

# WEAK QUASI-RANDOMNESS FOR UNIFORM HYPERGRAPHS

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ABSTRACT. We study *quasi-random* properties of  $k$ -uniform hypergraphs. Our central notion is uniform edge distribution with respect to large vertex sets. We will find several equivalent characterisations of this property and our work can be viewed as an extension of the well known Chung-Graham-Wilson theorem for quasi-random graphs.

Moreover, let  $K_k$  be the complete graph on  $k$  vertices and  $M(k)$  the line graph of the graph of the  $k$ -dimensional hypercube. We will show that the pair of graphs  $(K_k, M(k))$  has the property that if the number of copies of both  $K_k$  and  $M(k)$  in another graph  $G$  are as expected in the random graph of density  $d$ , then  $G$  is quasi-random (in the sense of the Chung-Graham-Wilson theorem) with density close to  $d$ .

## 1. INTRODUCTION

We study quasi-random properties of  $k$ -uniform hypergraphs,  $k$ -graphs for short. The systematic study of quasi-random or pseudo-random graphs was initiated by Thomason [32, 33]. Roughly speaking, Thomason studied deterministic graphs  $G_n$  of density  $p$  that “imitate” the binomial random graph  $G(n, p)$ , i.e., graphs  $G_n$  that share some important properties with  $G(n, p)$ . One of the key properties of  $G(n, p)$  is its uniform edge distribution and Thomason chose a quantitative version of this property, so-called *jumbledness*, to define pseudo-random graphs. Subsequently Chung, Graham and Wilson [8] (building on the work of others) considered a variation of jumbledness (see property  $P_4$  below) and showed that several other properties of  $G(n, p)$  are equivalent to it in a deterministic sense. In particular, those authors proved the following beautiful result.

**Theorem 1** (Chung, Graham, and Wilson). *For any sequence  $(G_n)_{n \in \mathbb{N}}$  of graphs with  $|V(G_n)| = n$  the following properties are equivalent:*

- $P_1$  : for all graphs  $F$  we have  $N_F^*(G_n) = (1/2)^{\binom{\ell}{2}} n^\ell + o(n^\ell)$ , where  $\ell = |V(F)|$  and  $N_F^*(G_n)$  denotes the number of labeled, induced copies of  $F$  in  $G_n$ ;
- $P_2$  :  $e(G_n) \geq \frac{1}{2} \binom{n}{2} - o(n^2)$  and  $N_{C_4}(G_n) \leq (n/2)^4 + o(n^4)$ , where  $C_4$  is the cycle on 4 vertices and  $N_{C_4}(G)$  denotes the number of labeled (not necessarily induced) copies of  $C_4$  in  $G_n$ ;
- $P_3$  :  $e(G_n) \geq \frac{1}{2} \binom{n}{2} - o(n^2)$ ,  $\lambda_1(G_n) = n/2 + o(n)$ , and  $|\lambda_2(G_n)| = o(n)$ , where  $\lambda_i(G_n)$  is the  $i$ -th largest eigenvalue of the adjacency matrix of  $G_n$  in absolute value;
- $P_4$  : for every subset  $U \subseteq V(G_n)$  we have  $e(U) = \frac{1}{2} \binom{|U|}{2} + o(n^2)$ ;

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- $P_5$  : for every subset  $U = \lfloor n/2 \rfloor$  we have  $e(U) = n^2/16 + o(n^2)$ ;  
 $P_6$  :  $\sum_{u,v} |s(u,v) - n/2| = o(n^3)$ , where for vertices  $u, v \in V(G_n)$  we set  $s(u,v) = |\{x \in V(G_n) : ux \in E(G_n) \Leftrightarrow vx \in E(G_n)\}|$ ;  
 $P_7$  :  $\sum_{u,v} |\text{codeg}(u,v) - n/4| = o(n^3)$ , where for vertices  $u, v \in V(G_n)$  we set  $\text{codeg}(u,v) = |\{x \in V(G_n) : ux \in E(G_n) \text{ and } vx \in E(G_n)\}|$ .  $\square$

Note that, e.g. due to property  $P_4$ , the density of  $G_n$  must tend to  $1/2$ . However, the properties  $P_1, \dots, P_7$  can be altered in a straightforward way and the analogue of Theorem 1 holds for all fixed, positive densities. Moreover, graphs satisfying one (and hence all) of the properties  $P_1, \dots, P_7$  are called *quasi-random* and  $P_1, \dots, P_7$  are quasi-random properties. The list of quasi-random properties was extended by several authors (see, e.g., [23, 24, 26, 27, 28, 29, 34]). Another result related to our work here is the following due to Simonovits and Sós [27].

**Theorem 2** (Simonovits and Sós). *For every  $d > 0$ , every graph  $F$  on  $\ell$  vertices containing at least one edge, and every  $\varepsilon > 0$  there exist  $\delta > 0$  and  $n_0$  such that the following is true. If  $G = (V, E)$  is a graph with  $|V| = n \geq n_0$  vertices such that  $N_F(U) = d^{e(F)}|U|^\ell \pm \delta n^\ell$  for every subset  $U \subseteq V$ , where  $N_F(U)$  denotes the number of labeled copies of  $F$  in the induced subgraph  $G[U]$ , then  $e(U) = d \binom{|U|}{2} \pm \varepsilon n^2$  for every subset  $U \subseteq V$ .  $\square$*

We consider extensions of Theorem 1 and Theorem 2 to  $k$ -graphs. Chung [2, 3], Chung and Graham [5, 6, 7] and Kohayakawa, Rödl, and Skokan [19] studied extensions of some of the properties  $P_1, \dots, P_7$  and showed their equivalences. In particular, for the following notion of quasi-randomness a generalisation of Theorem 1 was obtained: A  $k$ -graph  $H_n$  of density  $d$  is quasi-random, if the edges in  $H_n$  intersect a  $d$ -proportion of the cliques of order  $k$  of every  $(k-1)$ -graph on the same vertex set. In fact, this property can be viewed as a generalisation of  $P_4$  and as it turned out, this notion of quasi-randomness implies the natural analogue of  $P_1$  for  $k$ -graphs. On the other hand, for this notion of quasi-randomness there exist no appropriate extension of Szemerédi’s regularity lemma [31], i.e., there exists no lemma, which guarantees a decomposition of any given  $k$ -graph into relatively “few” blocks, such that most of them satisfy this notion of quasi-randomness. However, a variation of this notion together with a corresponding regularity lemma for  $k$ -graphs was found by Gowers [14, 15] and Rödl et al. [12, 22] (see, e.g., [20] for more details).

We study a simpler notion of uniform edge distribution, which only enforces similar densities induced on vertex sets. More precisely, we consider the following straightforward extension of  $P_4$ .

**DISC $_d(\delta)$** : We say a  $k$ -graph  $H_n$  on  $n$  vertices has DISC $_d(\delta)$  for  $d, \delta > 0$ , if

$$e(U) = d \binom{|U|}{k} \pm \delta n^k \quad \text{for all } U \subseteq V(H_n),$$

where by  $x = y \pm z$  we mean that  $x$  lies in the interval  $[y - z, y + z]$ .

Hypergraphs with property DISC $_d$  were studied in [2, 3, 18] and a straightforward generalisation of Szemerédi’s regularity lemma for this concept was observed to hold in [4, 11, 30] (see Theorem 23 below).

We will suggest extensions of properties  $P_1, P_2, P_6$ , and  $P_7$  to  $k$ -graphs which all turn out to be equivalent to DISC $_d$  (the analogue of  $P_4$  in this context). As a consequence we obtain a new generalisation of Theorem 1 to  $k$ -graphs, which we present in the next section, Section 1.1 (see Theorem 3). In Section 1.2 we will

discuss a consequence of this generalisation for graphs. In particular, we will show that for every integer  $k \geq 2$  the following is true: if the number of copies of the complete graph  $K_k$  and of the line graph of the  $k$ -dimensional hypercube  $M(k)$  are “right” in a given graph  $G$ , then  $G$  is quasi-random (see Corollary 4). We will also verify the equivalence of another property for  $k$ -graphs, which is inspired by Theorem 2 and which we discuss in Section 1.3 (see Theorem 5). Finally, we show the equivalence of several partite variants of  $\text{DISC}_d$  (see Theorem 6 in Section 1.4).

**1.1. Generalisation of Theorem 1.** We establish a generalisation of Theorem 1 for  $k$ -graphs which is based on  $\text{DISC}_d$ . Since  $\text{DISC}_d$  is the straightforward generalisation of  $P_4$ , we need to find generalisations of the other properties of Theorem 1, which are equivalent to  $\text{DISC}_d$ .

**1.1.1. Extension of  $P_1$ .** We start with property  $P_1$ . This property asserts that the number of induced copies of a fixed graph  $F$  in  $G_n$  is asymptotically the same as in the random graph  $G(n, 1/2)$ . It is well known that  $\text{DISC}_d$  does not imply such a property for  $k \geq 3$  as the following example shows: let  $H_n$  be the 3-graph whose edges are formed by the triangles of the random graph  $G(n, 1/2)$ . Chernoff type estimates show that  $H_n$  satisfies  $\text{DISC}_{1/8}$  with high probability. On the other hand, the number of labeled (not necessarily induced) copies of  $K_{1,1,2}^{(3)}$  (the 3-graph with two edges on four vertices) in  $H_n$  is  $\sim n^4/32$ , which is twice as much as the “right” number  $(1/8)^2 n^4$ . Moreover, the number of labeled, induced copies of  $K_{1,1,2}^{(3)}$  in  $H_n$  is  $\sim n^4/64$ , while the “right” number would be  $49n^4/64^2$ .

However, it was shown in [18], that  $k$ -graphs having  $\text{DISC}_d(\delta)$  for sufficiently small  $\delta$  must contain approximately the same number of copies of any fixed linear  $k$ -graph  $F$  as a genuine random  $k$ -graph of the same density. Here a *linear*  $k$ -graph  $F$  is defined as having no pair of edges which intersect in two or more vertices. In other words, the property  $\text{DISC}_d$  implies the following counting-lemma-type property,

$\text{CL}_d(F, \varepsilon)$ : We say a  $k$ -graph  $H_n$  on  $n$  vertices has  $\text{CL}_d(F, \varepsilon)$  for a given linear  $k$ -graph  $F$  on  $\ell$  vertices and  $d, \varepsilon > 0$ , if

$$N_F(H_n) = d^{e(F)} n^\ell \pm \varepsilon n^\ell,$$

where  $N_F(H)$  denotes the number of labeled copies of  $F$  in  $H$ .

For a property  $P_{x_1, \dots, x_p}(\alpha_1, \dots, \alpha_r)$  of  $k$ -graphs we say a sequence  $(H_n)_{n \in \mathbb{N}}$  of  $k$ -graphs with  $|V(H_n)| = n$  has or satisfies  $P_{x_1, \dots, x_p}$ , if for all choices of the parameters  $\alpha_1, \dots, \alpha_r$  there exists an  $n_0$  such that  $H_n$  satisfies  $P_{x_1, \dots, x_p}(\alpha_1, \dots, \alpha_r)$  for all  $n \geq n_0$ . Note that the parameters  $x_1, \dots, x_p$  are fixed for this definition and the fixed parameters always appear as subscripts on the name of the property. Moreover, the parameters  $x_1, \dots, x_p$  and  $\alpha_1, \dots, \alpha_r$  might be of different types, like  $k$ -graphs, integers, or real numbers. For example, in  $\text{CL}_d$  the parameter  $\alpha_1$  is an arbitrary linear  $k$ -graph, while  $x_1$  and  $\alpha_2$  are positive reals. Furthermore, for two properties  $P_{x_1, \dots, x_p}(\alpha_1, \dots, \alpha_r)$  and  $Q_{y_1, \dots, y_q}(\beta_1, \dots, \beta_s)$  we say  $P_{x_1, \dots, x_p}$  implies  $Q_{y_1, \dots, y_q}$  ( $P_{x_1, \dots, x_p} \Rightarrow Q_{y_1, \dots, y_q}$ ), if every sequence of  $k$ -graphs  $(H_n)_{n \in \mathbb{N}}$  that satisfies property  $P_{x_1, \dots, x_p}$  also satisfies property  $Q_{y_1, \dots, y_q}$ . Moreover, properties  $P_{x_1, \dots, x_p}$  and  $Q_{y_1, \dots, y_q}$  are called equivalent if  $P_{x_1, \dots, x_p} \Rightarrow Q_{y_1, \dots, y_q}$  and  $Q_{y_1, \dots, y_q} \Rightarrow P_{x_1, \dots, x_p}$ . With this notation, the aforementioned result from [18] states that

$$\text{DISC}_d \text{ implies } \text{CL}_d. \tag{1}$$

The discussion above suggests that the “right” extension of  $P_1$  in our context involves linear  $k$ -graphs, which leads to the following definition for the induced-counting-lemma-type property.

$\text{ICL}_d(F', F, \varepsilon)$ : We say a  $k$ -graph  $H_n$  on  $n$  vertices has  $\text{ICL}_d(F', F, \varepsilon)$  for given linear  $k$ -graphs  $F' \subseteq F$  with  $V(F') = V(F) = [\ell]$  and  $d, \varepsilon > 0$ , if

$$N_{F', F}^*(H_n) = d^{e(F')} (1 - d)^{e(F) - e(F')} n^\ell \pm \varepsilon n^\ell,$$

where  $N_{F', F}^*(H_n)$  denotes the number of labeled, induced copies of  $F'$  with respect to  $F$  in  $H_n$ , i.e.,  $N_{F', F}^*(H_n)$  is the number of injective mappings  $\varphi: V(F) \rightarrow V(H_n)$  such that for all edges  $e$  of the supergraph  $F$  we have  $\varphi(e) \in E(H_n)$  if and only if  $e$  is an edge of the subgraph  $F'$ .

The notion of induced copies with respect to a linear supergraph  $F$  may look a bit artificial. But it generalises the usual notion of induced graphs in the case of graphs, as may be seen by setting  $F = K_\ell$  to be the complete graph on the same vertex set. We will show that  $\text{ICL}_d$  is equivalent to  $\text{DISC}_d$  for  $k$ -graphs (see Theorem 3 below).

1.1.2. *Extension of  $P_2$* . Next we focus on a generalisation of  $P_2$ . For that we need to identify a  $k$ -graph which in some sense allows us to reverse the implication from (1). Note that there are  $k$ -graphs  $O$  known, which have the following property: if  $O$  appears asymptotically in the “right” frequency in  $H_n$ , then  $H_n$  must satisfy  $\text{DISC}_d$ . However, to our knowledge all known  $k$ -graphs  $O$  with this property are non-linear and, as shown for example in [18],  $\text{DISC}_d(\delta)$  never yields the “right” frequency for any non-linear  $k$ -graph  $O$ . Below we will define a linear  $k$ -graph  $M$  with the same property, i.e.,  $M$  plays the role of  $C_4$  for  $k \geq 3$ . (In fact, for  $k = 2$  the graph  $M$  will be equal to  $C_4$ .)

For a  $k$ -partite  $k$ -graph  $A$  with vertex classes  $X_1, \dots, X_k$  and  $i \in [k]$  we define the *doubling*  $\text{db}_i(A)$  of  $A$  around class  $X_i$  to be the  $k$ -graph obtained from  $A$  by taking two disjoint copies of  $A$  and identifying the vertices of  $X_i$ . More formally,  $\text{db}_i(A)$  is the  $k$ -partite  $k$ -graph with vertex classes  $Y_1, \dots, Y_k$ , where  $Y_i = X_i$  and for  $j \neq i$  we have  $Y_j = X_j \dot{\cup} \tilde{X}_j$  with  $\tilde{X}_j = \{\tilde{x} \mid x \in X_j\}$ . Thus  $\tilde{x}$  denotes the copy of  $x$ . Moreover, the edge set of  $\text{db}_i(A)$  is given by

$$E(\text{db}_i(A)) = E(A) \dot{\cup} \{\{\tilde{x}_1, \dots, \tilde{x}_{i-1}, x_i, \tilde{x}_{i+1}, \dots, \tilde{x}_k\} : \{x_1, x_2, \dots, x_k\} \in E(A)\}.$$

For the construction of the  $k$ -graph  $M$  we will start with a single hyperedge  $K_k$ , which can be seen as a  $k$ -partite  $k$ -graph with partition classes of size 1, and iteratively *double* this  $k$ -graph around the partition classes. More precisely,

$$M = \text{db}_k(\text{db}_{k-1}(\dots \text{db}_1(K_k) \dots)).$$

More generally, set

$$M_0 = K_k \quad \text{and} \quad M_j = \text{db}_j(M_{j-1}) \quad \text{for } j = 1, \dots, k,$$

so that  $M = M_k$ . We observe that for every  $j = 0, \dots, k$  we have

$$|V(M_j)| = j2^{j-1} + (k-j)2^j \quad \text{and} \quad |E(M_j)| = 2^j.$$

Moreover, for the vertex partition  $X_1 \dot{\cup} \dots \dot{\cup} X_k$  of  $M_j$  we have

$$|X_1| = \dots = |X_j| = 2^{j-1} \quad \text{and} \quad |X_{j+1}| = \dots = |X_k| = 2^j.$$

As already mentioned for graphs ( $k = 2$ ) the corresponding graph  $M$  is  $C_4$  and for  $k \geq 3$  the  $k$ -graph  $M$  will turn out to be the “right” generalisation for our purposes.

In fact, it follows from the Cauchy-Schwarz inequality that if  $H_n$  contains at least  $\alpha n^{|V(A)|}$  labeled copies of some given  $k$ -partite  $k$ -graph  $A$ , then  $H_n$  contains at least  $(\alpha^2 - o(1))n^{|V(\text{db}_i(A))|}$  labeled copies of  $\text{db}_i(A)$ . Consequently, every  $k$ -graph  $H_n$  with at least  $d \binom{n}{k} + o(n^k)$  edges contains at least  $(d^{2^k} - o(1))n^{k2^{k-1}}$  labeled copies of  $M$ . Hence, the random  $k$ -graph of density  $d$  contains approximately the minimum number of copies of  $M$  and as we will see  $k$ -graphs  $H_n$  having  $N_M(H_n)$  close to the minimum number will satisfy  $\text{DISC}_d$ . More precisely, we will show that  $\text{MIN}_d$  is another property equivalent to  $\text{DISC}_d$  (see Theorem 3 below), where  $\text{MIN}_d$  is defined as follows.

