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DECIDING POLYHEDRALITY OF SPECTRAHEDRA*

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Abstract. Spectrahedra are linear sections of the cone of positive semidefinite matrices which, as convex bodies, generalize the class of polyhedra. In this paper we investigate the problem of recognizing when a spectrahedron is polyhedral. We generalize and strengthen results of [M. V. Ramana, *Polyhedra, spectrahedra, and semidefinite programming*, in Topics in Semidefinite and Interior-Point Methods, Fields Inst. Commun. 18, AMS, Providence, RI, 1998, pp. 27–38] regarding the structure of spectrahedra, and we devise a normal form of representations of spectrahedra. This normal form is effectively computable and leads to an algorithm for deciding polyhedrality.

Key words. spectrahedra, polyhedra, semidefinite and polyhedral cone, optimization, polyhedral faces

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1. Introduction.

A *polyhedron* $\mathcal{P}$ is the intersection of the convex cone of nonnegative vectors $\mathbb{R}^n_{\geq 0}$ with an affine subspace. By choosing an affine basis for the subspace, we obtain a representation

$$\mathcal{P} \;=\; \left\{\mathbf{x} \in \mathbb{R}^{d-1} \,:\, b_i - \mathbf{a}_i^T\mathbf{x} \;\geq\; 0 \text{ for } i = 1, 2, \ldots, n\right\}$$

for some $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \in \mathbb{R}^{d-1}$ and $b_1, b_2, \ldots, b_n \in \mathbb{R}$. Polyhedra represent the geometry underlying linear programming [23] and, as a class of convex bodies, enjoy considerable interest throughout pure and applied mathematics. A proper superclass of convex bodies that inherits many of the favorable properties of polyhedra is the class of spectrahedra.

A *spectrahedron* $\mathcal{S}$ is the intersection of the convex cone of positive semidefinite matrices with an affine subspace. Identifying the affine subspace with $\mathbb{R}^{d-1}$, we write

$$\mathcal{S} \;=\; \left\{\mathbf{x} \in \mathbb{R}^{d-1} \,:\, x_1A_1 + \cdots + x_{d-1}A_{d-1} + A_d \;\succeq\; 0\right\}, \tag{1.1}$$

where $A_1, A_2, \ldots, A_d \in \mathbb{R}^{n\times n}$ are symmetric matrices. Thus, a spectrahedron is to a semidefinite program what a polyhedron is to a linear program. The associated map $A : \mathbb{R}^{d-1} \to \mathbb{R}^{n\times n}_{\mathsf{sym}}$ given by $A(\mathbf{x}) = x_1A_1 + \cdots + x_{d-1}A_{d-1} + A_d$ is called an affine (symmetric) *matrix map*. A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is *positive semidefinite* $A \succeq 0$ if $v^TAv \geq 0$ for all $v \in \mathbb{R}^n$. Hence, the set of points $\mathcal{S} \subseteq \mathbb{R}^{d-1}$ at which $A(\mathbf{x})$ is positive semidefinite is determined by a (quadratic) family of linear inequalities

$$l_v(\mathbf{x}) \;:=\; v^TA(\mathbf{x})v \;=\; x_1\,v^TA_1v \,+\, x_2\,v^TA_2v \,+\, \cdots \,+\, x_{d-1}\,v^TA_{d-1}v \,+\, v^TA_dv \;\geq\; 0$$

for $v \in \mathbb{R}^n$.

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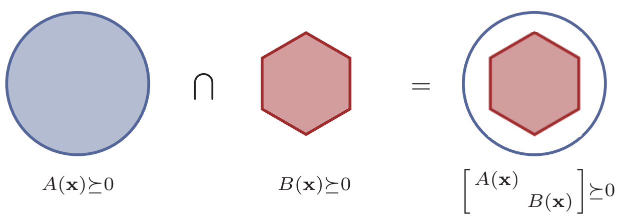
Spectrahedra and their projections have received considerable attention in the geometry of semidefinite optimization [15], polynomial optimization [7], and convex algebraic geometry [9]. To see that polyhedra are spectrahedra, observe that a diagonal matrix is positive semidefinite if and only if the diagonal is nonnegative. Thus, we have

$$\mathcal{P} \;=\; \big\{\mathbf{x} \in \mathbb{R}^{d-1} \,:\, D(\mathbf{x}) \succeq 0\big\},$$

where $D(\mathbf{x}) = \mathrm{Diag}(b_1 - \mathbf{a}_1^T\mathbf{x}, \ldots, b_n - \mathbf{a}_n^T\mathbf{x})$ is a diagonal matrix map.

It is a theoretically interesting and practically relevant question to recognize when a spectrahedron is a polyhedron. The diagonal embedding of $\mathbb{R}^n_{\geq 0}$ into the cone of positive semidefinite matrices suggests that a spectrahedron is a polyhedron if $A(\mathbf{x})$ can be diagonalized, i.e., $UA(\mathbf{x})U^{-1}$ is diagonal for some orthogonal matrix U. By basic linear algebra this is possible if and only if $A(\mathbf{p})$ and $A(\mathbf{q})$ commute for all $\mathbf{p}, \mathbf{q} \in \mathbb{R}^{d-1}$. While this is certainly a sufficient condition, observe that by Sylvester's law of inertia $\mathcal{S} = \big\{\mathbf{x} \,:\, LA(\mathbf{x})L^T \succeq 0\big\}$ for any nonsingular matrix L. In general, however, matrices in the image of $LA(\mathbf{x})L^T$ will not commute; see Example 2.6. A more serious situation is when a polyhedron is *redundantly* presented as the intersection of a proper "big" spectrahedron and a "small" polyhedron contained in it.



