_{p+1}$, of girth 3 and valency p, has chiral (i.e. regular but not reflexible) embeddings $\mathcal{M}$ with $\mathrm{Aut}\,\mathcal{M} \cong \mathrm{AGL}_1(\mathbf{F}_{p+1})$ and $\mathrm{Aut}\,\mathcal{G} \cong S_{p+1}$, see [**12**].

We can apply Lemma 8 to the Macbeath–Hurwitz dessins, which have valency 7 and girth 3, and are reflexible since the corresponding curves are defined over real fields.

Corollary 2 *Let p be a prime $\equiv \pm 1 \bmod 7$. Then the graphs underlying the three Galois conjugate Macbeath–Hurwitz dessins of type $(7,2,3)$ and automorphism group $\mathrm{PSL}_2(\mathbf{F}_p)$ are pairwise non–isomorphic.*

Proof. According to Lemma 8, any isomorphism between two of them would extend to an isomorphism between the corresponding dessins, contradiction. □

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