$\text{MIN}_d(\varepsilon)$ : We say a  $k$ -graph  $H_n$  on  $n$  vertices has  $\text{MIN}_d(\varepsilon)$  for  $d, \varepsilon > 0$ , if

$$e(H_n) \geq d \binom{n}{k} - \varepsilon n^k \quad \text{and} \quad N_M(H_n) \leq d^{2^k} n^{k2^{k-1}} + \varepsilon n^{k2^{k-1}}.$$

We did not find any interesting generalisation of property  $P_3$  from Theorem 1 to  $k$ -graphs for  $k \geq 3$ . Moreover, the extension property  $P_4$  in this work is  $\text{DISC}_d$  and the generalisation of  $P_5$  is straightforward (and the implication  $P_5 \Rightarrow P_4$  could be proved along the lines of [34]). Hence, we continue with the discussion of properties  $P_6$  and  $P_7$ .

1.1.3. *Extension of  $P_6$* . It was already noted in [6] that the property  $P_6$  is closely related to the appearance of subgraphs of  $C_4$ . More precisely, for a graph  $G_n$  let  $\text{EVEN}_{C_4}(G_n)$  be the sum of the number of labeled induced copies of subgraphs of  $C_4$  with an even number of edges, i.e.,

$$\text{EVEN}_{C_4}(G_n) = N_{\emptyset, C_4}^*(G_n) + 4N_{P_2, C_4}^*(G_n) + 2N_{2K_2, C_4}^*(G_n) + N_{C_4, C_4}^*(G_n),$$

where  $\emptyset$  is the subgraph of  $C_4$  without any edges,  $P_i$  is the path with  $i$  edges, and  $2K_2$  is a matching consisting of two edges. Note, that there are four different ways to select a path of length two within a  $C_4$  and there two different way to fix a matching of size two in any given  $C_4$ , while there is only one way to fix a  $C_4$  or an “empty  $C_4$ ” within a cycle of length four. Similarly, set

$$\text{ODD}_{C_4}(G_n) = 4N_{P_1, C_4}^*(G_n) + 4N_{P_3, C_4}^*(G_n).$$

We can rewrite  $\text{ODD}_{C_4}(G_n)$  and  $\text{EVEN}_{C_4}(G_n)$  in terms of  $s(u, v)$  (cf.  $P_6$  in Theorem 1) as follows

$$\text{EVEN}_{C_4}(G_n) = \sum_{u, v \in V} \left( s(u, v)^2 + (n - s(u, v))^2 \right) + o(n^4)$$

and

$$\text{ODD}_{C_4}(G_n) = 2 \sum_{u, v \in V} \left( s(u, v)(n - s(u, v)) \right) + o(n^4).$$

Hence, property  $P_6$  is, due to the Cauchy-Schwarz inequality, equivalent to the following property.

$$P_6': |\text{EVEN}_{C_4}(G_n) - \text{ODD}_{C_4}(G_n)| = \sum_{u, v \in V} (2s(u, v) - n)^2 = o(n^4).$$

For the extension of  $P_6'$  to  $k$ -graphs, we replace  $C_4$  by  $M$  from property  $\text{MIN}_d$  and in order to deal with arbitrary densities  $d > 0$  we need a different weight function for the subgraphs of  $M$ . For a  $k$ -graph  $H_n$  and  $1 \geq d > 0$  we define a weight function  $w: \binom{V(H_n)}{k} \rightarrow [-1, 1]$  and set for  $e \in \binom{V(H_n)}{k}$

$$w(e) = \begin{cases} 1 - d & \text{if } e \in E(H_n) \\ -d & \text{if } e \notin E(H_n). \end{cases}$$

For a labeled copy  $\tilde{A}$  of a given  $k$ -graph  $A$  in the complete  $k$ -graph on  $V(H_n)$  we set

$$w(\tilde{A}) = \prod_{e \in E(\tilde{A})} w(e).$$

It is easy to check that for a graph  $G_n$  and  $d = 1/2$  we have

$$|\text{EVEN}_{C_4}(G_n) - \text{ODD}_{C_4}(G_n)| = 16 \left| \sum_{\tilde{C}_4} w(\tilde{C}_4) \right| + o(n^4),$$

where the sum runs over all labeled copies  $\tilde{C}_4$  of  $C_4$  in the complete graph on  $V(G_n)$ .

With this in mind, we define the generalisation of  $P_6$  as follows, which may be viewed as a weighted form of  $\text{MIN}_d$ .

**DEV $_d(\varepsilon)$ :** We say a  $k$ -graph  $H_n$  on  $n$  vertices has  $\text{DEV}_d(\varepsilon)$  for  $d, \varepsilon > 0$ , if

$$\left| \sum_{\tilde{M}} w(\tilde{M}) \right| \leq \varepsilon n^{k2^{k-1}}$$

where the sum runs over all labeled copies  $\tilde{M}$  of  $M$  in the complete  $k$ -graph on  $V(H_n)$ .

Again Theorem 3 will show that  $\text{DEV}_d$  is equivalent to  $\text{DISC}_d$ .

1.1.4. *Extension of  $P_7$ .* The last property we consider here is  $P_7$ . Roughly speaking,  $P_7$  asserts that most pairs of vertices of  $G_n$  have approximately  $n/4$  neighbours and this implies, on the one hand, that the number of labeled  $C_4$ 's in  $G_n$  is close to  $n^4/16$ , while, on the other hand, for most vertices  $v$  the number of labeled  $C_4$ 's containing  $v$  satisfies  $\sum_{w \in V} (\text{codeg}(v, w))^2 \sim n \times (n/4)^2$  as well as  $\sum_{u, u' \in N(v)} \text{codeg}(u, u') \sim (\deg(v))^2 (n/4)$ , which yields  $\deg(v) \sim n/2$ . Consequently,  $P_7$  implies  $P_2$  and the reverse implication follows from the Cauchy-Schwarz inequality. From this point of view the obvious generalisation of  $P_7$  concerns the number of labeled copies of  $M_{k-1}$  attached to a fixed, labeled set of  $2^{k-1}$  vertices. We now make this precise.

Let  $H_n$  be a  $k$ -graph on  $n$  vertices. Let  $X_k$  be the (unique) largest vertex class of  $M_{k-1}$  and, for  $q = 2^{k-1}$ , let  $x_1, \dots, x_q$  be an arbitrary labeling of the vertices of  $X_k$ . For an ordered set  $\mathbf{u} = (u_1, \dots, u_q)$  of  $q$  vertices in  $V(H_n)$ , we denote by  $\text{ext}(M_{k-1}, H_n, \mathbf{u})$  the number of copies of  $M_{k-1}$  in  $H_n$  extending  $\mathbf{u}$  in a canonical way, i.e.,  $\text{ext}(M_{k-1}, H_n, \mathbf{u})$  is the number of injective, edge preserving mappings  $\varphi: V(M_{k-1}) \rightarrow V(H_n)$  with  $\varphi(x_i) = u_i$  for  $i = 1, \dots, q$ . The generalisation of  $P_7$  then reads as follows.

**MDEG $_d(\varepsilon)$ :** We say a  $k$ -graph  $H_n$  on  $n$  vertices has  $\text{MDEG}_d(\varepsilon)$  for  $d, \varepsilon > 0$ , if

$$\sum_{\mathbf{u}} \left| \text{ext}(M_{k-1}, H_n, \mathbf{u}) - d^{2^{k-1}} n^{(k-1)2^{k-2}} \right| \leq \varepsilon n^{(k+1)2^{k-2}}$$

where the sum runs over all ordered  $2^{k-1}$ -element subsets  $\mathbf{u}$  in  $V(H_n)$ .

After this discussion of the extension of properties  $P_1, P_2, P_6$ , and  $P_7$  we state our first result (for the proof see Section 2), which asserts that those generalisations are equivalent (recall the definition of equivalent properties in the paragraph above (1)).

**Theorem 3.** *For every integer  $k \geq 2$  and every  $d > 0$  the properties  $\text{DISC}_d, \text{CL}_d, \text{ICL}_d, \text{MIN}_d, \text{DEV}_d$ , and  $\text{MDEG}_d$  are equivalent.*

Note that, due to  $\text{MIN}_d$ , restricting  $\text{ICL}_d$  to all pairs of  $k$ -graphs  $F' \subseteq F$  for  $\ell \geq k2^{k-1}$  fixed is already equivalent to  $\text{DISC}_d$  and, in fact,  $P_1$  was stated in [8] in that form.

In the proof of Theorem 3 we will use (1) which was proved in [18]. We will include a direct proof of the implication from  $\text{DEV}_d$  to  $\text{CL}_d$  in Section 2.5.

**1.2. Forcing pairs for graphs.** Theorem 3, although a result about  $k$ -graphs, has an interesting consequence for graphs. Recall property  $P_2$  essentially says that if the density of a graph  $G$  is at least  $d - o(1)$  and the density of 4-cycles is at most  $d^4 + o(1)$ , then  $G$  is a quasi-random graph with density  $d$ . In other words, lower and upper bounds on the number of  $K_2$  and  $C_4$  in  $G$  imply that  $G$  is quasi-random and the question arises which other pairs of graphs replacing  $K_2$  and  $C_4$  have the same effect. Such pairs are called *forcing pairs* (note that our definition differs from [8], as we consider non-induced copies). For example, it follows from the work in [8] and [29] that  $C_4$  may be replaced by any even cycle or any complete bipartite graph  $K_{a,b}$  with  $a, b \geq 2$ . Moreover, it follows from the recent work of Hatami [16] that  $C_4$  can be replaced by  $Q_k$ , the graph of the  $k$ -dimensional hypercube for  $k \geq 2$ .

However, all known forcing pairs consist of bipartite graphs and it would be interesting to find forcing pairs involving non-bipartite graphs (see, e.g., [24]). Below, we will use Theorem 3 combined with Theorem 2 to verify certain forcing pairs involving cliques.

For an integer  $k$  let  $M(k)$  be the graph which we obtain if we replace every hyperedge of the  $k$ -graph  $M_k$  by a graph clique of order  $k$ . Since the  $k$ -graph  $M_k$  is linear, the graph  $M(k)$  consists of  $2^k$  graph cliques  $K_k$ , which intersect in at most one vertex. Hence,  $M(k)$  consists of  $k2^{k-1}$  vertices and  $2^k \binom{k}{2}$  edges. (Alternatively,  $M(k)$  is the graph we obtain from the  $k$ -dimensional hypercube, by letting  $V(M(k))$  be the edges of the hypercube and letting edges of  $M(k)$  connect two edges of the hypercube if they have a common end-vertex. In other words,  $M(k)$  is the line graph of the graph of the  $k$ -dimensional hypercube  $Q_k$ .) The following corollary of Theorem 3 shows that for every  $k \geq 2$  the pair of graphs  $K_k$  and  $M(k)$  is a forcing pair.

**Corollary 4.** *For every integer  $k \geq 2$ , every  $d > 0$ , and every  $\delta > 0$  there exist  $\varepsilon > 0$  and  $n_0$  such that the following is true. If  $G = (V, E)$  is a graph on  $|V| = n \geq n_0$  vertices that satisfies*

$$N_{K_k}(G) \geq d \binom{k}{2} n^k - \varepsilon n^k \quad \text{and} \quad N_{M(k)}(G) \leq d^{2^k} \binom{k}{2} n^{k2^{k-1}} + \varepsilon n^{k2^{k-1}},$$

*then  $G$  satisfies  $\text{DISC}_d(\delta)$ .*

*Proof.* We briefly sketch the proof of Corollary 4. From the given graph  $G$  we construct a  $k$ -graph  $H = H(G)$ , where the hyperedges of  $H$  correspond to the cliques  $K_k$  of  $G$ . Therefore we have a one-to-one correspondence between the hyperedges of  $H$  and the  $K_k$ 's of  $G$ , as well as, between the copies of  $M_k$  in  $H$  and the copies of  $M(k)$  in  $G$ . Hence, the assumption on  $G$  implies that  $H$  satisfies  $\text{MIN}_{d'}$  for  $k$ -graphs for  $d' = d \binom{k}{2}$  and from Theorem 3 we infer that  $H$  satisfies  $\text{DISC}_{d'}(\varepsilon')$  for  $k$ -graphs for some  $\varepsilon' = \varepsilon'(\varepsilon)$  with  $\varepsilon' \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . But  $\text{DISC}_{d'}(\varepsilon')$  for  $H$  implies that the assumption of Theorem 2 for the graphs  $F = K_k$  and  $G$  are met and, hence, Theorem 2 yields that  $G$  satisfies  $\text{DISC}_d(\delta)$  for graphs for some  $\delta = \delta(\varepsilon')$ .  $\square$

**1.3. Hereditary subgraphs properties.** From Theorem 3 we know that  $k$ -graphs containing the “right” number of copies of  $M$  are quasi-random. However, note that for characterising quasi-randomness the linear  $k$ -graph  $M$  cannot be replaced by an arbitrary (linear)  $k$ -graph. For example, in the case of graphs, the  $C_4$  in  $P_2$  cannot be replaced by a triangle, as the following example from [8] shows: partition the vertex set  $V(G_n)$  in four sets  $X_1 \dot{\cup} X_2 \dot{\cup} X_3 \dot{\cup} X_4 = V(G_n)$  as equal as possible and add the edges of the complete graph on  $X_1$ , of the complete graph on  $X_2$ , of the complete bipartite graph with vertex classes  $X_3$  and  $X_4$ , and of the random bipartite graph of density  $1/2$  with vertex classes  $X_1 \dot{\cup} X_2$  and  $X_3 \dot{\cup} X_4$ . Simple calculations show, that  $G_n$  defined this way has density  $1/2 + o(1)$  and contains  $n^3/8 + o(n^3)$  labeled triangles. On the other hand,  $G_n$  is not quasi-random, as it obviously violates  $P_4$ . Moreover, due to Theorem 1, a quasi-random graph must be *hereditarily* quasi-random, since if  $G_n$  satisfies  $P_4$ , then induced subgraphs  $G_n[U]$  for large subsets also satisfy  $P_4$  (with a bigger error). Consequently, any property equivalent to  $P_4$  must directly apply to induced subgraphs of linear sized subsets. (It is not obvious that all the properties in Theorem 1 indeed have this quality, but e.g. due to Theorem 1 it follows.) Returning to the example of triangles, we note that the “counterexample” shows that there are graphs which have globally the “right” number of triangles, but there are large subsets on which the number of triangles is wrong, e.g.  $G_n[X_1]$  contains too many (more than  $(n/4)^3/8$ ) triangles. In order to rule out this phenomenon Simonovits and Sós suggested a notion of hereditary properties and in [27] they showed that a graph  $G$  with density  $d$  is quasi-random if and only if every induced subgraph of  $G$  contains the right number of copies of a fixed graph  $F$  (see Theorem 2). This result has been extended to the case of induced copies of  $F$  by Simonovits and Sós [28] and by Shapira and Yuster [24]. We will continue this line of research and introduce hereditary properties for  $k$ -graphs, which are equivalent to  $\text{DISC}_d$ .

Let  $H_n$  be a  $k$ -graph on  $n$  vertices and let  $F$  be a  $k$ -graph with vertex set  $[\ell] = \{1, \dots, \ell\}$ . For pairwise disjoint sets  $U_1, \dots, U_\ell \subseteq V(H_n)$  let  $N_F(U_1, \dots, U_\ell)$  denote the number of partite-isomorphic, copies of  $F$  in  $H_n$ , i.e., the number of  $\ell$ -tuples  $(h_1, \dots, h_\ell)$  with  $h_1 \in U_1, \dots, h_\ell \in U_\ell$  such that  $\{h_{i_1}, \dots, h_{i_k}\}$  is an edge in  $H_n$  if  $\{i_1, \dots, i_k\}$  is an edge in  $F$ . We define the following properties and show that they are equivalent to  $\text{DISC}_d$ .

$\text{HCL}_{d,F,\alpha}(\varepsilon)$ : We say a  $k$ -graph  $H_n$  on  $n$  vertices has  $\text{HCL}_{d,F,\alpha}(\varepsilon)$  for a linear  $k$ -graph  $F$  with  $V(F) = [\ell]$ , a vector  $\alpha = (\alpha_1, \dots, \alpha_\ell) \in (0, 1)^\ell$  with  $\sum_{i=1}^\ell \alpha_i < 1$ , and  $d, \varepsilon > 0$ , if for all choices of pairwise disjoint subsets  $U_1, \dots, U_\ell \subset V(H_n)$  with  $|U_i| = \lfloor \alpha_i n \rfloor$  for all  $i \in [\ell]$  we have

$$N_F(U_1, \dots, U_\ell) = d^{e(F)} \prod_{i \in [\ell]} |U_i| \pm \varepsilon n^\ell.$$

$\text{HCL}_{d,F}(\varepsilon)$ : We say a  $k$ -graph  $H_n$  on  $n$  vertices has  $\text{HCL}_{d,F}(\varepsilon)$  for a linear  $k$ -graph  $F$  with  $V(F) = [\ell]$  and  $d, \varepsilon > 0$ , if  $H_n$  satisfies  $\text{HCL}_{d,F,\alpha}(\varepsilon)$  for every vector  $\alpha = (\alpha_1, \dots, \alpha_\ell) \in (0, 1)^\ell$  with  $\sum_{i=1}^\ell \alpha_i < 1$ .

**Theorem 5.** *For every integer  $k \geq 2$ , every linear  $k$ -graph  $F$  with at least one edge and  $V(F) = [\ell]$ , every  $d > 0$ , and every vector  $\alpha \in (0, 1)^\ell$  with  $\sum_{i=1}^\ell \alpha_i < 1$  the properties  $\text{DISC}_d$ ,  $\text{HCL}_{d,F}$ , and  $\text{HCL}_{d,F,\alpha}$  are equivalent.*

We prove Theorem 5 in Section 3. We also like to mention that the property  $\text{HCL}_{d,F}$  can be weakened in the graph case. In fact, Theorem 2 shows that it suffices



to ensure approximately the right number of copies of the fixed graph  $F$  in every subset  $U \subseteq V(G_n)$  of the vertices of  $G_n$  to make  $G_n$  quasi-random. We, however, need the assumption for all partitions of  $U$  into  $\ell$  classes. It seems quite plausible that this stronger looking assumption is not needed and, in fact, for 3-graphs this was proved recently by Dellamonica and Rödl [10].