In this case, the diagonalizability criterion is genuinely lost.

In this paper we consider the question of algorithmically telling polyhedra from spectrahedra. This question was first addressed by Ramana [18] with a focus on the computational complexity. Our results regarding the structure of spectrahedra strengthen and generalize those of [18], and we present a simple algorithm to test if a spectrahedron $S = \big\{x \,:\, A(\mathbf{x}) \succeq 0\big\}$ is a polyhedron. The algorithm we propose consists of two main components:

(Approximation) Calculate polyhedron $\widehat{\mathcal{S}} \;\supseteq\; \mathcal{S}$ from $A(\mathbf{x})$, and
(Containment) determine whether $\widehat{\mathcal{S}} \;\subseteq\; \mathcal{S}$.

Finding a *fast* algorithm is not to be expected: Ramana [18] showed that deciding whether a spectrahedron is polyhedral is NP-hard. As detailed later, the "containment" step, which is coNP-hard by the results in [10], is done by enumerating all vertices/rays of $\widehat{S}$. This is clearly not feasible in practice, and we make no claim that our algorithm is suitable for preprocessing semidefinite programs. However, as in the case of the "vertex enumeration problem" for polyhedra, it is of considerable interest to have a *practical* algorithm for exploration, experimentation, and hypothesis testing with spectrahedra. Our motivation arose in exactly this context. We nevertheless anticipate applications of our algorithm in the area of (combinatorial) optimization, in particular in connection with *semidefinite extended formulations*;[1]

[1]In this context it is also of interest to detect codimension-one faces of projections of spectrahedra. However, making statements about projections of spectrahedra is generally more challenging as they are geometrically less well behaved, and less (algebraic) information (such as polynomials vanishing on the boundary) is available.

see, for example, [5, 8]. In section 3 our algorithm is discussed in some detail and illustrated along an example. We close with some remarks regarding implementation and the complexity of the approximation step.

As for the approximation step, note that if there is a point $\mathbf{p} \in \mathcal{S}$ with $A(\mathbf{p})$ positive definite, then the *algebraic boundary*, the closure of $\partial\mathcal{S}$ in the Zariski topology, is contained in the vanishing locus of $f(\mathbf{x}) = \det A(\mathbf{x}) \not\equiv 0$. Thus, if $F \subset \mathcal{S}$ is a face of codimension one, the unique supporting hyperplane is a component of the algebraic boundary of $\mathcal{S}$ and hence yields a linear factor of f. Therefore, isolating linear factors in f gives rise to a polyhedral approximation $\widehat{\mathcal{S}}$ of $\mathcal{S}$. However, factoring a multivariate polynomial is computationally expensive, and an alternative is the use of *numerical algebraic geometry* such as Bertini [1] to isolate the codimension-one components of degree one (possibly with multiplicities). Our approach avoids calculating the determinant of the matrix map altogether by pursuing more algebro-geometric considerations. Ramana [18] showed that if $\mathcal{S}$ is a polyhedron, then the relevant linear factors can be read off a block-diagonal form of $A(\mathbf{x})$. The challenge is to find the block-diagonal form. In section 2 we recall and strengthen Ramana's results with very short proofs which highlight the underlying geometry. In particular, our proof emphasizes the role played by eigenspaces of the matrix map. From this, we define a normal form with stronger properties, and we prove that the polyhedral approximation can be obtained by essentially computing the *joint invariant subspace* of two generic points in the image of $A(\mathbf{x})$.

Convention. For reasons of clarity and elegance, we will work in a linear instead of an affine setting. That is, our main objects are exclusively *spectrahedral cones*, and hence all matrix maps are *linear* maps $\mathbb{R}^d \to \mathbb{R}^{n\times n}_{\mathsf{sym}}$. All results can be translated between the linear and affine setting. The spectrahedral cone S that we associate to the spectrahedron $\mathcal{S}$ above is

$$S \;=\; \big\{(\mathbf{x}, x_d) \in \mathbb{R}^d \,:\, x_1A_1 + x_2A_2 + \cdots + x_dA_d \;\succeq\; 0, x_d \geq 0\big\}. \tag{1.2}$$

The following proposition shows that it suffices to consider spectrahedral cones.

PROPOSITION 1.1. *The spectrahedron $\mathcal{S} \neq \varnothing$ given in* (1.1) *is a polyhedron if and only if the associated spectrahedral cone S given by* (1.2) *is a polyhedral cone.*

Proof. Note that for $\alpha > 0$

$$(\mathbf{x}, \alpha) \in S \quad\Longleftrightarrow\quad \mathbf{x} \in \alpha\mathcal{S}.$$

Indeed, $(\mathbf{x}, \alpha) \in S$ if and only if $\frac{1}{\alpha}\mathbf{x} \in \mathcal{S}$. In particular, if S is a polyhedral cone, then $\mathcal{S} \cong S \cap \{(\mathbf{x}, x_d) : x_d = 1\}$ is a polyhedron.

For the converse statement, we observe that the set

$$T \;:=\; \{(\mathbf{x}, \alpha) : \alpha > 0, \mathbf{x} \in \alpha\mathcal{S}\}$$

is a subset of S. Since S is closed, the closure $\overline{T}$ is contained in S as well. We claim that $\overline{T} = S$. Let $(\mathbf{x}, 0) \in S$. By convexity of S, we can pick a sequence $(\mathbf{x}_n, \alpha_n)_{n\geq 0} \in S$ with $\alpha_n > 0$ for all n and $(\mathbf{x}_n, \alpha_n) \xrightarrow{n\to\infty} (\mathbf{x}, 0)$. But this is a sequence in T, and therefore $(\mathbf{x}, 0) \in \overline{T}$. If $\mathcal{S}$ is a polyhedron, then, by the Minkowski–Weyl theorem (see [23, Thm. 1.2]), $\mathcal{S} = \{\mathbf{x} : A\mathbf{x} \leq b\}$ for some $A \in \mathbb{R}^{n\times(d-1)}$ and $b \in \mathbb{R}^n$. But then

$$T \;=\; \{(\mathbf{x}, x_d) : x_d > 0, A\mathbf{x} - x_d b \leq 0\},$$

and $\overline{T} = S$ is a polyhedral cone; see also [23, Prop. 1.14]. $\square$

2. Normal forms and joint invariant subspaces. Let $S = \{\mathbf{x} \in \mathbb{R}^d A(\mathbf{x}) \succeq 0\}$ be a full-dimensional spectrahedral cone given by a linear matrix map $A(\mathbf{x}) = x_1 A_1 + x_2 A_2 + \cdots + x_d A_d$. Throughout this section, we will assume that $A(\mathbf{x})$ is of full rank; i.e., there is a point $\mathbf{p} \in S$ with $A(\mathbf{p}) \succ 0$. As explained in the next section, this is not a serious restriction. We are interested in the codimension-one faces of S and how they manifest in the presentation of S given by $A(\mathbf{x})$. Let us recall the characterization of faces of a spectrahedral cone.

LEMMA 2.1 (see [16, Thm. 1]). *Let $S = \{\mathbf{x} : A(\mathbf{x}) \succeq 0\}$ be a full-dimensional spectrahedral cone. For every face $F \subseteq S$ there is an inclusion-maximal linear subspace $\mathcal{L}_F \subset \mathbb{R}^n$ such that*

$$F \;=\; \{\mathbf{p} \in S \,:\, \mathcal{L}_F \subseteq \ker A(\mathbf{p})\}.$$

For the case of faces of codimension one, this characterization in terms of kernels implies strong restrictions on the describing matrix map.

THEOREM 2.2. *Let $S = \{\mathbf{x} : A(\mathbf{x}) \succeq 0\}$ be a full-dimensional spectrahedral cone, and let $F \subset S$ be a face of codimension one. Then there is a nonsingular matrix $M \in \mathbb{R}^{n\times n}$ such that*

$$MA(\mathbf{x})M^T \;=\; \begin{bmatrix} A'(\mathbf{x}) & \\ & \ell(\mathbf{x})\mathrm{Id}_k \end{bmatrix},$$

where $k \geq 1$ and $\ell(\mathbf{x})$ is a supporting linear form such that $F = \{\mathbf{x} \in S : \ell(\mathbf{x}) = 0\}$.

Proof. Let $B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_d)$ be a basis of $\mathbb{R}^d$ such that $\mathbf{b}_1 \in \operatorname{int} S$ and $\mathbf{b}_2, \ldots, \mathbf{b}_d \in F$. By applying a suitable congruence, we can assume that $A(\mathbf{b}_1) = \mathrm{Id}$. In light of Lemma 2.1, let $U^T = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n) \in \mathbb{R}^{n\times n}$ be an orthonormal basis of $\mathbb{R}^n$ such that $\mathcal{L}_F$ is spanned by $\mathbf{u}_{n-k+1}, \ldots, \mathbf{u}_n$ with $k = \dim \mathcal{L}_F$. It is easily seen that $UA(B\mathbf{x})U^T$ is of the form

$$\begin{bmatrix} A'(B\mathbf{x}) & \\ & x_1\mathrm{Id}_k \end{bmatrix}.$$

Reverting to the original coordinates ($\mathbf{x} \mapsto B^{-1}\mathbf{x}$) replaces x_1 by $\ell(\mathbf{x})$. ◻

The form of the matrix map as given in the previous lemma expresses S as the intersection of a linear halfspace and a spectrahedral cone $S' = \{\mathbf{x} : A'(\mathbf{x}) \succeq 0\}$. Repeating the process for S' proves the following corollary.

COROLLARY 2.3. *Let $S = \{\mathbf{x} : A(\mathbf{x}) \succeq 0\}$ be a full-dimensional spectrahedral cone. Then there is a nonsingular matrix $M \in \mathbb{R}^{n\times n}$ such that*

$$MA(\mathbf{x})M^T \;=\; \begin{bmatrix} Q(\mathbf{x}) & \\ & D(\mathbf{x}) \end{bmatrix}, \tag{$\star$}$$

where $D(\mathbf{x})$ is a diagonal matrix map of order $m \geq 0$. Moreover, if $F \subset S$ is a face of codimension one, then $F = \{\mathbf{x} \in S : D_{ii}(\mathbf{x}) = 0\}$ for some $1 \leq i \leq m$.

If S is a polyhedral cone, then all inclusion-maximal faces have codimension one, and hence S is determined by $D(\mathbf{x})$ alone. This recovers Ramana's result.

COROLLARY 2.4 (see [18, Thm. 1]). *Let S be a full-dimensional spectrahedral cone. Then S is polyhedral if and only if there is a nonsingular matrix $M \in \mathbb{R}^{n\times n}$ such that*

$$MA(\mathbf{x})M^T \;=\; \begin{bmatrix} Q(\mathbf{x}) & \\ & D(\mathbf{x}) \end{bmatrix},$$

where $D(\mathbf{x})$ is a diagonal matrix map and $S = \{\mathbf{x} : D(\mathbf{x}) \succeq 0\}$.