**1.4. Partite versions of DISC.** Property  $P_4$  of Theorem 1 has a very natural bipartite version, stating that the number of edges between two subsets is close to half of all possible edges between those sets. More precisely, we may consider the following property.

$$P'_4: e(U, W) = |U||W|/2 + o(n^2) \text{ for all pairwise disjoint subsets } U, W \subseteq V(G_n), \text{ where } e(U, W) \text{ denotes the number of edges with one vertex in } U \text{ and one vertex in } W.$$

It is well known that in fact  $P_4$  and  $P'_4$  are equivalent. For example  $P_4$  implies  $P'_4$  due to the identity  $e(U, W) = e(U \cup W) - e(U) - e(W)$ , while  $P_4$  follows from  $P'_4$  by considering  $e(U', W')$  for a random partition  $U = U' \dot{\cup} W'$  of a given set  $U$  into classes of size  $|U|/2$ .

Below we introduce several partite variants of  $\text{DISC}_d$  for  $k$ -graphs, which will turn out to be equivalent. We start with some definitions. For integers  $1 \leq \ell \leq k$  we call  $\tau: [\ell] \rightarrow [k]$  an  $(\ell, k)$ -function if  $\sum_{i \in [\ell]} \tau(i) = k$ . The set of all  $(\ell, k)$ -functions will be denoted by  $T(\ell, k)$ . For a fixed  $\tau \in T(\ell, k)$  and  $\ell$  pairwise disjoint sets  $U_1, \dots, U_\ell \subset V$  of some vertex set  $V$  we say a  $k$ -set  $K \in \binom{V}{k}$  has *type*  $\tau$  (with respect to  $(U_1, \dots, U_\ell)$ ), if  $|K \cap U_i| = \tau(i)$  for all  $i \in [\ell]$ . The family of all  $k$ -sets having type  $\tau$  is denoted by

$$\text{Vol}_\tau(U_1, \dots, U_\ell) = \left\{ K \in \binom{V}{k} : K \text{ has type } \tau \right\}$$

and let  $\text{vol}_\tau(U_1, \dots, U_\ell) = |\text{Vol}_\tau(U_1, \dots, U_\ell)| = \prod_{i \in [\ell]} \binom{|U_i|}{\tau(i)}$ .

Alternatively  $\text{Vol}_\tau(U_1, \dots, U_\ell)$  can be considered the complete  $k$ -graph with respect to type  $\tau$ . The actual edges of a  $k$ -graph  $H_n$  with vertex set  $V$  of type  $\tau$  with respect to  $(U_1, \dots, U_\ell)$  will be denoted by

$$E_\tau(U_1, \dots, U_\ell) = E(H_n) \cap \text{Vol}_\tau(U_1, \dots, U_\ell)$$

and we set  $e_\tau(U_1, \dots, U_\ell) = |E_\tau(U_1, \dots, U_\ell)|$ . Note that for  $k = 2$  and  $\ell = 1, 2$  there exists only one  $(\ell, k)$ -function and edges of the corresponding type are considered in  $P_4$  ( $\ell = 1$ ) and in  $P'_4$  ( $\ell = 2$ ). For general  $k \geq 2$  we define the following property.

**DISC $_{d,\tau}(\varepsilon)$ :** We say a  $k$ -graph  $H_n$  on  $n$  vertices has  $\text{DISC}_{d,\tau}(\varepsilon)$  for some  $(\ell, k)$ -function  $\tau$ , and  $d, \varepsilon > 0$ , if

$$e_\tau(U_1, \dots, U_\ell) = d \cdot \text{vol}_\tau(U_1, \dots, U_\ell) \pm \varepsilon n^k$$

for all pairwise disjoint subsets  $U_1, \dots, U_\ell \subseteq V(H_n)$ .

Next, we define the notion of the  $\ell$ -partite sub- $k$ -graph with respect to the pairwise disjoint sets  $U_1, \dots, U_\ell \subset V(H_n)$ . The edge set of the complete  $\ell$ -partite  $k$ -graph with respect to the classes  $U_1, \dots, U_\ell$  is given by

$$\text{Vol}(U_1, \dots, U_\ell) = \bigcup_{\tau \in T(\ell, k)} \text{Vol}_\tau(U_1, \dots, U_\ell) \quad (2)$$

and the actual edge set of the  $\ell$ -partite sub- $k$ -graph on  $U_1, \dots, U_\ell$  is

$$E(U_1, \dots, U_\ell) = E(H_n) \cap \text{Vol}(U_1, \dots, U_\ell). \quad (3)$$

Finally, we consider the following notion of uniform edge distribution.

**DISC<sub>d,ℓ</sub>(ε):** We say a  $k$ -graph  $H_n$  on  $n$  vertices has DISC<sub>d,ℓ</sub>(ε) for some positive integer  $ℓ ≤ k$ , and  $d, ε > 0$ , if

$$e(U_1, \dots, U_\ell) = d \cdot \text{vol}(U_1, \dots, U_\ell) \pm \varepsilon n^k$$

for all pairwise disjoint subsets  $U_1, \dots, U_\ell \subseteq V(H_n)$ .

Note that for arbitrary  $k$  the properties DISC<sub>d</sub>, DISC<sub>d,1</sub>, and DISC<sub>d,(1)</sub> are the same and DISC<sub>d,k</sub> and DISC<sub>d,(1,...,1)</sub> are the same. Moreover, for  $k = 2$  these two properties are equivalent. The following result states that in fact any version of DISC defined above is equivalent to any other.

**Theorem 6.** *For all integer  $ℓ$  and  $k$  with  $1 ≤ ℓ ≤ k$ , every fixed  $(ℓ, k)$ -function  $τ$ , and every  $d > 0$  the properties DISC<sub>d</sub>, DISC<sub>d,ℓ</sub>, and DISC<sub>d,τ</sub> are equivalent.*

## 2. PROOF OF THEOREM 3

In this section we present the proof of Theorem 3. We have to show that for every  $k ≥ 2$  and every  $d > 0$  the properties DISC<sub>d</sub>, CL<sub>d</sub>, ICL<sub>d</sub>, MIN<sub>d</sub>, DEV<sub>d</sub>, and MDEG<sub>d</sub> are equivalent. As already noted in (1) it was shown in [18] that DISC<sub>d</sub> implies CL<sub>d</sub>. In Section 2.1 we will show the following obvious implications

$$\begin{array}{ccc} \text{CL}_d & \xrightarrow{\text{Fact 8}} & \text{ICL}_d \\ \text{Fact 7} \downarrow & & \downarrow \text{Fact 9} \\ \text{MIN}_d & & \text{DEV}_d \end{array} \quad (4)$$

and the proofs of the main implications

$$\text{MIN}_d \xrightarrow{\text{Lemma 10}} \text{DISC}_d \quad \text{and} \quad \text{DEV}_d \xrightarrow{\text{Lemma 13}} \text{DISC}_d$$

will be given in Sections 2.2 and 2.3. Finally, we prove the equivalence of MDEG<sub>d</sub> and MIN<sub>d</sub> in Section 2.4 (see Lemma 14), which concludes the proof of Theorem 3.

In addition in Section 2.5 we verify a more direct proof of the implication from DEV<sub>d</sub> to CL<sub>d</sub>.

**2.1. Simple facts.** In this section we verify the simple implications from (4). The first implication, CL<sub>d</sub> ⇒ MIN<sub>d</sub>, follows from the definition that a sequence  $(H_n)_{n \in \mathbb{N}}$  satisfies CL<sub>d</sub> if for every linear  $k$ -graph  $F$  and every  $ε > 0$  all but finitely many  $k$ -graphs  $H_n$  of the sequence satisfy CL<sub>d</sub>( $F, ε$ ).

**Fact 7.** *For every integer  $k ≥ 2$ , every  $d > 0$ , and every  $ε > 0$  there exists  $n_0$  such that the following is true. If  $H$  is a  $k$ -graph that satisfies CL<sub>d</sub>( $K_k, ε/2$ ) and CL<sub>d</sub>( $M, ε$ ), then  $H$  satisfies MIN<sub>d</sub>( $ε$ ).*

*Proof.* Clearly, satisfying CL<sub>d</sub>( $K_k, ε/2$ ) implies  $e(H_n) ≥ d \binom{n}{k} - \varepsilon n^k$  for sufficiently large  $n$  and satisfying CL<sub>d</sub>( $M, ε$ ) yields  $N_M(H) ≤ d^{|E(M)|} n^{|V(M)|} + \varepsilon n^{|V(M)|}$ , which gives MIN<sub>d</sub>( $ε$ ). □

A standard argument using the principle of inclusion and exclusion yields the implication from CL<sub>d</sub> to ICL<sub>d</sub>.

**Fact 8.** For every integer  $k \geq 2$ , every  $d > 0$ , all linear  $k$ -graphs  $F' \subseteq F$  with  $V(F') = V(F) = [\ell]$  for some integer  $\ell$ , and every  $\varepsilon > 0$ , there exists  $\delta > 0$  such that the following is true. If  $H$  is a  $k$ -graph that satisfies  $\text{CL}_d(\hat{F}, \delta)$  for every  $k$ -graph  $\hat{F}$  with  $F' \subseteq \hat{F} \subseteq F$ , then  $H$  satisfies  $\text{ICL}_d(F', F, \varepsilon)$ .

*Proof.* Let  $\delta = \varepsilon/2^{e(F)-e(F')}$  and  $H$  be a  $k$ -graph on  $n$  vertices. Note that by the principle of inclusion and exclusion we have

$$N_{F',F}^*(H) = \sum_{F' \subseteq \hat{F} \subseteq F} (-1)^{e(\hat{F})-e(F')} N_{\hat{F}}(H).$$

Since  $H$  satisfies  $\text{CL}_d(\hat{F}, \delta)$  for every  $k$ -graph  $\hat{F}$  with  $F' \subseteq \hat{F} \subseteq F$  we obtain

$$N_{F',F}^*(H) = d^{e(F')} (1-d)^{e(F)-e(F')} n^\ell \pm 2^{e(F)-e(F')} \delta n^\ell,$$

which shows that  $H$  satisfies  $\text{ICL}_d(F', F, \varepsilon)$ .  $\square$

We close this section by observing that  $\text{ICL}_d$  implies  $\text{DEV}_d$ .

**Fact 9.** For every integer  $k \geq 2$ , every  $d > 0$ , and every  $\varepsilon > 0$ , there exists  $\delta > 0$  such that the following is true. If  $H$  is a  $k$ -graph that satisfies  $\text{ICL}_d(M', M, \delta)$  for every  $k$ -graph  $M' \subseteq M$ , then  $H$  satisfies  $\text{DEV}_d(\varepsilon)$ .

*Proof.* Set  $\delta = \varepsilon/2^{2^k}$ . Let  $H$  be a  $k$ -graph on  $n$  vertices with vertex set  $V = V(H)$  satisfying  $\text{ICL}_d(M', M, \delta)$  for every  $M' \subseteq M$ . Recall that the edge weights  $w$  of the complete  $k$ -graph  $K_V$  on  $V$  are  $1-d$  for edges of  $H$  and  $-d$  for edges of the complement of  $H$ . Moreover,  $w(\tilde{A})$  for subgraph  $\tilde{A} \subseteq K_V$  is  $\prod_{e \in E(\tilde{A})} w(e)$ . Summing over all copies  $\tilde{M}$  of  $M$  in  $K_V$  we obtain

$$\sum_{\tilde{M}} w(\tilde{M}) = \sum_{M' \subseteq M} (1-d)^{e(M')} (-d)^{2^k - e(M')} N_{M',M}^*(H).$$

Applying the assumption that  $H$  satisfies  $\text{ICL}_d(M', M, \delta)$  for all  $k$ -graphs  $M' \subseteq M$  we get

$$\begin{aligned} \sum_{\tilde{M}} w(\tilde{M}) &= \sum_{M' \subseteq M} (1-d)^{e(M')} (-d)^{2^k - e(M')} \left( d^{e(M')} (1-d)^{2^k - e(M')} \pm \delta \right) n^{|V(M)|} \\ &= \sum_{j=0}^{2^k} \binom{2^k}{j} \left( d(1-d) \right)^j \left( (-d)(1-d) \right)^{2^k - j} n^{|V(M)|} \pm 2^{2^k} \delta n^{|V(M)|}. \end{aligned}$$

Consequently, the binomial theorem and the choice of  $\delta$  yields  $\text{DEV}_d$ ,

$$\left| \sum_{\tilde{M}} w(\tilde{M}) \right| \leq \varepsilon n^{|V(M)|}.$$

$\square$

**2.2. MIN implies DISC.** In this section we focus on one of the central implications of Theorem 3 and prove the following lemma, which asserts that  $\text{MIN}_d$  implies  $\text{DISC}_d$ .

**Lemma 10.** For every integer  $k \geq 2$ , every  $d > 0$ , and every  $\varepsilon > 0$ , there exists  $\delta > 0$  and  $n_0$  such that the following is true. If  $H$  is a  $k$ -graph on  $n \geq n_0$  vertices that satisfies  $\text{MIN}_d(\delta)$ , then  $H$  satisfies  $\text{DISC}_d(\varepsilon)$ .

Before we prove Lemma 10 we introduce a bit of notation, which will be also useful for the proof of Lemma 13. It will be convenient to consider the number of homomorphisms from certain  $k$ -graphs  $A$  to some  $k$ -graph  $H$ , instead of the number of labeled copies of  $A$  in  $H$ . Recall that a homomorphism from  $A$  to  $H$  is a (not necessarily injective) mapping from  $V(A)$  to  $V(H)$  that preserves edges. Note that the difference of the number of homomorphisms and the number of labeled copies of  $A$  in  $H$  is  $o(|V(H)|^{|V(A)|})$ , which is inessential for the properties considered in Theorem 3.

Let  $A$  be a  $k$ -partite  $k$ -graph given with its partition classes  $X_1, \dots, X_k$  and let  $U_1, \dots, U_k$  be (not necessarily pairwise disjoint) subsets of  $V(H)$  and set  $\mathcal{U} = (U_1, \dots, U_k)$ . We denote by  $\text{Hom}(A, H, \mathcal{U})$  those homomorphisms  $\varphi$  from  $A$  to  $H$  that map every  $X_i$  into  $U_i$ , i.e.  $\varphi(X_i) \subseteq U_i$  for all  $i \in [k]$ . Furthermore, let  $\text{hom}(A, H, \mathcal{U}) = |\text{Hom}(A, H, \mathcal{U})|$ .

Moreover, let  $X_i = \{x_{i,1}, \dots, x_{i,|X_i|}\}$  be a labeling of the vertices of the partition class  $X_i$ . Then, for an  $|X_i|$ -tuple  $\mathbf{u}_i = (u_1, \dots, u_{|X_i|}) \in U_i^{|X_i|}$  denote by  $\text{Hom}(M, H, \mathcal{U}, i, \mathbf{u}_i)$  those homomorphisms  $\varphi$  from  $\text{Hom}(M, H, \mathcal{U})$ , that map the  $j$ -th vertex in the ordering of  $X_i$  to  $u_j$ , i.e.,  $\varphi(x_{i,j}) = u_j$ . Similarly, let  $\text{hom}(A, H, \mathcal{U}, i, \mathbf{u}_i) = |\text{Hom}(A, H, \mathcal{U}, i, \mathbf{u}_i)|$ .

The following well known fact (see, e.g. [29]) will be useful for the proof of Lemma 10.

**Fact 11.** *For every  $\gamma > 0$  there exists  $\eta > 0$  such that for all non-negative reals  $a_1, \dots, a_N$  and  $a$  satisfying  $\sum_{i=1}^N a_i \geq (1 - \eta)aN$  and  $\sum_{i=1}^N a_i^2 \leq (1 + \eta)a^2N$ , we have  $|\{i \in [N]: |a - a_i| < \gamma a\}| > (1 - \gamma)N$ .  $\square$*

*Proof of Lemma 10.* We first make a few observations (see Claim 12 below). For that let  $H$  be a  $k$ -graph with vertex set  $V = V(H)$  and let  $U_1, \dots, U_k$  be arbitrary, not necessarily disjoint, subsets of  $V$ . Set  $\mathcal{U} = (U_1, \dots, U_k)$ . For every  $j \in [k]$  the Cauchy-Schwarz inequality yields

$$\begin{aligned} \sum_{\mathbf{u}_j \in U_j^{2^j-1}} (\text{hom}(M_{j-1}, H, \mathcal{U}, j, \mathbf{u}_j))^2 \\ \geq \frac{1}{|U_j|^{2^j-1}} \left( \sum_{\mathbf{u}_j \in U_j^{2^j-1}} \text{hom}(M_{j-1}, H, \mathcal{U}, j, \mathbf{u}_j) \right)^2. \end{aligned} \quad (5)$$

Furthermore note, that  $M_j = \text{db}_j(M_{j-1})$ , i.e.,  $M_j$  arises from  $M_{j-1}$  by ‘‘fixing’’ the vertices from the  $j$ -th partition class of  $M_{j-1}$ , denoted by  $X_j(M_{j-1})$ , and ‘‘doubling’’ all other vertices of  $M_{j-1}$  and the corresponding edges. Thus, this definition yields the following identity for every  $j \in [k]$ .

$$\begin{aligned} \text{hom}(M_j, H, \mathcal{U}) &= \sum_{\mathbf{u}_j \in U_j^{2^j-1}} \text{hom}(M_j, H, \mathcal{U}, j, \mathbf{u}_j) \\ &= \sum_{\mathbf{u}_j \in U_j^{2^j-1}} (\text{hom}(M_{j-1}, H, \mathcal{U}, j, \mathbf{u}_j))^2. \end{aligned} \quad (6)$$

Combining (5) and (6), we get

$$\begin{aligned} \text{hom}(M_j, H, \mathcal{U}) &\stackrel{(6)}{=} \sum_{\mathbf{u}_j \in U_j^{2^{j-1}}} (\text{hom}(M_{j-1}, H, \mathcal{U}, j, \mathbf{u}_j))^2 \\ &\stackrel{(5)}{\geq} \frac{1}{|U_j|^{2^{j-1}}} \left( \sum_{\mathbf{u}_j \in U_j^{2^{j-1}}} \text{hom}(M_{j-1}, H, \mathcal{U}, j, \mathbf{u}_j) \right)^2 \\ &= \frac{1}{|U_j|^{2^{j-1}}} (\text{hom}(M_{j-1}, H, \mathcal{U}))^2. \end{aligned}$$

Iterating the last estimate  $j - \ell + 1$  times for some  $1 \leq \ell \leq j$  we get the following line of inequalities for every integer  $r$  between  $\ell$  and  $j$

$$\text{hom}(M_j, H, \mathcal{U}) = \sum_{\mathbf{u}_j \in U_j^{2^{j-1}}} (\text{hom}(M_{j-1}, H, \mathcal{U}, j, \mathbf{u}_j))^2 \quad (7)$$

$$\begin{aligned} &\geq \left( \frac{1}{|U_j|} \right)^{2^{j-1}} \left( \sum_{\mathbf{u}_j \in U_j^{2^{j-1}}} \text{hom}(M_{j-1}, H, \mathcal{U}, j, \mathbf{u}_j) \right)^2 \\ &\dots \\ &\geq \left( \prod_{i=r+1}^j \frac{1}{|U_i|} \right)^{2^{j-1}} \left( \sum_{\mathbf{u}_r \in U_r^{2^{r-1}}} (\text{hom}(M_{r-1}, H, \mathcal{U}, r, \mathbf{u}_r))^2 \right)^{2^{j-r}} \end{aligned} \quad (8)$$

$$\geq \left( \prod_{i=r}^j \frac{1}{|U_i|} \right)^{2^{j-1}} \left( \sum_{\mathbf{u}_r \in U_r^{2^{r-1}}} \text{hom}(M_{r-1}, H, \mathcal{U}, r, \mathbf{u}_r) \right)^{2^{j-r+1}} \quad (9)$$

$$\begin{aligned} &\dots \\ &= \left( \prod_{i=\ell}^j \frac{1}{|U_i|} \right)^{2^{j-1}} \left( \text{hom}(M_{\ell-1}, H, \mathcal{U}) \right)^{2^{j-\ell+1}}. \end{aligned} \quad (10)$$

Combining the last line of inequalities with Fact 11 yields the following claim.