We want to utilize Corollary 2.3 for computations, but the block-diagonal form ($\star$) is not canonical. This is due to the fact that $Q(\mathbf{x})$ might be further block-diagonalized, giving additional linear parts. The natural idea is to prevent this from happening. Let us call a matrix map $Q(\mathbf{x})$ *proper* if there is no $v \in \mathbb{R}^n$ such that v is an eigenvector of $Q(\mathbf{p})$ for all $\mathbf{p} \in \mathbb{R}^d$. It is clear that if $Q(\mathbf{x})$ is not proper, then there is an orthogonal matrix U such that $UQ(\mathbf{x})U^t$ is block-diagonal with a block of order 1.

DEFINITION 2.5. *A matrix map $A(\mathbf{x})$ is in* normal form *if*

$$A(\mathbf{x}) \;=\; \begin{bmatrix} Q(\mathbf{x}) & \\ & D(\mathbf{x}) \end{bmatrix}$$

with $Q(\mathbf{x})$ proper and $D(\mathbf{x})$ diagonal.

Thus for a spectrahedral cone S with $A(\mathbf{x})$ in normal form, we are guaranteed to find all linear forms defining codimension-one faces of S among the linear forms in $D(\mathbf{x})$. In the rest of this section we will be concerned with the question of how to compute the normal form. Let us start with an example where we can do that by hand.

Example 2.6. The two dimensional spectrahedral cone given by

$$A(x, y) \;=\; x\begin{bmatrix} 2 & \\ & 1 \end{bmatrix} \;+\; y\begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \;\succeq\; 0$$

is the polyhedral cone generated by the two vectors $(1, \pm\sqrt{2})$. A congruence that brings $A(x, y)$ into normal form is given by

$$M \;=\; \begin{bmatrix} \frac{1}{\sqrt{2}} & 1 \\ \frac{1}{\sqrt{2}} & -1 \end{bmatrix}.$$

The transformation M is unique up to left-multiplication with $\mathrm{Diag}(a, b)$ and $a, b \in \mathbb{R} \setminus \{0\}$. $\square$

The example shows that the congruence M that brings $A(\mathbf{x})$ into normal form is not necessarily an orthogonal transformation and thus not directly related to the eigenstructure of the matrices in the image of $A(\mathbf{x})$. It turns out that we can assume that M is orthogonal under an additional assumption. As we will see, this is key to the computation of the normal form. A matrix map $A(\mathbf{x})$ is *unital* if $A(\mathbf{p}_0) = \mathrm{Id}$ for some $\mathbf{p}_0 \in \mathbb{R}^d$.

PROPOSITION 2.7. *Let $A(\mathbf{x})$ be a unital matrix map. Then there is an orthogonal $n \times n$ matrix U such that $UA(\mathbf{x})U^T$ is in normal form.*

Proof. If A is a positive definite matrix, then, from a Cholesky decomposition, we get a matrix $L \in \mathbb{R}^{n\times n}$ such that $LAL^T = \mathrm{Id}_n$. We call L a Cholesky inverse of A. It is unique up to left-multiplication by an orthogonal matrix; i.e., if L' also satisfies the condition, then $(L')^{-1}L$ is orthogonal.

Let M be such that $MA(\mathbf{x})M^T$ is in normal form, and, since $Q(\mathbf{p}_0)$ and $D(\mathbf{p}_0)$ are both positive definite, let L_Q and L_D be respective Cholesky inverses such that L_D is diagonal. Now,

$$L = \begin{bmatrix} L_Q & \\ & L_D \end{bmatrix} M$$

also brings $A(\mathbf{x})$ into normal form and is a Cholesky inverse for $A(\mathbf{p}_0)$. However, a Cholesky inverse for $A(\mathbf{p}_0) = \mathrm{Id}_n$ is given by $L' = \mathrm{Id}_n$, and by the above remark, we see that $L = (L')^{-1}L$ is orthogonal. $\square$

This result gives us a way to compute the normal form: For a unital matrix map we seek the *joint invariant subspace*, that is, the largest linear subspace $\mathcal{N} \subseteq \mathbb{R}^n$ such that for all $u \in \mathcal{N}$ and all $\mathbf{p}, \mathbf{q} \in \mathbb{R}^d$ we have $A(\mathbf{p})u \in \mathcal{N}$ (invariant subspace) and $A(\mathbf{p})A(\mathbf{q})u = A(\mathbf{q})A(\mathbf{p})u$. Indeed, $\mathcal{N}$ is then the largest invariant subspace restricted to which $A(\mathbf{x})$ can be simultaneously diagonalized. This will yield the diagonal part $D(\mathbf{x})$.

At this point one could think that the joint invariant subspace of a unital matrix map $A(\mathbf{x})$ can be computed by diagonalizing a single (generic) element $A(\mathbf{p})$. That this is unfortunately not the case is the content of the next example.

Example 2.8. The spectrahedral cone S given by

$$A(x, y, t) \;=\; \begin{bmatrix} t & x & y & \\ x & t & & \\ y & & t & \\ & & & t \end{bmatrix} \;\succeq\; 0$$

is the redundant intersection of the *second order cone* $\{(x, y, t) \,:\, t \geq 0, t^2 \geq x^2 + y^2\}$ and the halfspace $\{(x, y, t) \,:\, t \geq 0\}$.