**Claim 12.** *For all integers  $k \geq j \geq \ell \geq 1$  and every  $\gamma_{j,\ell} > 0$  there exists  $\eta_{j,\ell} > 0$  such that for all  $\mathcal{U} = (U_1, \dots, U_k)$  with  $U_i \subseteq V$  the following is true. If*

- (a)  $\text{hom}(M_{\ell-1}, H, \mathcal{U}) \geq (1 - \eta_{j,\ell}) d^{2^{\ell-1}} \prod_{i=1}^{\ell-1} |U_i|^{2^{\ell-2}} \prod_{i=\ell}^k |U_i|^{2^{\ell-1}}$  and
- (b)  $\text{hom}(M_j, H, \mathcal{U}) \leq (1 + \eta_{j,\ell}) d^{2^j} \prod_{i=1}^j |U_i|^{2^{j-1}} \prod_{i=j+1}^k |U_i|^{2^j}$

*hold, then for every  $r$  with  $\ell \leq r \leq j$  the following holds. For all but at most  $\gamma_{j,\ell} |U_r|^{2^{r-1}}$  tuples  $\mathbf{u}_r = (u_1, \dots, u_{2^{r-1}})$  from  $U_r^{2^{r-1}}$  we have*

$$\text{hom}(M_{r-1}, H, \mathcal{U}, r, \mathbf{u}_r) = (1 \pm \gamma_{j,\ell}) d^{2^{r-1}} \prod_{i=1}^{r-1} |U_i|^{2^{r-2}} \prod_{i=r+1}^k |U_i|^{2^{r-1}}.$$

*Proof of Claim 12.* Note that the assumptions (a) and (b) of the claim yield a lower bound for the right-hand side of (10) and an upper bound for the left-hand

side in (7). Consequently, for every  $r$  between  $\ell$  and  $j$  we obtain from (8) and (9)

$$\sum_{\mathbf{u}_r \in U_r^{2^{r-1}}} (\text{hom}(M_{r-1}, H, \mathcal{U}, r, \mathbf{u}_r))^2 \leq (1 + \eta_{j,\ell})^{1/2^{j-r}} d^{2^r} \prod_{i=1}^r |U_i|^{2^{r-1}} \prod_{i=r+1}^k |U_i|^{2^r}$$

and

$$\sum_{\mathbf{u}_r \in U_r^{2^{r-1}}} \text{hom}(M_{r-1}, H, \mathcal{U}, r, \mathbf{u}_r) \geq (1 - \eta_{j,\ell})^{2^{r-\ell}} d^{2^{r-1}} \prod_{i=1}^{r-1} |U_i|^{2^{r-2}} \prod_{i=r}^k |U_i|^{2^{r-1}}.$$

Hence, a sufficiently small choice of  $\eta_{j,\ell} > 0$  yields the conclusion of Claim 12 due to Fact 11 applied with  $N = |U_r|^{2^{r-1}}$  and  $a = d^{2^{r-1}} \prod_{i=1}^{r-1} |U_i|^{2^{r-2}} \prod_{i=r+1}^k |U_i|^{2^{r-1}}$ .  $\square$

After those preparations we finally prove Lemma 10. Let  $k$ ,  $d$ , and  $\varepsilon$  be given. We determine  $\delta > 0$  as follows: Set  $\gamma_{1,1} = \varepsilon/4$  and for  $j = 2, \dots, k$  let

$$\gamma_{j,1} = \frac{1}{2}(d\varepsilon)^{2^{j-1}} \eta_{j-1,1},$$

where  $\eta_{j-1,1}$  is given by Claim 12 applied for  $j-1$ ,  $\ell = 1$  with  $\gamma_{j-1,1}$ . We then set  $\delta = \eta_{k,1}/2$  and let  $n_0$  be sufficiently large.

Suppose the  $k$ -graph  $H$  with vertex set  $V$  satisfies  $\text{MIN}_d(\delta)$ . We have to show that  $H$  satisfies  $\text{DISC}_d(\varepsilon)$ . For that fix an arbitrary set  $U \subseteq V$ . We have to show that

$$e(U) = d \binom{|U|}{k} \pm \varepsilon n^k. \quad (11)$$

This claim is trivial for sets  $U$  of size at most  $\varepsilon n$ , so we assume  $|U| \geq \varepsilon n$ .

We are going to apply Claim 12  $k$  times. We start with  $j = k$ ,  $\ell = 1$ , and  $\mathcal{U}_k = (U_{k,1}, \dots, U_{k,k})$ , where all sets  $U_{k,i}$  are equal to  $V$  for  $i = 1, \dots, k$ . Note that the property  $\text{MIN}_d(\delta)$  shows that for sufficiently large  $n$  the assumptions (a) and (b) of Claim 12 are satisfied by  $H$ . Recall, that  $M_0 = K_k$  consists of one edge and

$$\text{hom}(M_0, H, (V, \dots, V)) = k!e(H)$$

here. Now the conclusion of Claim 12 for  $r = k$  shows that, due to the choice of  $\gamma_{k,1}$  and  $|U| \geq \varepsilon n$ , the assumption (b) of Claim 12 for  $j = k-1$ ,  $\ell = 1$ , and  $\mathcal{U}_{k-1} = (U_{k-1,1}, \dots, U_{k-1,k})$  with  $U_{k-1,i} = V$  for  $i = 1, \dots, k-1$  and  $U_{k-1,k} = U$  is met.

Moreover, noting that in general if  $U_1 = U_i$ , then  $\text{hom}(M_0, H, \mathcal{U}, 1, (u)) = \text{hom}(M_0, H, \mathcal{U}, i, (u))$  for every  $u \in U_1 = U_i$ , we see that conclusion of Claim 12 for  $r = 1$  applied for  $j = k$ ,  $\ell = 1$ , and  $\mathcal{U}_k$ , yields the assumption (a) of Claim 12 for  $j = k-1$ ,  $\ell = 1$ , and  $\mathcal{U}_{k-1}$ .

In general we apply Claim 12 for  $j = k, \dots, 1$ , always with  $\ell = 1$ , and  $\mathcal{U}_j = (U_{j,1}, \dots, U_{j,k})$ , where  $U_{j,1} = \dots = U_{j,j} = V$  and  $U_{j,j+1} = \dots = U_{j,k} = U$  and observe, as above, that the conclusion of Claim 12 for  $j$  yield the assumptions for  $j-1$ .

This way the conclusion of the last application of Claim 12 for  $j = \ell = 1$  and  $r = 1$  gives a lower and an upper bound for  $\text{hom}(M_0, H, (V, U, \dots, U), 1, (u))$  for all but at most  $\gamma_{1,1}|V|$  vertices of  $u \in V$ . Consequently,

$$\begin{aligned} k!e(U) &= \sum_{u \in U} \text{hom}(M_0, H, (V, U, \dots, U), 1, (u)) \\ &= |U|(1 \pm \gamma_{1,1})d|U|^{k-1} \pm \gamma_{1,1}|V||U|^{k-1} = d|U|^k \pm \frac{\varepsilon}{2}n^k, \end{aligned}$$

which yields (11) for sufficiently large  $n$ .  $\square$

**2.3. DEV implies DISC.** In this section we verify another of the key implications of Theorem 3, by showing that  $\text{DEV}_d$  implies  $\text{DISC}_d$ .

**Lemma 13.** *For every integer  $k \geq 2$ , every  $d > 0$ , and every  $\varepsilon > 0$ , there exists  $\delta > 0$  and  $n_0$  such that the following is true. If  $H$  is a  $k$ -graph on  $n \geq n_0$  vertices that satisfies  $\text{DEV}_d(\delta)$ , then  $H$  satisfies  $\text{DISC}_d(\varepsilon)$ .*

*Proof.* For given  $k, d$  and  $\varepsilon$  we set  $\delta = (\varepsilon/4)^{2^k}$  and  $n_0$  sufficiently large. Let  $H$  be a  $k$ -graph with vertex set  $V = V(H)$  and  $|V| = n \geq n_0$ , which satisfies  $\text{DEV}_d(\delta)$ . We want to verify  $\text{DISC}_d(\varepsilon)$  and for that let  $U \subseteq V$  be a subset of vertices. Again we may assume without loss of generality that  $|U| \geq \varepsilon n$ .

Again, as in Section 2.2, we consider homomorphisms of  $M$  (and its subhypergraphs) instead of labeled copies. Additionally to the notation from Section 2.2, we denote by  $\mathcal{V} = (V, \dots, V)$  the vector which contains the vertex set  $V$   $k$  times. Moreover, we denote by  $K_V$  the complete  $k$ -graph with vertex set  $V$ . Recall that  $w: E(K_V) \rightarrow [-1, 1]$ , where  $w(e) = 1 - d$  if  $e \in E(H)$  and  $w(e) = -d$  otherwise. We introduce  $f(M_j, H, U)$ , which is a short hand notation for the total weight of all homomorphisms of  $M_j$  into  $K_V$  with the property that the “last”  $k - j$  vertex classes  $X_{j+1}(M_j), \dots, X_k(M_j)$  of  $M_j$  are mapped into  $U$ . More precisely, for  $j = 0, \dots, k$  we set

$$f(M_j, H, U) = \sum_{\varphi \in \text{Hom}(M_j, K_V, \mathcal{V})} \prod_{e \in E(M_j)} w(\varphi(e)) \prod_{i=j+1}^k \prod_{x \in X_i(M_j)} \mathbb{1}_U(\varphi(x)), \quad (12)$$

where  $\mathbb{1}_U$  denotes the indicator function of  $U$ . Fixing first the image of  $X_{j+1}(M_j)$  and summing over all homomorphisms  $\varphi$  which extend this choice to a full homomorphism of  $M_j$ , we can rewrite  $f(M_j, H, U)$  as follows

$$\sum_{\mathbf{v} \in V^{2^j}} \prod_{i=1}^{2^j} \mathbb{1}_U(v_i) \sum_{\varphi \in \text{Hom}(M_j, K_V, \mathcal{V}, j+1, \mathbf{v})} \prod_{e \in E(M_j)} w(\varphi(e)) \prod_{i=j+2}^k \prod_{x \in X_i(M_j)} \mathbb{1}_U(\varphi(x)).$$

Recalling, that  $M_{j+1} = \text{db}_{j+1}(M_j)$ , i.e.,  $M_{j+1}$  arises from  $M_j$  by fixing the  $(j+1)$ -st vertex class  $X_{j+1}(M_j)$  of  $M_j$  and “doubling” all the edges together with the remaining vertices, and applying the Cauchy-Schwarz inequality to  $f(M_j, H, U)$  (to the form stated above), we obtain

$$(f(M_j, H, U))^2 \leq |U|^{2^j} f(M_{j+1}, H, U)$$

for every  $j \in \{0, \dots, k-1\}$  and, consequently,

$$(f(M_j, H, U))^{2^{k-j}} \leq |U|^{2^{k-1}} (f(M_{j+1}, H, U))^{2^{k-j-1}}.$$

Applying the last inequality inductively for  $j = 0, \dots, k-1$  we obtain

$$|f(M_0, H, U)|^{2^k} \leq |U|^{k2^{k-1}} |f(M_k, H, U)|. \quad (13)$$

Since  $M_0$  consists of a single edge we have

$$f(M_0, H, U) = k!e(U) - dk! \binom{|U|}{k} = k!e(U) - d|U|^k \pm \delta n^k,$$

since  $|U| \geq \varepsilon n$  and  $n$  is sufficiently large. On the other hand, since  $M_k = M$  we have for sufficiently large  $n$

$$f(M_k, H, U) = \sum_{\varphi \in \text{Hom}(M, K_V, \mathcal{V})} \prod_{e \in E(M)} w(\varphi(e)) = \sum_{\tilde{M}} \prod_{e \in E(\tilde{M})} w(\varphi(e)) \pm \delta n^{|V(M)|},$$

where the sum runs over all copies  $\tilde{M}$  of  $M$  in  $K_V$ . Since  $H$  satisfies  $\text{DEV}_d(\delta)$  we obtain for sufficiently large  $n$

$$|f(M_k, H, U)| \leq 2\delta n^{|V(M)|}$$

and consequently (13) yields

$$|k!e(U) - d|U|^k| \leq (\delta + (2\delta)^{1/2^k})n^k$$

which implies

$$e_H(U) = d \binom{|U|}{k} \pm \varepsilon n^k,$$

for sufficiently large  $n$  by our choice of  $\delta$ .  $\square$

**2.4. Equivalence of MIN and MDEG.** In this section we verify the equivalence of  $\text{MIN}_d$  and  $\text{MDEG}_d$ . As we will see the implication from  $\text{MIN}_d$  to  $\text{MDEG}_d$  is quite straightforward. Moreover, the reverse implication would be trivial, if  $\text{MDEG}_d$  would comprise the assumption that  $e(H) \geq d \binom{n}{k} - o(n^k)$ . In fact, in the main part of the proof we will deduce that  $k$ -graphs having  $\text{MDEG}_d$  must have the right density.

**Lemma 14.** *For every integer  $k \geq 2$ , every  $d > 0$ , and every  $\varepsilon, \varepsilon' > 0$ , there exists  $\delta, \delta' > 0$  and  $n_0$  such that the following is true.*

- (i) *If  $H$  is a  $k$ -graph on  $n \geq n_0$  vertices that satisfies  $\text{MIN}_d(\delta)$ , then  $H$  satisfies  $\text{MDEG}_d(\varepsilon)$ .*
- (ii) *If  $H$  is a  $k$ -graph on  $n \geq n_0$  vertices that satisfies  $\text{MDEG}_d(\delta')$ , then  $H$  satisfies  $\text{MIN}_d(\varepsilon')$ .*

*Proof.* We start with the proof of (i). Let  $k, d$  and  $\varepsilon$  be given. We set  $\gamma_{k,1} = \varepsilon/4$  and we let  $\eta_{k,1}$  be given by Claim 12 applied with  $j = k$  and  $\gamma_{k,1}$ . Then set  $\delta = \eta_{k,1}/2$  and let  $n_0$  be sufficiently large.

Let  $H$  be a  $k$ -graph on  $n$  vertices satisfying  $\text{MIN}_d(\delta)$ , i.e.,  $e(H) \geq d \binom{n}{k} - \delta n^k$  and  $N_M(H) \leq d^{e(M)} n^{|V(M)|} + \delta n^{|V(M)|}$  and, consequently, for sufficiently large  $n$  we have

$$\text{hom}(M_0, H, \mathcal{V}) \geq dn^k - 2\delta n^k$$

and

$$\text{hom}(M_k, H, \mathcal{V}) \leq d^{e(M_k)} n^{|V(M_k)|} + 2\delta n^{|V(M_k)|}.$$

Hence, the conclusion of Claim 12 implies that

$$\text{ext}(M_{k-1}, H, \mathbf{u}) = \text{hom}(M_{k-1}, H, \mathcal{V}, k, \mathbf{u}) \pm \frac{\varepsilon}{4} n^{(k-1)2^{k-2}} = (d^{2^{k-1}} \pm \frac{\varepsilon}{2}) n^{(k-1)2^{k-2}}$$

for all but at most  $\gamma_{k,1} n^{2^{k-1}}$  labeled subsets  $\mathbf{u}_k = (u_1, \dots, u_{2^{k-1}})$  of  $2^{k-1}$  vertices in  $V$ . Therefore, from our choice of  $\gamma_{k,1} \leq \varepsilon/4$  we obtain

$$\sum_{\mathbf{u}} \left| \text{ext}(M_{k-1}, H, \mathbf{u}) - d^{2^{k-1}} n^{(k-1)2^{k-2}} \right| \leq \varepsilon n^{(k+1)2^{k-2}},$$

where the sum runs over all labeled  $2^{k-1}$ -element subsets  $\mathbf{u}$  of  $V$ . This shows that  $H$  satisfies  $\text{MDEG}_d(\varepsilon)$  and concludes the proof of (i) from the lemma.



For the second implication of the lemma, we first note that, due to

$$N_M(H) \leq \sum_{\mathbf{u}} (\text{ext}(M_{k-1}, H, \mathbf{u}))^2$$

property  $\text{MDEG}_d(\delta')$ , for sufficiently small choice of  $\delta'$ , immediately implies

$$N_M(H) \leq d^{2^k} n^{k2^{k-1}} + \varepsilon' n^{k2^{k-1}}.$$

Consequently, we have to show that  $\text{MDEG}_d(\delta')$  also implies  $e(H) \geq d \binom{n}{k} - \varepsilon' n^k$ . For that we will verify the following claim.