The matrix map is unital ($A(0, 0, 1) = \mathrm{Id}$), and we claim that $A(x, y, t)$ is already in normal form. Let $Q(x, y, t)$ be the principal submatrix given by the first three rows and columns. We need to argue that $Q(x, y, t)$ is proper. To this end, let $B_1 = A(1, 0, 0)$ and $B_2 = A(0, 1, 0)$. It is easily seen that all eigenspaces of B_1 and B_2 are one dimensional and that no two eigenspaces intersect nontrivially. Hence, there is no $v \neq 0$ that is an eigenvector for both B_1 and B_2. If $Q(x, y, t)$ were not proper, then such a common eigenvector would exist.

Up to scaling, the only eigenvector for all specializations of $A(x, y, t)$ is $(0, 0, 0, 1)$ with eigenvalue $\lambda = t$. But for each specialization, the eigenspace for $\lambda = t$ is of dimension ≥ 2. Hence, $(0, 0, 0, 1)$ is not a distinguished basis vector for the eigenspace corresponding to $\lambda = t$. It is therefore not possible to check if $A(x, y, t)$ is in normal form by analyzing a (generic) point in the image. ☐

The next result shows that the joint invariant subspace of $A(\mathbf{x})$ can be computed from *two* generic points in the image. Here *generic points* refer to points not satisfying a certain polynomial condition (that is implicitly given in the proof).

THEOREM 2.9. *Let $A(\mathbf{x})$ be a unital matrix map, and let $\mathbf{p}, \mathbf{q} \in \mathbb{R}^d$ be two distinct generic points. Let $\mathcal{N} \subset \mathbb{R}^n$ be the smallest subspace containing all eigenvectors common to $A(\mathbf{p})$ and $A(\mathbf{q})$. Then $\mathcal{N}$ is invariant under any matrix in the image of $A(\mathbf{x})$, and $\mathcal{N}^\perp$ is the largest invariant subspace on which $A(\mathbf{x})$ restricts to a proper matrix map.*

Proof. Let us assume that $A(\mathbf{x})$ is already in normal form. Then the joint invariant subspace $\mathcal{N}$ of $A(\mathbf{x})$ can be directly read off, and we have to show that $A(\mathbf{p})$ and $A(\mathbf{q})$ do not have a common eigenvector outside $\mathcal{N}$. That is, we have to consider the situation when $Q(\mathbf{p})$ and $Q(\mathbf{q})$ have a common eigenvector.

The set $\mathcal{V} \subset (\mathbb{C}^{n\times n})^2$ of pairs of matrices (B_1, B_2) such that B_1 and B_2 have a common eigenvector is an algebraic variety. Hence, $\mathcal{V}$ is nowhere dense, and any generic pair of matrices will fail to be in $\mathcal{V}$. To see that it is an algebraic variety, we can argue that the set of tuples $(B_1, \lambda_1, B_2, \lambda_2, v)$ where v is an eigenvector of B_1 and B_2 with eigenvalue λ_1 and λ_2, respectively, is clearly a projective algebraic variety. Using elimination theory (cf. [3, Ch. 3]), we can project onto (B_1, B_2). The result is a proper subvariety of $(\mathbb{C}^{n\times n})^2$ that is equal to $\mathcal{V}$. Since $Q(\mathbf{x})$ is proper, it follows that the image of $Q(\mathbf{x})$ meets $\mathcal{V}$ in a nowhere-dense set.

Alternatively, we can appeal to Theorem 3.3 below: There is an eigenvector common to both $Q(\mathbf{p})$ and $Q(\mathbf{q})$ if and only if

$$\bigcap_{i,j=1}^{n} \ker[Q(\mathbf{p})^i, Q(\mathbf{q})^j] \;\neq\; \{0\},$$

Writing out this condition states that a certain matrix with entries being polynomials in $\mathbf{p}$ and $\mathbf{q}$ does not have full rank. This, in turn, can be checked by calculating a determinant which then has a nonzero polynomial in the entries of $\mathbf{p}$ and $\mathbf{q}$. For generic $\mathbf{p}$ and $\mathbf{q}$ this determinant does not vanish. ∎

3. The algorithm. In this section we describe an algorithm for recognizing polyhedrality of a spectrahedral cone

$$S \;=\; \{\mathbf{x} \in \mathbb{R}^d \,:\, A(\mathbf{x}) \;\succeq\; 0\},$$

where $A(\mathbf{x})$ is a linear, symmetric matrix map of order n. As already stated in the introduction, the algorithm consists of two steps: An "approximation" step constructs an outer polyhedral approximation $\widehat{S}$ from the matrix map $A(\mathbf{x})$ that coincides with S whenever S is polyhedral. This is then verified in the "containment" step.

For the approximation step note that if $A(\mathbf{x})$ is in normal form, then S is presented as the intersection of a spectrahedron without codimension one faces and a polyhedron (both of which can be trivial).

Proposition 3.1. *Let $S = \{\mathbf{x} \in \mathbb{R}^d \,:\, A(\mathbf{x}) \succeq 0\}$ be a full-dimensional spectrahedral cone with*

$$A(\mathbf{x}) \;=\; \begin{bmatrix} Q(\mathbf{x}) & \\ & D(\mathbf{x}) \end{bmatrix}$$

in normal form. Then $\widehat{S} = \{\mathbf{x} \,:\, D(\mathbf{x}) \geq 0\}$ is a polyhedral cone with $S \subseteq \widehat{S}$.

Proof. Let $\mathbf{p} \in S$ be a point. By definition, if $A(\mathbf{p})$ is positive semidefinite, then $A(\mathbf{p})_{ii} \geq 0$ for all i. In particular, $D(\mathbf{p})_{ii} \geq 0$ for all i, which implies that $\mathbf{p} \in \widehat{S}$. ∎

Toward a procedure to bring $A(\mathbf{x})$ into normal form, we need to ensure that S is full-dimensional and $A(\mathbf{x})$ is of full rank. Lemma 2.1 implies that faces of the positive semidefinite cone are embeddings of lower-dimensional positive semidefinite cones into subspaces parametrized by kernels. Recall that the *linear hull* $\operatorname{lin}(C)$ of a convex cone C is the intersection of all linear spaces containing C, and C is full-dimensional relative to $\operatorname{lin}(C)$.

Proposition 3.2 (see [16, Cor. 5]). *Let $S = \{\mathbf{x} \,:\, A(\mathbf{x}) \succeq 0\}$ be a spectrahedral cone, and let $\mathbf{p} \in \operatorname{relint} S$ be a point in the relative interior. Then the linear hull of S is given by*

$$\operatorname{lin}(S) \;=\; \{\mathbf{x} \in \mathbb{R}^d \,:\, \ker A(\mathbf{p}) \subseteq \ker A(\mathbf{x})\}.$$

If $\bar{A}(\mathbf{x})$ is the restriction of $A(\mathbf{x})$ to $(\ker A(\mathbf{p}))^{\perp}$, then

$$S \;=\; \{\mathbf{x} \in \operatorname{lin}(S) \,:\, \bar{A}(\mathbf{x}) \;\succeq\; 0\}$$

and $\bar{A}(\mathbf{p}) \succ 0$.

In concrete terms this means that if M is a basis for the kernel of $A(\mathbf{p})$ at a relative interior point $\mathbf{p} \in S$, then $\operatorname{lin}(S)$ is the kernel for all points in the image of $MA(\mathbf{x})M^T$. The map $\bar{A}(\mathbf{x})$ is given by $M_0 A(\mathbf{x}) M_0^T$ up to a choice of basis M_0 for the orthogonal complement of ker $A(\mathbf{p})$. Since $\bar{A}(\mathbf{p})$ is positive definite, we can choose M_0 so that $\bar{A}(\mathbf{p}) = \mathrm{Id}$ and hence is unital. This, for example, can be achieved by taking advantage of the Cholesky decomposition. By choosing a basis B for $\mathrm{lin}(S)$, we identify $\mathrm{lin}(S) \cong \mathbb{R}^k$ for $k = \dim S$, which ensures that $S \subset \mathbb{R}^k$ is full-dimensional. The resulting spectrahedral cone

$$\bar{S} \;=\; \left\{\mathbf{z} \in \mathbb{R}^k \;:\; \bar{A}(B\mathbf{z}) \;\succeq\; 0\right\}$$

is linearly isomorphic to S (via B).

In actual computations, a point in the relative interior of S may be found by interior point algorithms. In case the spectrahedral cone S is *strictly feasible*, i.e., a point $\mathbf{p} \in \mathbb{R}^d$ with $A(\mathbf{p}) \succ 0$ exists, an interior point algorithm finds a point arbitrarily close to the analytic center of a suitable dehomogenization of S. Viewed as a linear section of the cone of positive semidefinite matrices, S is not strictly feasible if the linear subspace only meets the boundary of $\{X \succeq 0\}$. These are subtle but well-studied cases in which techniques from semidefinite and cone programming such as self-dual embeddings [22, Ch. 5], facial reduction [2], and an iterative procedure analogous to [11, Remark. 4.15] can be used to obtain a point $\mathbf{p} \in \mathrm{relint}\, S$. Independent of the chosen strategy, the computation of a point $\mathbf{p} \in \mathrm{relint}\, S$ is potentially numerically delicate and has to be handled with care. For the purpose of this paper, we will simply follow the first approach as detailed in the implementation remarks below. After applying the above procedure and possibly after a change of basis and a transformation of the matrix map $A(\mathbf{x})$, we may assume that the spectrahedral cone is indeed full-dimensional and described by a unital matrix map.

Utilizing Theorem 2.9, we compute the normal form of the unital matrix map $A(\mathbf{x})$ by determining an orthonormal basis for the joint invariant subspace $\mathcal{N}$. The joint invariant subspace is given as the smallest subspace containing all eigenvectors common to matrices $A(\mathbf{p})$ and $A(\mathbf{q})$ for generically chosen $\mathbf{p}, \mathbf{q} \in \mathbb{R}^d$. It can be computed either by pairwise intersecting eigenspaces of $A(\mathbf{p})$ and $A(\mathbf{q})$ or, somewhat more elegantly, by employing the following result followed by a diagonalization step.

THEOREM 3.3 (see [20, Thm. 3.1]). *Let A and B be two symmetric matrices. Then the smallest subspace containing all common eigenvectors is given by*

$$\mathcal{N} \;=\; \bigcap_{i,j=1}^{n-1} \ker\,[A^i, B^j],$$

where $[A, B] = AB - BA$ is the commutator.