**Claim 15.** *For all integers  $k-1 \geq j \geq 1$ , every  $d > 0$  and every  $\gamma_j > 0$ , there exists  $\eta_j \geq 0$  such that the following is true. If*

$$\sum_{\mathbf{u}_{j+1} \in V^{2^j}} \left| \text{hom}(M_j, H, \mathcal{V}, j+1, \mathbf{u}_{j+1}) - d^{2^j} n^{|V(M_j)|-2^j} \right| \leq \eta_j n^{|V(M_j)|}$$

for  $\mathcal{V} = (V, \dots, V)$ , then

$$\sum_{\mathbf{u}_j \in V^{2^{j-1}}} \left| \text{hom}(M_{j-1}, H, \mathcal{V}, j, \mathbf{u}_j) - d^{2^{j-1}} n^{|V(M_{j-1})|-2^{j-1}} \right| \leq \gamma_j n^{|V(M_{j-1})|}.$$

Before we verify Claim 15, we deduce part (ii) of Lemma 14 from the claim. For given  $\varepsilon' > 0$  let  $\gamma_1 = \varepsilon'/2$  and for  $j = 1, \dots, k-1$  let  $\eta_j$  be given by Claim 15 applied with  $\gamma_j$  and set  $\gamma_{j+1} = \eta_j$ . Finally, set  $\delta' = \eta_{k-1}/2$  and let  $n_0$  be sufficiently large. From the assumption  $\text{MDEG}_d(\delta')$  standard calculations show that the assumption of Claim 15 for  $j = k-1$  is satisfied and the conclusion yields the assumption for the claim with  $j = k-2$ . Repeating this argument for  $j = k-2, \dots, 1$  we infer

$$\sum_{u \in V} \left| \text{hom}(M_0, H, \mathcal{V}, 1, (v)) - dn^{k-1} \right| \leq \gamma_1 n^k = \frac{\varepsilon'}{2} n^k,$$

which yields  $e(H) = d \binom{n}{k} \pm \varepsilon' n^k$  for sufficiently large  $n$ .  $\square$

*Proof of Claim 15.* For given  $\gamma_j$  let  $\eta_j$  be sufficiently small, determined later. For  $\mathbf{u}_j \in V^{2^{j-1}}$  set

$$\text{hom}(M_{j+1}, H, \mathcal{V}, j+1, \mathbf{u}_j) = \sum_{\mathbf{u}'_j \in V^{2^{j-1}}} \text{hom}(M_{j+1}, H, \mathcal{V}, j+1, (\mathbf{u}_j, \mathbf{u}'_j)),$$

i.e.,  $\text{hom}(M_{j+1}, H, \mathcal{V}, j+1, \mathbf{u}_j)$  denotes the number of homomorphisms  $\varphi$  from  $M_{j+1}$  to  $H$ , where the “first”  $2^{j-1}$  vertices of  $X_{j+1}(M_{j+1})$  are mapped to  $\mathbf{u}_j$ . Here we have to clarify what mean by “first”  $2^{j-1}$  vertices. By that we mean those vertices in  $X_{j+1}(M_{j+1})$  which form  $X_{j+1}(M_{j-1})$ , i.e., the originals before the  $j$ -th “doubling” step. First we observe

$$\text{hom}(M_{j+1}, H, \mathcal{V}, j+1, \mathbf{u}_j) = \sum_{\mathbf{u}'_j \in V^{2^{j-1}}} (\text{hom}(M_j, H, \mathcal{V}, j+1, (\mathbf{u}_j, \mathbf{u}'_j)))^2 \quad (14)$$

and the assumption of the claim enables us to control the right-hand side of (14). Indeed, due to the assumption of the claim we know that for all but at most  $\sqrt[4]{\eta_j} n^{2^{j-1}}$  vectors  $\mathbf{u}_j \in V^{2^{j-1}}$  there exist at most  $\sqrt[4]{\eta_j} n^{2^{j-1}}$  vectors  $\mathbf{u}'_j \in V^{2^{j-1}}$  such that

$$\left| \text{hom}(M_j, H, \mathcal{V}, j+1, (\mathbf{u}_j, \mathbf{u}'_j)) - d^{2^j} n^{|V(M_j)|-2^j} \right| \geq \sqrt{\eta_j} n^{|V(M_j)|-2^j}$$

and we call such vectors  $\mathbf{u}_j \in V^{2^{j-1}}$  *deviant*. For a non-deviant vector  $\mathbf{u}_j \in V^{2^{j-1}}$  we infer from (14)

$$\begin{aligned} \text{hom}(M_{j+1}, H, \mathcal{V}, j+1, \mathbf{u}_j) &= n^{2^{j-1}} d^{2^{j+1}} n^{2|V(M_j)|-2^{j+1}} \pm (3\sqrt{\eta_j} + \sqrt[4]{\eta_j}) n^{2^{j-1}} n^{2|V(M_j)|-2^{j+1}} \\ &= (d^{2^{j+1}} \pm 4\sqrt[4]{\eta_j}) n^{2|V(M_j)|-2^{j+1}+2^{j-1}}. \end{aligned} \quad (15)$$

On the other hand, for all  $\mathbf{u}_j \in V^{2^{j-1}}$ , we have

$$\text{hom}(M_{j+1}, H, \mathcal{V}, j+1, \mathbf{u}_j) = \text{hom}(M_{j+1}, H, \mathcal{V}, j, \mathbf{u}_j), \quad (16)$$

where  $\text{hom}(M_{j+1}, H, \mathcal{V}, j, \mathbf{u}_j)$  denotes the number of homomorphisms  $\varphi$  from  $M_{j+1}$  to  $H$ , where the “first”  $2^{j-1}$  vertices of  $X_j(M_{j+1})$  are mapped to  $\mathbf{u}_j$ . Again, by “first”  $2^{j-1}$  vertices we mean those vertices in  $X_j(M_{j+1})$  which form  $X_j(M_{j-1}) = X_j(M_j)$ , i.e., those vertices which are fixed in the  $j$ -th “doubling” step. Now, we further rewrite  $\text{hom}(M_{j+1}, H, \mathcal{V}, j, \mathbf{u}_j)$  and observe that it equals

$$\begin{aligned} \text{hom}(M_{j+1}, H, \mathcal{V}, j, \mathbf{u}_j) &= \sum_{(\varphi, \varphi')} \text{hom}(M_j, H, \mathcal{V}, j+1, (\varphi(X_{j+1}(M_{j-1})), \varphi'(X_{j+1}(M_{j-1})))) \end{aligned} \quad (17)$$

where the sum is indexed by all pairs of homomorphisms

$$(\varphi, \varphi') \in (\text{Hom}(M_{j-1}, H, \mathcal{V}, j, \mathbf{u}_j))^2,$$

i.e., over all those pairs of homomorphism each of which extends  $\mathbf{u}_j$  to a homomorphic image of  $M_{j-1}$ . The identity simply says that we obtain all homomorphic images of  $M_{j+1}$  which extend  $\mathbf{u}_j$  as the first  $2^{j-1}$  vertices in  $X_j(M_{j+1})$  by taking two homomorphic extensions of  $\mathbf{u}_j$  to  $M_{j-1}$  (to obtain a homomorphic image of  $M_j$ ) and attaching another homomorphic image of  $M_j$  to the image to the thereby fixed images of  $X_{j+1}(M_j)$ . From (15) we obtain another possibility to apply the assumption of the claim and more importantly to connect it with the conclusion. Note that, given the fixed choice of  $\mathbf{u}_j$  and  $X_{j+1}(M_j)$ , there are at most  $n^{|V(M_j)|-2^{j-1}-2^j}$  ways to attach such a copy of  $M_j$ . Therefore, the assumption combined with (17) yields

$$\begin{aligned} \text{hom}(M_{j+1}, H, \mathcal{V}, j, \mathbf{u}_j) &= (\text{hom}(M_{j-1}, H, \mathcal{V}, j, \mathbf{u}_j))^2 \times d^{2^j} n^{|V(M_j)|-2^j} \\ &\quad \pm n^{|V(M_j)|-2^{j-1}-2^j} \times \eta_j n^{|V(M_j)|}. \end{aligned} \quad (18)$$

Combining (15), (16), and (18), we obtain, for non-deviant vectors  $\mathbf{u}_j \in V^{2^{j-1}}$ ,

$$(\text{hom}(M_{j-1}, H, \mathcal{V}, j, \mathbf{u}_j))^2 = (d^{2^j} \pm (4\sqrt[4]{\eta_j} + \eta_j)/d^{2^j}) n^{|V(M_j)|-2^{j-1}}$$

and, consequently, for sufficiently small choice of  $\eta_j$  (compared to  $\gamma_j$  and  $d$ ) we have

$$\left| \text{hom}(M_{j-1}, H, \mathcal{V}, j, \mathbf{u}_j) - d^{2^{j-1}} n^{|V(M_{j-1})|-2^{j-1}} \right| \leq \frac{\gamma_j}{2} n^{|V(M_{j-1})|-2^{j-1}}$$

for non-deviant  $\mathbf{u}_j \in V^{2^{j-1}}$ . Summing over all  $\mathbf{u}_j \in V^{2^{j-1}}$  we get

$$\begin{aligned} \sum_{\mathbf{u}_j \in V^{2^{j-1}}} \left| \text{hom}(M_{j-1}, H, \mathcal{V}, j, \mathbf{u}_j) - d^{2^{j-1}} n^{|V(M_{j-1})|-2^{j-1}} \right| &\leq \frac{\gamma_j}{2} n^{|V(M_{j-1})|} + \sqrt[4]{\eta_j} n^{|V(M_{j-1})|} \leq \gamma_j n^{|V(M_{j-1})|} \end{aligned}$$

as claimed.  $\square$

**2.5. DEV implies CL.** In this section we give a direct proof of  $\text{DEV}_d \Rightarrow \text{CL}_d$ . For that we will introduce another version of  $\text{DISC}_d$  called  $\text{FDISC}_d$ , which is motivated by the quasi-random functions introduced by Gowers in [14, see Section 3]. It will turn out that  $\text{DEV}_d$  implies  $\text{FDISC}_d$  (see Lemma 16) and the implication from  $\text{FDISC}_d$  to  $\text{CL}_d$  will follow by similar arguments to those from [14] (see Lemma 17).

Before we define  $\text{FDISC}_d$ , we will generalise the weight function  $w$  defined in Section 1. For a  $k$ -graph  $H$  with vertex set  $V$  and some  $d \in (0, 1]$ , we define the weight function  $w: \binom{V}{\leq k} = \bigcup_{j=1}^k \binom{V}{j} \rightarrow [-1, 1]$  as follows: for a set  $X \subseteq V$  of cardinality at most  $k$  we set

$$w(X) = \begin{cases} 1 - d & \text{if } X \in E(H), \\ -d & \text{otherwise.} \end{cases}$$

Our weight function is now applicable also to subsets of cardinality smaller than  $k$ . This generalisation will simplify the notation. Moreover, we will again use homomorphism instead of copies of  $k$ -graphs. In this section we study the following properties.

$\text{FDISC}_d(\varepsilon)$ : We say a  $k$ -graph  $H$  on  $n$  vertices has  $\text{FDISC}_d(\varepsilon)$  for  $d, \varepsilon > 0$ , if

$$\left| \sum_{\varphi: [k] \rightarrow V(H)} w(\varphi([k])) \prod_{i=1}^k g_i(\varphi(i)) \right| \leq \varepsilon n^k$$

for all families of functions  $g_i: V(H) \rightarrow [-1, 1]$  with  $i \in [k]$ .

For convenience we will work with the following version of  $\text{DEV}_d$ .

$\text{DEV}'_d(\varepsilon)$ : We say a  $k$ -graph  $H_n$  on  $n$  vertices has  $\text{DEV}'_d(\varepsilon)$  for  $d, \varepsilon > 0$ , if

$$\left| \sum_{\varphi: V(M) \rightarrow V} \prod_{e \in E(M)} w(e) \right| \leq \varepsilon n^{k2^{k-1}}.$$

This definition, though formally different to the definition of  $\text{DEV}_d$ , is equivalent to it. For  $\text{DEV}_d$  we were summing over all labeled copies of  $M$  in  $K_V$ , and here we sum over all mappings from  $V(M)$  to  $V$  (note that we extended  $w$  to  $\binom{V}{\leq k}$  for that). By doing this, we get at most an additional additive error term of  $O(n^{k2^{k-1}-1}) = o(n^{k2^{k-1}})$ , which is asymptotically negligible.

**Lemma 16.** *For every integer  $k \geq 2$ , every  $d > 0$ , and every  $\varepsilon > 0$  there exist  $\delta > 0$  and  $n_0$  such that the following is true. If  $H$  is a  $k$ -graph on  $n \geq n_0$  vertices that satisfies  $\text{DEV}'_d(\delta)$ , then  $H$  satisfies  $\text{FDISC}_d(\varepsilon)$ .*

*Proof.* The assertion  $\text{DEV}_d \Rightarrow \text{FDISC}_d$  is a simple generalisation of the proof of Lemma 13. We only have to replace  $\mathbb{1}_U(\varphi(x))$  for  $x \in X_i(M_j)$  by  $g_i(\varphi(x))$ . Thus, applying each time the Cauchy-Schwarz inequality we will square  $g_i(\varphi(x))$ , and we then only have to upper bound  $(g_i(\varphi(x)))^2$  by 1. We also now have to sum over all functions  $\varphi: V(M_j) \rightarrow V$  (instead over all homomorphisms  $\varphi \in \text{Hom}(M_j, K_V, \mathcal{V})$ ). With those adjustments the proof works verbatim.  $\square$

We close this section with the proof of the implication  $\text{FDISC}_d \Rightarrow \text{CL}_d$ .

**Lemma 17.** *For every integer  $k \geq 2$ , every  $d > 0$ , every linear  $k$ -graph  $F$  on  $\ell$  vertices, and every  $\varepsilon > 0$ , there exists  $\delta > 0$  and  $n_0$  such that the following is true. If  $H$  is a  $k$ -graph on  $n \geq n_0$  vertices that satisfies  $\text{FDISC}_d(\delta)$ , then  $H$  satisfies  $\text{CL}_d(F, \varepsilon)$ .*

*Proof.* We may assume  $E(F) \neq \emptyset$  and let us fix an edge  $f \in E(F)$ . It suffices to verify an estimate on

$$\text{hom}(F, H) = \sum_{\varphi \in \text{Hom}(F, K_V, \mathcal{V})} \prod_{e \in E(F)} \mathbb{1}_{E(H)}(\varphi(e)). \quad (19)$$

the number of homomorphism from  $F$  into  $H$ . Here, again, we may further enlarge the sum by going over all functions  $\varphi: V(F) \rightarrow V$ . However, for every  $\varphi$  which is not a homomorphism, there will be an  $f \in E(F)$  with  $|\varphi(f)| < k$ , and thus  $\varphi$  will contribute 0 to the total sum. Noting furthermore that  $\mathbb{1}_{E(H)}(\varphi(e)) = w(\varphi(e)) + d$  for every  $\varphi(e) \in \binom{V}{\leq k}$  we may rewrite (19) as

$$\begin{aligned} \text{hom}(F, H) &= \sum_{\varphi: V(F) \rightarrow V} \prod_{e \in E(F)} (w(\varphi(e)) + d) \\ &= \sum_{\varphi': V(F) \setminus \{f\} \rightarrow V} \sum_{\substack{\varphi: V(F) \rightarrow V \\ \varphi|_{V(F) \setminus \{f\}} = \varphi'}} \prod_{e \in E(F)} (w(\varphi(e)) + d). \end{aligned}$$

Now we may concentrate on the inner sum. We first multiply out the product  $\prod_{e \in E(F)} (w(\varphi(e)) + d)$ , and consider the inner sum. We obtain the leading term  $d^{e(F)} n^k$ , while each of the other terms from the product can be interpreted as functions  $g_i$  (for every vertex  $i$  of  $f$  since  $F$  is linear). Therefore we apply  $\text{FDISC}_d(\delta)$  to each term from the inner sum to obtain an estimate for the sum. Therefore, setting  $\delta = \varepsilon/2^{e(F)+1}$ , we have shown that the inner sum is  $d^{e(F)} n^k \pm \varepsilon n^k/2$  and, hence,

$$\text{hom}(F, H) = d^{e(F)} n^\ell \pm \varepsilon n^\ell,$$

which implies  $\text{CL}_d(F, \varepsilon)$  for sufficiently large  $n$ .  $\square$

### 3. PROOF OF THEOREM 5

In this section we present the proof of Theorem 5. We have to show that for every  $k \geq 2$ , every linear  $k$ -graph  $F$  with at least one edge and  $V(F) = [\ell]$  for some integer  $\ell$ , every  $d > 0$ , and every vector  $\alpha \in (0, 1]^\ell$  the properties  $\text{DISC}_d$ ,  $\text{HCL}_{d,F,\alpha}$  and  $\text{HCL}_{d,F}$  are equivalent. In Section 3.1 we show the simple implication

$$\text{HCL}_{d,F,\alpha} \xrightarrow{\text{Fact 18}} \text{HCL}_{d,F}.$$

The main part of this section is devoted to the proof of  $\text{HCL}_{d,F} \Rightarrow \text{DISC}_d$ . For that we will introduce another property  $\text{REG}_d$ , which will turn out to be equivalent to  $\text{DISC}_d$  and we then show  $\text{HCL}_{d,F} \Rightarrow \text{REG}_d$  in Section 3.2

$$\text{HCL}_{d,F} \xrightarrow{\text{Lemma 25}} \text{REG}_d \xleftarrow{\text{Fact 24}} \text{DISC}_d.$$

Finally, in Section 3.3 we verify

$$\text{DISC}_d \xrightarrow{\text{Fact 27}} \text{HCL}_{d,F,\alpha}.$$

3.1.  $\text{HCL}_{d,F,\alpha}$  implies  $\text{HCL}_{d,F}$ . The following observation yields the implication from  $\text{HCL}_{d,F,\alpha}$  to  $\text{HCL}_{d,F}$ .

**Fact 18.** *For every integer  $k \geq 2$ , every  $d > 0$ , every linear  $k$ -graph  $F$  with at least one edge and  $V(F) = [\ell]$  for some integer  $\ell$ , all vectors  $\alpha \in (0, 1)^\ell$  with  $\sum_{i=1}^\ell \alpha_i < 1$ , and every  $\varepsilon > 0$ , there exists  $\delta > 0$  and  $n_0$  such that the following is true. If  $H$  is a  $k$ -graph on  $n \geq n_0$  vertices that satisfies  $\text{HCL}_{d,F,\alpha}(\delta)$ , then, for all  $\beta \in (0, 1)^\ell$  with  $\sum_{i=1}^\ell \beta_i < 1$ ,  $H$  satisfies  $\text{HCL}_{d,F,\beta}(\varepsilon)$ .*

*Proof.* Note that it suffices to consider the case when  $\alpha = (\alpha_1, \dots, \alpha_\ell)$  and  $\beta = (\beta_1, \dots, \beta_\ell)$  differ in at most one entry, i.e., there is an  $i \in [\ell]$  such that  $\alpha_i \neq \beta_i$  and for all  $j \neq i$  we have  $\alpha_j = \beta_j$ . Without loss of generality we may assume that  $i = \ell$ . For given  $k, d, F, \alpha$ , and  $\varepsilon > 0$  we set  $\delta = \varepsilon \min\{\alpha_\ell, 1 - \sum_{i \in [\ell]} \alpha_i\}/7$  and let  $n_0$  be sufficiently large. We then verify the fact for given  $\beta \in (0, 1)^\ell$ .

First, we prove the claim for all  $\beta = (\beta_1, \dots, \beta_{\ell-1}, \gamma)$  with  $\gamma \geq \alpha_\ell$ . Let  $U_1, \dots, U_\ell \subseteq V(H)$  be subsets satisfying  $|U_i| = \lfloor \beta_i n \rfloor$  for  $i \in [\ell-1]$ ,  $|U_\ell| = \lfloor \gamma n \rfloor$  and  $\mathcal{P} = \{W \subseteq U_\ell : |W| = \lfloor \alpha_\ell n \rfloor\}$ . Since  $H$  satisfies  $\text{HCL}_{d,F,\alpha}(\delta)$  and  $\beta_j = \alpha_j$  for all  $j \in [\ell-1]$  we infer

$$N_F(U_1, \dots, U_{\ell-1}, W) = d^{e(F)} \lfloor \alpha_\ell n \rfloor \prod_{i \in [\ell-1]} |U_i| \pm \delta n^\ell$$

for all  $W \in \mathcal{P}$ . Hence, having each copy of  $F$  counted  $\binom{\lfloor \gamma n \rfloor - 1}{\lfloor \alpha_\ell n \rfloor - 1}$  times, we obtain, for  $n \geq 1/\alpha_\ell$ ,

$$\begin{aligned} N_F(U_1, \dots, U_\ell) &= \binom{\lfloor \gamma n \rfloor - 1}{\lfloor \alpha_\ell n \rfloor - 1}^{-1} \sum_{W \in \mathcal{P}} N_F(U_1, \dots, U_{\ell-1}, W) \\ &= \binom{\lfloor \gamma n \rfloor - 1}{\lfloor \alpha_\ell n \rfloor - 1}^{-1} \binom{\lfloor \gamma n \rfloor}{\lfloor \alpha_\ell n \rfloor} d^{e(F)} \left( \lfloor \alpha_\ell n \rfloor \prod_{i \in [\ell]} |U_i| \pm \delta n^\ell \right) \\ &= d^{e(F)} \prod_{i \in [\ell]} |U_i| \pm \frac{2\delta}{\alpha_\ell} n^\ell, \end{aligned}$$

which by our choice of  $\delta$  yields the fact for this case.

Suppose  $\beta_\ell < \alpha_\ell$ . Without loss of generality we may assume that  $\sum_{i \in [\ell]} \beta_i + \alpha_\ell < 1$ . (Otherwise, first choose  $\beta'_\ell = (1 - \sum_{i \in [\ell]} \alpha_i)/2$  and then use the proof from above to finish the claim for  $\beta_\ell$  with appropriately chosen  $\delta$ .) Let  $U_1, \dots, U_\ell \subseteq V(H)$  be pairwise disjoint with  $|U_i| = \lfloor \beta_i n \rfloor$ ,  $i \in [\ell]$ . Considering  $W \subseteq V \setminus U_\ell$  of size  $|W| = \lfloor \alpha_\ell n \rfloor$  we infer from  $\text{HCL}_{d,F,\alpha}(\delta)$  and the case considered above

$$N_F(U_1, \dots, U_{\ell-1}, U_\ell \dot{\cup} W) = d^{e(F)} (\lfloor \alpha_\ell n \rfloor + \lfloor \beta_\ell n \rfloor) \prod_{i \in [\ell-1]} |U_i| \pm \frac{2\delta}{\alpha_\ell} n^\ell$$

and

$$N_F(U_1, \dots, U_{\ell-1}, W) = d^{e(F)} \lfloor \alpha_\ell n \rfloor \prod_{i \in [\ell-1]} |U_i| \pm \delta n^\ell.$$

Hence, we have

$$\begin{aligned} N_F(U_1, \dots, U_\ell) &= N_F(U_1, \dots, U_{\ell-1}, U_\ell \dot{\cup} W) - N_F(U_1, \dots, U_{\ell-1}, W) \\ &= d^{e(F)} \prod_{i \in [\ell]} |U_i| \pm \frac{3\delta}{\alpha_\ell} n^\ell, \end{aligned}$$

which concludes the proof of the fact by the choice of  $\delta$ .  $\square$

**3.2. HCL $_{d,F}$  implies DISC $_d$ .** In this section we verify the implication from HCL $_{d,F}$  to DISC $_d$ . The proof is based on ideas of Shapira and Yuster [24], the main tools being the theorem of Gottlieb [13] on the rank of the inclusion matrices and the weak regularity lemma for hypergraphs. In the next section, Section 3.2.1, we introduce the result of Gottlieb and its consequences. In Section 3.2.2 we introduce the weak regularity lemma for hypergraphs and another quasi-random property REG $_d$ , which is equivalent to DISC $_d$ . Finally, in Section 3.2.3 we prove that HCL $_{d,F}$  implies REG $_d$ .

**3.2.1. Tools from linear algebra.** For positive integers  $r \geq \ell \geq k$  the inclusion matrix  $I(r, \ell, k)$  is an  $\binom{r}{\ell} \times \binom{r}{k}$  matrix defined as follows. For  $L \in \binom{[r]}{\ell}$  and  $K \in \binom{[r]}{k}$  the entry of  $I_{L,K}$  is given by

$$I_{L,K} = \begin{cases} 1 & \text{if } K \subset L \\ 0 & \text{otherwise} \end{cases}$$

Note that we implicitly assume fixed orderings on the set of subgraphs  $\binom{[r]}{\ell}$  and on the edge set  $\binom{[r]}{k}$ . This does not effect the rank of  $I(r, \ell, k)$  which is at most  $\binom{r}{k}$  and in fact it was shown by Gottlieb [13], that  $I(r, \ell, k)$  has full rank if  $r \geq \ell + k$ .

**Theorem 19 (Gottlieb).** *For all positive integers  $\ell \geq k$  and  $r \geq \ell + k$  the inclusion matrix  $I(r, \ell, k)$  has rank  $\binom{r}{k}$ .*  $\square$

Note that the rows of  $I(r, \ell, k)$  can be interpreted as incidence vectors of the edges of copies of the complete  $k$ -graph  $K_\ell$  in  $K_r$ . For our purposes, it will be convenient to consider a similar matrix, where the rows correspond to incidence vectors of the edges of the given  $k$ -graph  $F$ . To this end, for a  $k$ -graph  $F$  on  $\ell$  vertices, we define the matrix  $A(r, F, k)$  as follows. The rows of  $A(r, F, k)$  are indexed by the labelled copies of  $F$  in  $K_r$  and the columns are indexed, as above, by the  $k$ -element subsets of  $[r]$ . Now for a labeled copy  $\tilde{F}$  of  $F$  in  $K_r$  and a  $k$ -set  $e \in \binom{[r]}{k}$  the entry  $A_{\tilde{F},e}$  is given by

$$A_{\tilde{F},e} = \begin{cases} 1 & \text{if } e \in E(\tilde{F}) \\ 0 & \text{otherwise.} \end{cases}$$

Thus  $A(r, F, k)$  is a  $N_F(K_r) \times \binom{[r]}{k}$  and Theorem 19 determines the rank of  $A(r, F, k)$ .

**Corollary 20.** *For all positive integers  $\ell \geq k$ ,  $r \geq \ell + k$  and all non-empty  $k$ -graphs  $F$  on  $\ell$  vertices the matrix  $A(r, F, k)$  has rank  $\binom{r}{k}$ .*

*Proof.* The proof of Corollary 20 is identical to the proof of Lemma 3.1 in [23] and follows from the observation that the rows of  $A(r, F, k)$  span the rows of  $I(r, \ell, k)$ . Indeed, summing all rows of  $A(r, F, k)$  that correspond to copies  $\tilde{F}$  of  $F$  with the same vertex set  $L \in \binom{[r]}{\ell}$  we obtain a multiple of the row in  $I(r, \ell, k)$  indexed by  $L$ .  $\square$

From Corollary 20 we deduce the key lemma of this section, Lemma 21 below. In Lemma 21 we consider complete, weighted  $k$ -graphs on  $r$  vertices. Let  $w: E(K_r) \rightarrow (0, 1]$  be an arbitrary weight function and  $F$  be a fixed  $k$ -graph on  $\ell$  vertices. We set the weight of a labeled copy  $\tilde{F}$  of  $F$  in  $K_r$ , as before, to the product of the weights of the edges of  $\tilde{F}$ , i.e.,

$$w(\tilde{F}) = \prod_{e \in E(\tilde{F})} w(e).$$

Lemma 21 states that if  $w(\tilde{F})$  is “almost” the same for all copies of  $F$ , then  $w$  must be almost constant.

**Lemma 21.** *For all integers  $\ell \geq k \geq 2$  and  $r \geq \ell + k$ , every  $d > 0$ , every  $k$ -graph  $F$  on  $\ell$  vertices with at least one edge, and every  $\varepsilon > 0$ , there exists  $\delta > 0$  such that if  $w: E(K_r) \rightarrow (0, 1]$  satisfies*

$$w(\tilde{F}) = d^{e(F)} \pm \delta$$

for all labeled copies  $\tilde{F}$  of  $F$  in  $K_r$ , then  $w(e) = d \pm \varepsilon$  for all  $e \in E(K_r)$ .

*Proof.* Let  $\ell, k, r, d, F$ , and  $\varepsilon$  be given. Due to the continuity of the function  $2^x$  we can choose  $\varepsilon' > 0$  such that if  $|x - \log_2 d| \leq \varepsilon'$  then  $|2^x - d| \leq \varepsilon$ . Next we fix an ordering  $e_1, \dots, e_m$ ,  $m = \binom{r}{k}$  of the edges of the  $K_r$  and an ordering  $\tilde{F}_1, \dots, \tilde{F}_t$  for  $t = r(r-1) \dots (r-\ell+1)$  of all labeled copies of  $F$  in  $K_r$ . This defines the matrix  $A = A(r, F, k)$  which, by Corollary 20, has rank  $\binom{r}{k}$ . Thus  $A: \mathbb{R}^{\binom{r}{k}} \rightarrow \mathbb{R}^t$  is an injective and linear function and consequently there exists a  $\delta' > 0$  such that the following holds: if  $A\mathbf{y} = \mathbf{b}$  and  $A\mathbf{x} = \mathbf{c}$  with  $\|\mathbf{b} - \mathbf{c}\|_\infty \leq \delta'$  then  $\|\mathbf{y} - \mathbf{x}\|_\infty \leq \varepsilon'$ . Further, due to the continuity of the function  $\log_2 x$  we can choose  $\delta > 0$  such that if  $|2^b - d^{e(F)}| \leq \delta$ , then  $|b - e(F) \log_2 d| \leq \delta'$  and we fix the  $\delta$  for Lemma 21 this way.

Now let  $w: E(K_r) \rightarrow (0, 1]$  satisfy the assumption of the lemma. Therefore, we have for every copy  $\tilde{F}$  of  $F$  in  $K_r$

$$\sum_{e \in E(\tilde{F})} \log_2(w(e)) = \log_2(d^{e(F)} \pm \delta). \quad (20)$$

Let  $\mathbf{y} = ((y(e_1), \dots, y(e_m))) \in \mathbb{R}^m$  be given by

$$y(e_i) = \log_2 w(e_i)$$

for  $i = 1, \dots, m$ . Then (20) is equivalent to  $A\mathbf{y} = \mathbf{b}$  where  $\mathbf{b} = (b_1, \dots, b_t)$  with  $b_i = \log_2(d^{e(F)} \pm \delta)$  for all  $i \in [t]$ .

On the other hand, by Corollary 20 we know that  $A$  has rank  $\binom{r}{k}$  and, hence, the system of linear equations  $A\mathbf{x} = \mathbf{c}$  for  $\mathbf{c} = (e(F) \log d)\mathbf{1}_t$  for the all ones vector  $\mathbf{1}_t = \{1\}^t$  has at most one solution. Since the everywhere  $\log d$  vector  $(\log_2 d)\mathbf{1}_m$  is a solution to this system of equations, it must be the unique solution  $\mathbf{x}$ .

From our choice of  $\delta$  we infer  $\|\mathbf{b} - \mathbf{c}\|_\infty \leq \delta'$  and, consequently, due to the choice of  $\delta'$  we have  $\|\mathbf{y} - \mathbf{x}\|_\infty \leq \varepsilon'$ . In other words,  $|\log_2(w(e_i)) - \log_2(d)| \leq \varepsilon'$  for every  $i = 1, \dots, m$  and the choice of  $\varepsilon'$  yields  $|w(e) - d| \leq \varepsilon$  for all edges  $e \in E(K_r)$ .  $\square$

**3.2.2. Weak hypergraph regularity lemma.** For the proof of  $\text{HCL}_{d,F} \Rightarrow \text{DISC}_d$  we will use the so-called weak regularity lemma for  $k$ -graphs, which is a straightforward extension of Szemerédi’s regularity lemma for graphs [31]. Roughly speaking, the property  $\text{HCL}_{d,F}$  will imply that for the weighted cluster-hypergraph of a regular

partition the assumption of Lemma 21 hold. Consequently, the densities of all  $k$ -tuples of the regular partition will be close to  $d$  and from this we will infer  $\text{DISC}_d$ . Below we introduce the weak hypergraph regularity lemma and a few related results.

Let  $H = (V, E)$  be a  $k$ -graph and let  $U_1, \dots, U_k$  be pairwise disjoint non-empty subsets of  $V$ . Recall that  $e(U_1, \dots, U_k)$  denotes the number of edges with one vertex in each  $U_i$ ,  $i \in [k]$  and the *density* of  $(U_1, \dots, U_k)$  is defined to be

$$d(U_1, \dots, U_k) = \frac{e(U_1, \dots, U_k)}{|U_1| \cdots |U_k|}.$$

We say the  $k$ -tuple  $(V_1, \dots, V_k)$  of pairwise disjoint subsets  $V_1, \dots, V_k \subseteq V$  is  $\varepsilon$ -regular if

$$|d(U_1, \dots, U_k) - d(V_1, \dots, V_k)| \leq \varepsilon$$

for all  $k$ -tuples of subsets  $U_1 \subset V_1, \dots, U_k \subset V_k$  satisfying  $|U_1| \geq \varepsilon|V_1|, \dots, |U_k| \geq \varepsilon|V_k|$ .

Though the notion of weak regularity is not sufficient to imply a general counting lemma it was shown in [18] that it is strong enough to imply a counting lemma for linear  $k$ -graphs:

**Lemma 22** (Counting lemma for linear hypergraphs). *For all integers  $\ell \geq k \geq 2$  and every  $\gamma$ , there exist  $\varepsilon = \varepsilon(\ell, k, \gamma) > 0$  and  $m_0 = m_0(\ell, k, \gamma)$  so that the following holds.*

*Let  $F = ([\ell], E(F))$  be a linear  $k$ -graph and let  $H = (V_1 \dot{\cup} \dots \dot{\cup} V_\ell, E)$  be an  $\ell$ -partite,  $k$ -graph where  $|V_1|, \dots, |V_\ell| \geq m_0$ . Suppose, moreover, that for all edges  $f \in E(F)$ , the  $k$ -tuple  $(V_i)_{i \in f}$  is  $(\varepsilon, d_f)$ -regular. Then the following holds:*

$$N_F(V_1, \dots, V_\ell) = \prod_{f \in E(F)} d_f \prod_{i \in [\ell]} |V_i| \pm \gamma \prod_{i \in [\ell]} |V_i|.$$

□

A partition  $V_1 \dot{\cup} \dots \dot{\cup} V_t$  of  $V(H)$  will be called a  $t$ -equipartition if  $|V_1| \leq |V_2| \leq \dots \leq |V_t| \leq |V_1| + 1$  and such an equipartition will be called  $\varepsilon$ -regular if all but at most  $\varepsilon \binom{t}{k}$  of the  $k$ -tuples  $(V_{i_1}, \dots, V_{i_k})$  are  $\varepsilon$ -regular. The proof of the following theorem follows the lines of the original proof of Szemerédi (see, e.g., [4, 11, 30]).

**Theorem 23** (Weak hypergraph regularity lemma). *For all  $k, t_0 \in \mathbb{N}$  and all  $\varepsilon > 0$  there is a  $T_0 = T_0(t_0, \varepsilon)$  and an  $n_0$  such that for all  $n \geq n_0$  and all  $k$ -graphs  $H$  on  $n$  vertices there is an  $\varepsilon$ -regular,  $t$ -equipartition of  $H$  with  $t$  satisfying  $t_0 \leq t \leq T_0$ . □*

In case of graphs, it was noted by Simonovits and Sós [26] that there is a close relationship between quasi-randomness and the Szemerédi regular partition. Indeed, it is easily shown that a graph  $G$  is quasi-random in the sense of Theorem 1 if and only if  $G$  permits a partition such that almost all pairs of partition classes are regular and have roughly the same density. This generalises to  $k$ -graphs in a straightforward manner.

It will be convenient to consider the property  $\text{REG}_d$  defined as follows.

$\text{REG}_d(\varepsilon)$ : We say a  $k$ -graph  $H$  on  $n$  vertices has  $\text{REG}_d(\varepsilon)$  for  $d, \varepsilon > 0$ , if there exists an  $\varepsilon$ -regular,  $t$ -equipartition  $V(H) = V_1 \dot{\cup} \dots \dot{\cup} V_t$  of  $H$  with  $g(d, \varepsilon) \geq t \geq 1/\varepsilon$  for some arbitrary function  $g(d, \varepsilon) \geq 1/\varepsilon$  independent of  $H$  and  $n$  such that  $d(V_{i_1}, \dots, V_{i_k}) = d \pm \varepsilon$  for all but at most  $\varepsilon t^k$  tuples  $\{i_1, \dots, i_k\} \in \binom{[t]}{k}$ .

It is easy to see that  $\text{DISC}_d$  and  $\text{REG}_d$  are equivalent (see, e.g. [4]) and we omit the proof here.



**Fact 24.** For every integer  $k \geq 2$  and every  $d > 0$  the properties  $\text{DISC}_d$  and  $\text{REG}_d$  are equivalent.  $\square$

3.2.3.  $\text{HCL}_{d,F}$  implies  $\text{REG}_d$ . In this section we deduce  $\text{REG}_d$  from  $\text{HCL}_{d,F}$  by proving the following lemma.