These techniques originate from the theory of finite dimensional $C*$-algebras and have been used in block-diagonalizations of semidefinite programs; see [4, 14]. After all $(n-1)^2$ commutators have been computed, the intersection of their kernels can be computed effectively by means of simple linear algebra. By Theorem 2.9, the restriction of $A(\mathbf{x})$ to $\mathcal{N}$ is a map of pairwise commuting matrices; there is an orthogonal transformation U such that

$$UA(\mathbf{x})U^T = \begin{bmatrix} Q(\mathbf{x}) & \\ & D(\mathbf{x}) \end{bmatrix}$$

has the desired normal form with $Q(\mathbf{x})$ proper and $D(\mathbf{x})$ diagonal. The outer polyhedral approximation of S obtained from $A(\mathbf{x})$ is given by

$$\widehat{S} \;=\; \left\{\mathbf{x} \in \mathbb{R}^d \;:\; D(\mathbf{x}) \geq 0\right\}.$$

It remains to check that $\widehat{S} \subseteq S$. While deciding containment of general (spectrahedral) cones is difficult, we exploit here the finite generation of polyhedral cones.

THEOREM 3.4 (see [23, Thm. 1.3]). *For every polyhedral cone C there is a finite set $R = R(C) \subseteq C$ such that*

$$C \;=\; \left\{ \sum_{\mathbf{r}\in R} \lambda_{\mathbf{r}}\mathbf{r} : \lambda_{\mathbf{r}} \geq 0 \textit{ for all } \mathbf{r} \in R \right\}.$$

Thus, if $R(\widehat{S}) \subseteq S$, we infer that $\widehat{S} \subseteq S \subseteq \widehat{S}$, and hence S is polyhedral. Let us remark that computationally expensive polyhedral computations may be avoided by inspecting the lineality spaces of S and $\widehat{S}$ first. The lineality space of S, i.e., the largest linear subspaces contained in S, is given by the kernel of the linear map $A(\mathbf{x})$. The complete procedure is given in Algorithm 1. As a certificate the algorithm returns the collection of generators $R(\widehat{S})$. As we assume that $A(\mathbf{x})$ is in normal form, it can be easily checked if S is polyhedral or not.

Algorithm 1 *Recognizing polyhedrality of a spectrahedral cone.*

Input: Spectrahedral cone $S = \{\mathbf{x} \in \mathbb{R}^d \,:\, A(\mathbf{x}) \succeq 0\}$ given by a linear matrix map $A(\mathbf{x})$.

1: Generate point $\mathbf{a} \in \mathbb{R}^d$ in the relative interior of S.

2: Compute unital matrix map $\bar{A}(\mathbf{z})$ of order m and linear isomorphism B such that

$$S \;=\; \{B\mathbf{z} \,:\, \bar{A}(\mathbf{z}) \;\succeq\; 0\}.$$

3: Determine the joint invariant subspace $\mathcal{N} = \bigcap_{i,j=1}^{n-1} \ker [\bar{A}(\mathbf{p})^i, \bar{A}(\mathbf{q})^j]$ for two generic points $\mathbf{p}, \mathbf{q} \in \mathbb{R}^k$.

4: Compute an orthonormal basis U corresponding to the decomposition $\mathbb{R}^k = \mathcal{N}^\perp \oplus \mathcal{N}$, and compute

$$U\bar{A}(\mathbf{z})U^T \;=\; \begin{bmatrix} Q(\mathbf{z}) & \\ & D'(\mathbf{z}) \end{bmatrix}.$$

5: Obtain diagonal map $D(\mathbf{z}) = VD'(\mathbf{z})V^T$ via an orthogonal transformation that diagonalizes $D'(\mathbf{p})D'(\mathbf{q})$.

6: Compute the extreme rays $R = R(\widehat{S})$ of the polyhedral cone

$$\widehat{S} \;=\; \{\mathbf{z} \in \mathbb{R}^k \,:\, D(\mathbf{z})_{ii} \geq 0 \text{ for all } i = 1, \ldots, \dim \mathcal{N}\}.$$

7: S is polyhedral if and only if $Q(\mathbf{r}) \succeq 0$ for all $\mathbf{r} \in R$.

Implementation details. The algorithm is implemented in MATLAB using the free optimization package YALMIP [13] and is available as part of the convex algebraic geometry toolbox *Bermeja* [19]. The semidefinite programming solver chosen for the computation of an interior point is SeDuMi [21], which implements a self dual embedding strategy and is thus guaranteed to find a point in the relative interior, even if the spectrahedral cone is not full-dimensional. Extreme rays of $\widehat{S}$ are computed using the software `cdd/cddplus` [6].

In order to illustrate the algorithm, we consider the following example involving a variant of the elliptope $\mathcal{E}_3$ (also known as the "Samosa"); cf. [12].

Example 3.5. The spectrahedral cone $S = \{\mathbf{x} \in \mathbb{R}^4 : A(\mathbf{x}) \succeq 0\}$ with

$$A(\mathbf{x}) = \begin{bmatrix} 4x_4 & 2x_4 + 2x_1 & 2x_4 & 0 & 2x_3 \\ 2x_4 + 2x_1 & 2x_4 + 2x_1 & x_4 + x_1 & 0 & x_3 + x_2 \\ 2x_4 & x_4 + x_1 & 2x_4 + x_1 & x_3 - x_2 & x_3 \\ 0 & 0 & x_3 - x_2 & x_4 + x_1 & 0 \\ 2x_3 & x_3 + x_2 & x_3 & 0 & x_4 \end{bmatrix}$$

is to be analyzed. Since the spectrahedral cone in context is full-dimensional and $A(\mathbf{x})$ is of full rank, i.e., $A(\mathbf{p}) \succ 0$ with $\mathbf{p} = (0, 0, 0, 1)$, the algorithm proceeds by first making the matrix map unital. This is facilitated by applying the Cholesky inverse, computed at the interior point $\mathbf{p}$. The congruence transformation U thus obtained yields the unital matrix map $\bar{A}(\mathbf{z})$, allowing the use of orthogonal transformations thereafter.

The next step involves separating the invariant subspace from its orthogonal complement. This step is carried out using Theorem 3.3 by means of computing all commutator matrices and then intersecting their kernel. The following step involves (simultaneous) diagonalization of the commuting part of the matrix (here the lower right 2×2 block) in order to arrive at the desired normal form. This transformation matrix V may be computed by diagonalizing any generic matrix in the image, restricted to the commuting part. The corresponding unital matrix map $UA(\mathbf{x})U^T$ and its normal form

$$MA(\mathbf{x})M^T \quad \text{with} \quad M = \begin{bmatrix} I & \\ & V \end{bmatrix} U$$

are depicted below:

$$UA(\mathbf{x})U^T = \begin{bmatrix} x_4 & x_1 & x_3 & 0 & 0 \\ x_1 & x_4 & x_2 & 0 & 0 \\ x_3 & x_2 & x_4 & 0 & 0 \\ 0 & 0 & 0 & x_4 + x_1 & x_3 - x_2 \\ 0 & 0 & 0 & x_3 - x_2 & x_4 + x_1 \end{bmatrix},$$

$$MA(\mathbf{x})M^T = \begin{bmatrix} x_4 & x_1 & x_3 & 0 & 0 \\ x_1 & x_4 & x_2 & 0 & 0 \\ x_3 & x_2 & x_4 & 0 & 0 \\ 0 & 0 & 0 & x_4 + x_3 - x_2 + x_1 & 0 \\ 0 & 0 & 0 & 0 & x_4 - x_3 + x_2 + x_1 \end{bmatrix}.$$

The normal form clearly shows that the spectrahedral cone has two polyhedral faces.

The algorithm eventually terminates by confirming existence of a lineality space in the corresponding polyhedral cone, even though the initial spectrahedral cone was pointed. This ensures the nonpolyhedrality of S. Figure 1 shows a dehomogenization $(x_4 = 1)$ of S with its two polyhedral facets.

A word about complexity. Calculating the joint invariant subspace of a matrix map by way of Theorem 2.9 requires the generation of two generic points. In practice picking random points works very well but is not guaranteed to give generic points. Alternatively Theorem 3.3 can be used to compute the joint invariant subspace $\mathcal{N}_{ij}$ of A_i and A_j for all $i < j$ and then take $\mathcal{N} = \bigcap_{ij} \mathcal{N}_{ij}$. Either way, calculating the

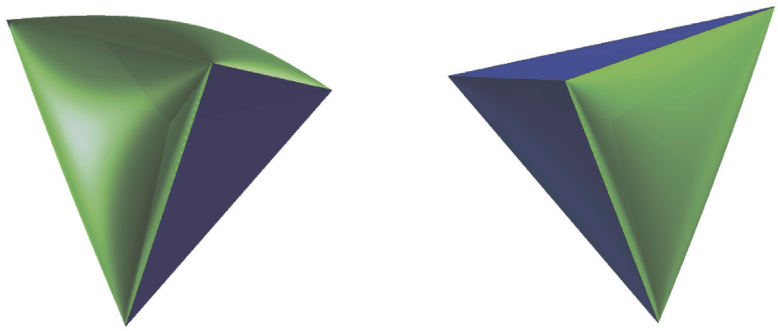


FIG. 1. *Dehomogenization $\mathcal{S}$ (at $x_4 = 1$) of the spectrahedral cone S.*

joint invariant subspace of a matrix map can be done in polynomial time. The transformation of $A(\mathbf{x})$ to a unital matrix map is more involved. The following example, adapted from [17, Example 23], shows that any such procedure may involve numbers with doubly exponential bit complexity.

Example 3.6. Consider the family of spectrahedral cones

$$S_i \;=\; \left\{ \mathbf{x} \in \mathbb{R}^{d+1} : \begin{bmatrix} x_{i+1} & 2x_i \\ 2x_i & x_0 \end{bmatrix} \;\succeq\; 0 \right\} \;=\; \left\{ \mathbf{x} \in \mathbb{R}^{d+1} : \begin{array}{rcl} x_0 & \geq & 0 \\ x_0 x_{i+1} & \geq & 4x_i^2 \end{array} \right\}$$

for $i = 0, \ldots, d-1$. The intersection $S = S_0 \cap S_1 \cap \cdots \cap S_{d-1}$ is strictly contained in the cone $\{\mathbf{x} \in \mathbb{R}^{d+1} : x_i \geq 2^{2^{i-1}} x_0\}$. Denote by $A(\mathbf{x})$ the matrix map for S. Now assume that $B(\mathbf{x})$ is a matrix map for S such that $B(\mathbf{p}) = \mathrm{Id}$ for some $\mathbf{p} \in \operatorname{int} S$. Then $B(\mathbf{x}) = UL\, A(\mathbf{x})\, (UL)^T$, where L is the Cholesky inverse of $A(\mathbf{p})$ and U is an orthogonal matrix. Denote by $\mathbf{l} = (QL)_1$ the first column of QL. From the definition of the Cholesky decomposition we infer that $0 < \|\mathbf{l}\|^2 = L_{11}^2 = \frac{1}{\mathbf{p}_n}$ has doubly exponential bit complexity, and hence for $\mathbf{q} = (1, 0, \ldots, 0)$ we have that $B(\mathbf{q}) = \mathbf{l}\mathbf{l}^T$ has doubly exponential bit complexity. $\square$

We currently do not know if in the computation of the normal form the unital matrix map can be avoided.

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