**Lemma 25.** For every integer  $k \geq 2$ , every  $d > 0$ , every linear  $k$ -graph  $F$  containing at least one edge, and every  $\varepsilon > 0$ , there exists  $\delta > 0$  and  $n_0$  such that the following is true. If  $H$  is a  $k$ -graph on  $n \geq n_0$  vertices that satisfies  $\text{HCL}_{d,F}(\delta)$ , then  $H$  satisfies  $\text{REG}_d(\varepsilon)$ .

Besides the results from Sections 3.2.1 and 3.2.2 we will also need the following consequence of a packing result of Rödl [21].

**Lemma 26.** For all integers  $r \geq k \geq 2$  and every  $\gamma > 0$  there exists an integer  $t_0$  such that for all  $t \geq t_0$  the following holds. If  $R$  is a  $k$ -graph on  $t$  vertices with  $e(R) \geq (1 - \gamma) \binom{t}{k}$  edges, then there exist at least  $(1 - \gamma r^k) \binom{t}{k}$  edges in  $R$  each of which belong to at least one copy of  $K_r$  in  $R$ .

*Proof.* We choose  $t_0$  large enough to guarantee that the packing result of Rödl [21] is applicable for  $t \geq t_0$  and  $r, k$ , and  $\gamma$ . Given a  $k$ -graph  $R$  on  $t$  vertices which contains at least  $(1 - \gamma) \binom{t}{k}$  edges we first consider the complete  $k$ -graph  $K_t$  on the same vertex set. From Rödl's theorem we infer that  $K_t$  contains at least  $(1 - \gamma) \binom{t}{k} / \binom{r}{k}$  edge disjoint copies of the  $K_r$ . Taking the same copies of  $K_r$  we see that at most  $\gamma \binom{t}{k} = \gamma \binom{r}{k} \binom{t}{k} / \binom{r}{k}$  of them fail to be a subgraph of  $R$  since  $R$  contains at least  $(1 - \gamma) \binom{t}{k}$  edges. This implies that  $R$  contains at least  $(1 - \gamma - \gamma \binom{r}{k}) \binom{t}{k} / \binom{r}{k}$  edge disjoint copies of  $K_r$  which implies that all but at most  $\gamma r^k \binom{t}{k}$  edges of  $R$  are contained in a copy of a  $K_r$  in  $R$ .  $\square$

*Proof of Lemma 25.* For given  $k, d$ , linear  $k$ -graph  $F$  with at least one edge and  $V(F) = [\ell]$ , and  $\varepsilon > 0$ , we first apply Lemma 21 with  $\ell, k$ , and  $r = \ell + k, d, F$ , and  $\varepsilon$  and obtain  $\delta_{\text{GL}} > 0$ . Then we apply the counting lemma, Lemma 22, with  $\ell, k$ , and  $\gamma_{\text{CL}} = \delta_{\text{GL}}/2$  to obtain  $\varepsilon_{\text{CL}}$  and  $m_{\text{CL}}$ . Further, we apply Lemma 26 with  $r, k$  and  $\gamma_{\text{PL}} = \varepsilon/(2r^k)$  to obtain  $t_{\text{PL}}$ . Applying the weak regularity lemma, Theorem 23, with

$$\varepsilon_{\text{RL}} = \min\{\varepsilon_{\text{CL}}, \varepsilon/(2r^k)\} \quad \text{and} \quad t_0 = \max\{1/\varepsilon_{\text{RL}}, t_{\text{PL}}\}$$

we obtain  $T_0$ . Finally, we choose  $\delta = \delta_{\text{GL}} d^{e(F)} / (2^{\ell+2} T_0^\ell)$  and  $n_0 \geq T_0 m_{\text{CL}}$  sufficiently large to satisfy the equations needed.

Let  $H$  be a  $k$ -graph on  $n$  vertices with  $n \geq n_0$  which satisfies  $\text{HCL}_{d,F}(\delta)$ . We have to show that there exists a partition  $V_1 \dot{\cup} \dots \dot{\cup} V_t = V(H)$  such that

- (i)  $1/\varepsilon \leq t \leq T_0$  (note that  $T_0 = T_0(d, \varepsilon, F)$  is independent of  $H$  and  $n$ ),
- (ii)  $||V_i| - |V_j|| \leq 1$  for all  $i, j \in [t]$
- (iii) all but at most  $\varepsilon t^k$   $k$ -tuples  $(V_{i_1}, \dots, V_{i_k})$  are  $\varepsilon$ -regular and have density  $d \pm \varepsilon$ .

To this end, we first apply Theorem 23 with  $\varepsilon_{\text{RL}}$  and  $t_0$  to obtain a partition  $V(H) = V_1 \dot{\cup} \dots \dot{\cup} V_t$ , which already satisfies (i) and (ii) and the first part of (iii), i.e., all but at most  $\varepsilon_{\text{RL}} \binom{t}{k} \leq \frac{1}{2} \varepsilon t^k$   $k$ -tuples  $(V_{i_1}, \dots, V_{i_k})$  are  $\varepsilon$ -regular. Thus, it remains to show that all but at most  $\frac{1}{2} \varepsilon t^k$  of the  $k$ -tuples  $(V_{i_1}, \dots, V_{i_k})$  have density  $d \pm \varepsilon$ .

We consider the reduced (or cluster)  $k$ -graph  $R$ , i.e., the  $k$ -graph on the vertex set  $\{1, \dots, t\}$  with  $\{i_1, \dots, i_k\}$  being an edge if and only if  $(V_{i_1}, \dots, V_{i_k})$  is  $\varepsilon_{\text{RL}}$ -regular. Then  $R$  is a  $k$ -graph on  $t$  vertices which contains at least  $(1 - \varepsilon_{\text{RL}}) \binom{t}{k}$  edges and we assign to each edge  $\{i_1, \dots, i_k\}$  the weight

$$w(i_1, \dots, i_k) = d(V_{i_1}, \dots, V_{i_k}).$$

Applying Lemma 26 to  $R$  we know that all but at most  $\gamma_{\text{PL}} r^k \binom{t}{k} < \frac{1}{2} \varepsilon t^k$  edges belong to a copy of  $K_r$  in  $R$ . Thus, it is sufficient to show that every edge contained in a copy of  $K_r$  has weight  $d \pm \varepsilon$ .

For that fix a copy of  $K_r$  in  $R$  and without loss of generality we may assume that  $V_1, \dots, V_r$  are the vertices of that copy. Recall that  $H$  satisfies  $\text{HCL}_{d,F}(\delta)$  and as a consequence we have for every injective map  $\varphi: [\ell] \rightarrow [r]$

$$N_F(V_{\varphi(1)}, \dots, V_{\varphi(\ell)}) = d^{e(F)} \prod_{i \in [\ell]} |V_{\varphi(i)}| \pm \delta n^\ell.$$

Since each set  $V_{\varphi(j)}$  has size at least  $n/(2T_0)$  and  $\delta = \delta_{\text{GL}}/(2^{\ell+2}T_0^\ell)$ , we obtain

$$N_F(V_{\varphi(1)}, \dots, V_{\varphi(\ell)}) = \left( d^{e(F)} \pm \delta_{\text{GL}}/2 \right) \prod_{i \in [\ell]} |V_{\varphi(i)}|. \quad (21)$$

On the other hand, applying the counting lemma, Lemma 22, we obtain

$$N_F(V_{\varphi(1)}, \dots, V_{\varphi(\ell)}) = \left( \prod_{e \in E(F)} w(\varphi(e)) \pm \gamma_{\text{CL}} \right) \prod_{i \in [\ell]} |V_{\varphi(i)}|. \quad (22)$$

Combining (21) and (22) with the choice of  $\gamma_{\text{CL}} = \delta_{\text{GL}}/2$  we conclude that

$$\prod_{e \in E(F)} w(\varphi(e)) = d^{e(F)} \pm \delta_{\text{GL}}$$

for all injective mappings  $\varphi: [\ell] \rightarrow [r]$ . By applying Lemma 21 we derive that all edges  $\{i_1, \dots, i_k\}$  have weight  $d \pm \varepsilon$  and, therefore,  $d(V_{i_1}, \dots, V_{i_k}) = d \pm \varepsilon$  which finishes the proof of Lemma 25.  $\square$

**3.3.  $\text{DISC}_d$  implies  $\text{HCL}_{d,F,\alpha}$ .** In this section we deduce  $\text{HCL}_{d,F,\alpha}$  from  $\text{DISC}_d$  by proving the following lemma.

**Fact 27.** *For every integer  $k \geq 2$ , every  $d > 0$ , every linear  $k$ -graph  $F$  with at least one edge and  $V(F) = [\ell]$  for some integer  $\ell$ , and every vector  $\alpha \in (0, 1]^\ell$ , there exists  $\delta > 0$  and  $n_0$  such that the following is true. If  $H$  is  $k$ -graph on  $n \geq n_0$  vertices that satisfies  $\text{DISC}_d(\delta)$ , then  $H$  satisfies  $\text{HCL}_{d,F,\alpha}(\varepsilon)$ .*

*Proof.* The fact is a simple consequence of the counting lemma, Lemma 22. Indeed for given  $k$ ,  $d > 0$ ,  $F$ ,  $\alpha \in (0, 1]^\ell$ , and  $\varepsilon > 0$ , set  $\delta$  to be sufficiently small, so that  $\text{DISC}_d(\delta)$  implies  $\text{DISC}_{d,k}(\delta')$  (see Theorem 6) for  $\delta' = (\delta_{\text{CL}} d \min_{i \in [\ell]} \alpha_i)^k$ , where  $\delta_{\text{CL}}$  is given by Lemma 22 applied for  $F$  and  $\gamma_{\text{CL}} = \varepsilon/2$  and we may assume  $\delta_{\text{CL}} \leq \varepsilon/2$ . Let  $n_0$  be sufficiently large and  $H$  be a  $k$ -graph on  $n \geq n_0$  vertices which satisfies  $\text{DISC}_d(\delta)$ .

Let  $U_1, \dots, U_\ell \subseteq V(H)$  with  $|U_i| = \lfloor \alpha_i n \rfloor$  be pairwise disjoint sets. We consider the induced  $\ell$ -partite  $k$ -graph  $H[U_1, \dots, U_\ell]$ . Since  $H$  satisfies  $\text{DISC}_d(\delta)$ , by Theorem 6 we infer that  $H$  satisfies  $\text{DISC}_{d,k}(\delta')$ . Moreover, since  $(\delta')^{1/k} / \min_{i \in [\ell]} \alpha_i \leq$

$\delta_{\text{CL}}$  we have that  $(U_{i_1}, \dots, U_{i_k})$  is  $\delta_{\text{CL}}$ -regular with density  $d \pm \delta_{\text{CL}}$  for every choice  $1 \leq i_1 < \dots < i_k \leq \ell$ . Consequently, Lemma 22 implies

$$N_F(U_1, \dots, U_\ell) = (d^{e(F)} \pm (\delta_{\text{CL}} + \gamma_{\text{CL}})) \prod_{i \in [\ell]} |U_i| = d^{e(F)} \prod_{i \in [\ell]} |U_i| \pm \varepsilon n^\ell,$$

which concludes the proof of the fact.  $\square$

#### 4. PROOF OF THEOREM 6

This section concerns the proof of Theorem 6. We have to show that for  $k \geq \ell \geq 2$ , every  $(\ell, k)$ -function  $\tau$ , and every  $d > 0$  the properties  $\text{DISC}_d$ ,  $\text{DISC}_{d,\ell}$ , and  $\text{DISC}_{d,\tau}$  are equivalent. The equivalence will follow from the implication

$$\text{DISC}_{d,\ell} \xrightarrow{\text{Fact 28}} \text{DISC}_{d,\ell+1},$$

which holds for every  $\ell = 1, \dots, k-1$  and the equivalence

$$\text{DISC}_{d,k} \xrightarrow{\text{Fact 32}} \text{DISC}_{d,\tau} \xrightarrow{\text{Fact 30}} \text{DISC}_{d,k},$$

which holds for every  $\ell = 1, \dots, k$  and every  $(\ell, k)$ -function  $\tau$ . Theorem 6 then follows, since Fact 28 applied for all  $\ell = 1, \dots, k-1$  gives

$$\text{DISC}_d = \text{DISC}_{d,1} \Rightarrow \dots \Rightarrow \text{DISC}_{d,\ell} \Rightarrow \text{DISC}_{d,\ell+1} \Rightarrow \dots \Rightarrow \text{DISC}_{d,k}$$

and Fact 32 applied for the unique  $(1, k)$ -function  $\tau = (1)$  gives

$$\text{DISC}_{d,k} \Rightarrow \text{DISC}_{d,(1)} = \text{DISC}_d.$$

Finally, due to Fact 30 and Fact 32 we have

$$\text{DISC}_{d,k} \Leftrightarrow \text{DISC}_{d,\tau}$$

for every  $\ell = 1, \dots, k$  and every  $(\ell, k)$ -function  $\tau$ . We prove Fact 28, Fact 30, and Fact 32 in the next section.

**4.1. Equivalence of different versions of DISC.** We first deduce  $\text{DISC}_{d,\ell+1}$  from  $\text{DISC}_{d,\ell}$  in a straightforward way.

**Fact 28.** *For all integers  $1 \leq \ell < k$ , every  $d > 0$ , and every  $\varepsilon > 0$  the following holds. If  $H$  is a  $k$ -graph that satisfies  $\text{DISC}_{d,\ell}(\varepsilon/3)$ , then  $H$  satisfies  $\text{DISC}_{d,\ell+1}(\varepsilon)$ .*

*Proof.* Let  $U_1, \dots, U_{\ell+1} \subset V(H)$  be pairwise disjoint sets. Then

$$\begin{aligned} \text{vol}(U_1, \dots, U_{\ell-1}, U_\ell, U_{\ell+1}) &= \text{vol}(U_1, \dots, U_{\ell-1}, U_\ell \dot{\cup} U_{\ell+1}) \\ &\quad - \text{vol}(U_1, \dots, U_{\ell-1}, U_\ell) - \text{vol}(U_1, \dots, U_{\ell-1}, U_{\ell+1}). \end{aligned}$$

and

$$\begin{aligned} e(U_1, \dots, U_{\ell-1}, U_\ell, U_{\ell+1}) &= e(U_1, \dots, U_{\ell-1}, U_\ell \dot{\cup} U_{\ell+1}) \\ &\quad - e(U_1, \dots, U_{\ell-1}, U_\ell) - e(U_1, \dots, U_{\ell-1}, U_{\ell+1}). \end{aligned}$$

Since  $H$  satisfies  $\text{DISC}_{d,\ell}(\varepsilon/3)$  we have

$$e(U_1, \dots, U_{\ell-1}, X) = d \text{vol}(U_1, \dots, U_{\ell-1}, X) \pm \varepsilon n^k/3$$

for all  $X \in \{U_\ell, U_{\ell+1}, U_\ell \dot{\cup} U_{\ell+1}\}$  and, consequently

$$e(U_1, \dots, U_\ell, U_{\ell+1}) = d \text{vol}(U_1, \dots, U_\ell, U_{\ell+1}) \pm \varepsilon n^k.$$

$\square$

We continue with the following observation, which is a direct consequence of the principle of inclusion and exclusion.

**Fact 29.** *Let  $t, \ell$ , and  $k$  be positive integers with  $t + \ell \leq k + 1$  and let  $\tau \in T(\ell, k)$  be an  $(\ell, k)$ -function with  $\tau(\ell) = t$ . Let  $\tau'$  be the  $(\ell + t - 1, k)$ -function given by*

$$\tau'(i) = \begin{cases} \tau(i) & \text{if } i < \ell \\ 1 & \text{if } i \geq \ell. \end{cases}$$

*Then for every  $k$ -graph  $H$  and all  $\ell + t - 1$  pairwise disjoint sets  $U_1, \dots, U_{\ell-1}, U_\ell^1, \dots, U_\ell^t \subset V(H)$  we have*

$$e_{\tau'}(U_1, \dots, U_{\ell-1}, U_\ell^1, \dots, U_\ell^t) = \sum_{\emptyset \neq J \subseteq [t]} (-1)^{t-|J|} e_\tau(U_1, \dots, U_{\ell-1}, \bigcup_{j \in J} U_\ell^j).$$

*Proof.* Let  $K \subset \bigcup_{j \in [\ell-1]} U_j \dot{\cup} \bigcup_{j \in [t]} U_\ell^j$  be a set of size  $k$  such that  $K \cap U_i = \tau(i)$  for all  $i < \ell$  and let  $I_K = \{i : |K \cap U_\ell^i| > 0\}$ . Note that  $K$  appears in  $e_{\tau'}(U_1, \dots, U_{\ell-1}, U_\ell^1, \dots, U_\ell^t)$  if and only if  $|I_K| = t$ . Moreover, the contribution of  $K$  to the right-hand side is

$$\sum_{I_K \subseteq J \subseteq [t]} (-1)^{t-|J|} = \sum_{j=0}^{t-|I_K|} \binom{t-|I_K|}{j} (-1)^{t-(|I_K|+j)} = \begin{cases} 1 & \text{if } |I_K| = t \\ 0 & \text{otherwise.} \end{cases}$$

□

**Fact 30.** *For all integers  $1 \leq \ell \leq k$ , every  $d > 0$ , every  $(\ell, k)$ -function  $\tau$ , and every  $\varepsilon > 0$  the following holds. If  $H$  is a  $k$ -graph that satisfies  $\text{DISC}_{d,\tau}(\varepsilon/2^{k^2/2})$ , then  $H$  satisfies  $\text{DISC}_{d,k}(\varepsilon)$ .*

*Proof.* Recall first that  $\text{DISC}_{d,k}(\varepsilon) = \text{DISC}_{d,\sigma}(\varepsilon)$  if  $\sigma$  is the everywhere 1-function or equivalently the unique  $(k, k)$ -function. For a given  $\tau$  we call  $|\{i : \tau(i) \geq 2\}|$  the defect of  $\tau$ . Since the everywhere 1-function  $\sigma$  is the only  $(\ell, k)$ -function, for any  $\ell$ , with defect 0, the fact follows from at most  $\lfloor k/2 \rfloor$  applications of the following claim. □

**Claim 31.** *Suppose  $\tau$  is an  $(\ell, k)$ -function with defect  $s \geq 1$ . Then there is a  $\tau'$  with defect  $s - 1$  such that if  $H$  satisfies  $\text{DISC}_{d,\tau}(\varepsilon/2^k)$ , then  $H$  satisfies  $\text{DISC}_{d,\tau'}(\varepsilon)$ .*

*Proof.* Claim 31 follows from Fact 29. For a given  $\tau \in T(\ell, k)$  with defect  $s \geq 1$  we may assume without loss of generality that  $\tau(\ell) = t \geq 2$ . We define the  $(\ell + t - 1, k)$ -function  $\tau'$  by

$$\tau'(i) = \begin{cases} \tau(i) & \text{if } i < \ell \\ 1 & \text{if } i \geq \ell. \end{cases} \quad (23)$$

Then  $\tau'$  has defect  $s - 1$  and from Fact 29 we infer

$$e_{\tau'}(U_1, \dots, U_{\ell-1}, U_\ell^1, \dots, U_\ell^t) = \sum_{\emptyset \neq J \subseteq [t]} (-1)^{t-|J|} e_\tau(U_1, \dots, U_{\ell-1}, \bigcup_{j \in J} U_\ell^j)$$

and

$$\text{vol}_{\tau'}(U_1, \dots, U_{\ell-1}, U_\ell^1, \dots, U_\ell^t) = \sum_{\emptyset \neq J \subseteq [t]} (-1)^{t-|J|} \text{vol}_\tau(U_1, \dots, U_{\ell-1}, \bigcup_{j \in J} U_\ell^j)$$

for any choice of pairwise disjoint sets  $U_1, \dots, U_{\ell-1}, U_\ell^1, \dots, U_\ell^t \subset V(H)$ . Since  $H$  satisfies  $\text{DISC}_{d,\tau}(\varepsilon/2^k)$  we have

$$e_\tau(U_1, \dots, U_{\ell-1}, \bigcup_{j \in J} U_\ell^j) = d \text{vol}_\tau(U_1, \dots, U_{\ell-1}, \bigcup_{j \in J} U_\ell^j) \pm \varepsilon n^k / 2^k$$

for all  $\emptyset \neq J \subseteq [t]$  and, hence,

$$\begin{aligned}
e_{\tau'}(U_1, \dots, U_{\ell-1}, U_\ell^1, \dots, U_\ell^t) &= \sum_{\emptyset \neq J \subseteq [t]} (-1)^{t-|J|} (d \operatorname{vol}_\tau(U_1, \dots, U_{\ell-1}, \bigcup_{j \in J} U_\ell^j) \pm \varepsilon n^k / 2^k) \\
&= d \sum_{\emptyset \neq J \subseteq [t]} (-1)^{t-|J|} \operatorname{vol}_\tau(U_1, \dots, U_{\ell-1}, \bigcup_{j \in J} U_\ell^j) \pm 2^{t-k} \varepsilon n^k \\
&= d \operatorname{vol}_{\tau'}(U_1, \dots, U_{\ell-1}, U_\ell^1, \dots, U_\ell^t) \pm \varepsilon n^k.
\end{aligned}$$

□

The last observation in this section reverses the implication of Fact 30.

**Fact 32.** *For all integers  $1 \leq \ell \leq k$ , every  $d > 0$ , every  $(\ell, k)$ -function  $\tau$ , and every  $\varepsilon > 0$  there is an  $n_0$  such that the following holds. If  $H$  is a  $k$ -graph on  $n \geq n_0$  vertices that satisfies  $\operatorname{DISC}_{d,k}(\varepsilon/3^{k^2})$ , then  $H$  satisfies  $\operatorname{DISC}_{d,\tau}(\varepsilon)$ .*

*Proof.* We choose  $n_0$  sufficiently large and by induction on  $\ell = k, \dots, 1$  we prove that if  $H$  satisfies  $\operatorname{DISC}_{d,k}(\varepsilon/3^{(k-\ell)k})$  then  $H$  also satisfies  $\operatorname{DISC}_{d,\tau}(\varepsilon)$  for an arbitrary  $(\ell, k)$ -function  $\tau$ .

For  $\ell = k$  there is only one  $(\ell, k)$ -function  $\tau$  which is the everywhere 1-function. Then  $\operatorname{DISC}_{d,\tau}(\varepsilon) = \operatorname{DISC}_{d,k}(\varepsilon)$  and the implication is obviously true.

So suppose by induction that for every  $(\ell+1, k)$ -function  $\tau'$  every  $k$ -graph  $H$  on  $n$  vertices with the property  $\operatorname{DISC}_{d,k}(\varepsilon/3^{(k-\ell)k})$  also satisfies  $\operatorname{DISC}_{d,\tau'}(\varepsilon/3^k)$ .

Let  $\tau$  be an arbitrary  $(\ell, k)$ -function and let  $U_1, \dots, U_\ell \subseteq V(H)$  be pairwise disjoint sets. Without loss of generality we assume that  $\tau(\ell) = t \geq 2$  and we define an  $(\ell+1, k)$ -function  $\tau'$  by

$$\tau'(i) = \begin{cases} \tau(i) & \text{if } i < \ell \\ \tau(i) - 1 & \text{if } i = \ell \\ 1 & \text{if } i = \ell + 1. \end{cases} \quad (24)$$

Further let  $\mathcal{P}(U_\ell)$  be the family of all ordered bipartitions of  $U_\ell$  into two equitable sets, i.e. all pairs  $(W_1, W_2)$  with  $U_\ell = W_1 \dot{\cup} W_2$  and  $|W_1| = \lfloor |U_\ell|/2 \rfloor = w$ . Then

$$\operatorname{vol}_{\tau'}(U_1, \dots, U_{\ell-1}, W_1, W_2) = \binom{w}{t-1} (|U_\ell| - w) \prod_{i \in [\ell-1]} \binom{|U_i|}{\tau(i)}$$

holds for all bipartitions  $(W_1, W_2) \in \mathcal{P}(U_\ell)$ . Since  $H$  satisfies  $\operatorname{DISC}_{d,\tau'}(\varepsilon/3^k)$  we have

$$e_{\tau'}(U_1, \dots, U_{\ell-1}, W_1, W_2) = d \operatorname{vol}_{\tau'}(U_1, \dots, U_{\ell-1}, W_1, W_2) \pm \varepsilon n^k / 3^k.$$

Summing over all bipartitions in  $\mathcal{P}(U_\ell)$  every edge in  $E_\tau(U_1, \dots, U_\ell)$  is counted exactly  $t \binom{|U_\ell| - t}{w - (t-1)}$  times. Thus, we infer

$$\begin{aligned}
e_\tau(U_1, \dots, U_\ell) &= \frac{1}{t \binom{|U_\ell| - t}{w - (t-1)}} \sum_{(W_1, W_2) \in \mathcal{P}(U_\ell)} e_{\tau'}(U_1, \dots, U_{\ell-1}, W_1, W_2) \\
&= \frac{|\mathcal{P}(U_\ell)|}{t \binom{|U_\ell| - t}{w - (t-1)}} \left( d \binom{w}{t-1} (|U_\ell| - w) \prod_{i \in [\ell-1]} \binom{|U_i|}{\tau(i)} \pm \varepsilon n^k / 3^k \right).
\end{aligned}$$

With  $|\mathcal{P}(U_\ell)| = \binom{|U_\ell|}{w}$  and

$$\frac{|\mathcal{P}(U_\ell)|}{t \binom{|U_\ell|-t}{w-(t-1)}} \binom{w}{t-1} (|U_\ell| - w) = \binom{|U_\ell|}{t}$$

and since  $|\mathcal{P}(U_\ell)| \leq 3^k t \binom{|U_\ell|-t}{w-(t-1)}$  we obtain

$$e_\tau(U_1, \dots, U_\ell) = d \prod_{i \in [\ell]} \binom{|U(i)|}{\tau(i)} \pm \varepsilon n^k.$$

□

## 5. CONCLUDING REMARKS

**5.1. Extension of  $P_3$ .** For Theorem 3 we extended properties  $P_1, P_2, P_4, P_6$ , and  $P_7$ . While the extension of  $P_5$  is straightforward and its equivalence to  $\text{DISC}_d$  follows along the lines of [34], we did not find an interesting generalisation of  $P_3$  for  $k$ -graphs and leave this open.

**5.2. Uniform edge distribution with respect to  $i$ -sets.** We studied quasi-random properties equivalent to uniform edge distribution of  $k$ -graphs with respect to large vertex sets. A natural generalisation concerns the edge distribution with respect to large subsets of  $i$ -tuples.

$i$ - $\text{DISC}_d(\varepsilon)$ : We say a  $k$ -graph  $H = (V, E)$  on  $n$  vertices has  $i$ - $\text{DISC}_d(\varepsilon)$  for  $1 \leq i \leq k-1$ ,  $d, \varepsilon > 0$ , if

$$|E(H) \cap \mathcal{K}_k(G^{(i)})| = d|\mathcal{K}_k(G^{(i)})| \pm \varepsilon n^k,$$

for any  $i$ -graph  $G^{(i)}$  with vertex set  $V$ , where  $\mathcal{K}_k(G^{(i)})$  denotes the set of all  $k$ -sets  $K$  in  $\binom{V}{k}$  which span a copy of  $K_k^{(i)}$  (the complete  $i$ -graph on  $k$  vertices) in  $G^{(i)}$ .

Clearly,  $i$ - $\text{DISC}_d$  for  $i = 1$  coincides with  $\text{DISC}_d$  and for  $i = k-1$  this is the central concept of quasi-randomness studied in [19]. The general notion  $i$ - $\text{DISC}_d$  was first studied by Frankl and Rödl [11] and Chung [2, 3, 4]. We believe that Theorem 3 can be extended for general  $i$ . As 1- $\text{DISC}_d$  is characterised by the subgraph frequencies of linear  $k$ -graphs,  $i$ - $\text{DISC}_d$  is closely related to the appearance of partial Steiner  $(i+1, k)$ -systems, i.e.,  $k$ -graphs for which every two hyperedges intersect in at most  $i$  vertices. In this context the natural generalisation of the “doubling” operation from Section 1.1 seems to be the following. Let  $A$  be a  $k$ -partite  $k$ -graph with vertex classes  $X_1, \dots, X_k$  and let  $I \in \binom{[k]}{i}$  be an  $i$ -set, then the doubling  $\text{db}_I(A)$  of  $A$  is obtained by taking two copies of  $A$  and identifying the vertices in the classes  $X_i$  for all  $i \in I$ . Again starting with a single edge and applying consecutively  $\text{db}_I$  for every  $I \in \binom{[k]}{i}$  (in some arbitrary order) we will get a  $k$ -partite  $k$ -graph, which seems likely to be of similar importance for  $i$ - $\text{DISC}_d$  as  $M$  had in Theorem 3. In fact, for  $i = k-1$ , this way we obtain the  $k$ -graph of the octahedron  $K_{2, \dots, 2}^{(k)}$  which was already studied in connection with  $(k-1)$ - $\text{DISC}_d$  in [5, 19].

A related line of research concerns the connection to extensions of Szemerédi’s regularity lemma. While there is a regularity lemma which decomposes any given  $k$ -graph into relatively few “blocks” such that most of them satisfy a  $k$ -partite version 1- $\text{DISC}_d$  (i.e.,  $\text{DISC}_{d,k}$ ), for  $i \geq 2$  the notion of  $i$ - $\text{DISC}$  seems too strong and likely no regularity lemma compatible for this notion exists. Instead, one needs to work

with “relative” versions of  $i$ -DISC. For  $i = k - 1$ , this notion of quasi-randomness was introduced in the work on hypergraph regularity by Rödl et al. [12, 22] and Gowers [14, 15], and for  $k = 3$  the equivalence was studied in [20]. It would be interesting to further investigate those connections for general  $i$  and we intend to return to this in the near future.

**5.3. Extension of Corollary 4.** In Corollary 4 we showed that for every  $k \geq 2$  the complete graph  $K_k$  and the line graph of the  $k$ -dimensional hypercube  $M(k)$  (which alternatively can be obtained from the  $k$ -graph  $M_k$  by replacing every hyperedge of  $M_k$  with a graph clique  $K_k$ ) is a forcing pair. The construction of  $M(k)$  can be easily extended from cliques to arbitrary graphs  $F$ . For a graph  $F$  with vertex set  $[k]$  let  $M(F)$  be the graph obtained from the  $k$ -graph  $M_k$  with vertex classes  $X_1, \dots, X_k$  by replacing every hyperedge by a copy of  $F$  such that the vertex representing vertex  $i \in [k] = V(F)$  lies in  $X_i$ . It seems possible that  $(F, M(F))$  is a forcing pair for every graph  $F$ . Indeed the following observation supports this belief.

While the notion of forcing pairs is closely related to the property  $\text{MIN}_d$ , we may also consider the following version of  $\text{DEV}_d$  for graphs.

$\text{DEV}_{d,F}(\varepsilon)$ : We say a graph  $G = (V, E)$  on  $n$  vertices has  $\text{DEV}_{d,F}(\varepsilon)$  for a graph  $F$  with vertex set  $[k]$  and  $d, \varepsilon > 0$ , if

$$\left| \sum_{\tilde{M}} \prod_{\tilde{F} \subseteq \tilde{M}} \left( \prod_{e \in E(\tilde{F})} \mathbb{1}_E(e) \right) - d^{e(F)} \right| \leq \varepsilon n^{k2^{k-1}},$$

where the sum runs over all copies  $\tilde{M}$  of  $M(F)$  in the complete graph  $K_V$  on vertex set  $V$  and the outer product runs over the  $2^k$  copies  $\tilde{F}$  of  $F$  (corresponding to the hyperedges of  $M_k$ ).

Following closely the lines of the proof of Lemma 13 it can be shown that for every  $d > 0$  and every graph  $F$  with at least one edge, a graph  $G$  satisfying  $\text{DEV}_{d,F}(\varepsilon)$  also satisfies the assumptions of Theorem 2 and consequently such graphs are quasi-random with density  $d$ . Moreover, it can be shown that quasi-random graphs with density  $d$  satisfy  $\text{DEV}_{d,F}$  for every fixed graph  $F$  (for details see [17]).

**5.4. Strengthening of Theorem 5.** It would be interesting to strengthen Theorem 5. We believe the partite assumption of  $\text{HCL}_{d,F}$  is not needed and it suffices that a given  $k$ -graph  $H$  contains approximately the “right” number of copies of  $F$  on every subset  $U \subseteq V(H)$ . Indeed, for graphs Theorem 2 and for 3-graphs the recent work of Dellamonica and Rödl [10] imply such an assertion.

**5.5. Algorithmic considerations.** Since  $\text{DEV}_d$ ,  $\text{MIN}_d$ , and  $\text{MDEG}_d$  can be easily checked in polynomial time, in fact in  $O(n^{k2^{k-1}})$ , we obtain by Theorem 3 an efficient algorithm which can approximately check whether a given  $k$ -graph has  $\text{DISC}_d$ . More precisely, for any given  $d$  and  $\varepsilon > 0$  there exists some positive  $\varepsilon' < \varepsilon$  such that the algorithm can distinguish in polynomial time, whether a given  $k$ -graph  $H$  satisfies  $\text{DISC}_d(\varepsilon')$  or fails to satisfy  $\text{DISC}_d(\varepsilon)$ . In some sense we cannot hope for an efficient algorithm, which decides  $\text{DISC}_d(\varepsilon)$  precisely, since it was shown in [1] that deciding  $\text{DISC}_d(\varepsilon)$  for graphs is co-NP complete.

Likely such an approximation algorithm can be used for an algorithmic version of the weak hypergraph regularity lemma, Theorem 23. Such an algorithm would

find an  $\varepsilon$ -regular partition in  $O(n^{k2^{k-1}})$ . However, a more efficient algorithm, with running time  $O(n^{2k-1} \log^2 n)$  was found by Czygrinow and Rödl [9].

Moreover, since the proof of the implication  $\text{DEV}_d \Rightarrow \text{DISC}_d$ , Lemma 13 extends to sparse  $k$ -graphs, i.e., for the case  $d = o(1)$  as long as  $d \gg n^{-(k-1)/2}$ , we obtain a sufficient, efficiently verifiable condition for checking  $\text{DISC}_d$  for sparse  $k$ -graphs. We believe it would be interesting to investigate this problem further. For example, we are not aware of a property which is equivalent to  $\text{DISC}_d$  as long as  $d \gg n^{-k+1}$  and which can be verified in polynomial time.

**5.6. Non forcing pairs.** In this section we show that there exists no minimal configuration for 3-graphs with 6 or less vertices. In other words for 3-graphs the 3-graph  $M$  from property  $\text{MIN}_d$  with 8 edges and 12 vertices can not be replaced by a 3-graph on at most 6 vertices. Hence, for every linear 3-graph  $F$  on six vertices we have to construct 3-graphs of density  $d > 0$  such that they contain the right number of copies of  $F$ , but fail to be weak quasi-random, i.e., fail to satisfy  $\text{DISC}_d$ . There are, up to isomorphism, 6 such 3-graphs  $F$ : the one with no edge, with a single edge, with two disjoint edges, with two edges sharing a vertex, the (6, 3)-configuration (the unique linear 3-graph with 3 edges on six vertices), and the Pasch-configuration (the unique linear 3-graph with 4 edges on six vertices). It is simple to see that for  $F$  being one of the first four of those configuration the property that  $H$  contains  $\sim (2/9)^{e(F)} n^{|V(F)|}$  labeled copies of  $F$  does not imply that  $H$  has  $\text{DISC}_{2/9}$  as for example the complete, 3-partite 3-graph on vertex classes of size  $n/3$  shows. Hence we will focus on the (6, 3)- and the Pasch-configuration.

**5.6.1. The (6, 3)-configuration.** We denote by  $C$  the (6, 3)-configuration, which is the 3-graph with  $V(C) = [6]$  and  $E(C) = \{\{1, 2, 3\}, \{3, 4, 5\}, \{5, 6, 1\}\}$ . We consider the complete 3-partite 3-graph  $H = H(\alpha)$  on  $n$  vertices with vertex classes  $V_1, V_2, V_3$  such that  $|V_1| = |V_2| = (1-\alpha)n/2$  and  $|V_3| = \alpha n$  for some  $\alpha \in (0, 1/3]$ . The density of  $H$  is  $\frac{3}{2}\alpha(1-\alpha)^2 - o(1)$ , while simple calculations show that

$$N_C(H) = \left( \frac{3}{8}\alpha^2(1-\alpha)^4 + o(1) \right) n^6,$$

since any copy of  $C$  in  $H$  must distribute the copies of the vertices 1, 3, 5 over all three distinct classes, and after fixing the vertex classes of the copies of 1, 3, and 5 the vertex classes of the other three vertices are fixed. Now we need to chose  $\alpha > 0$  in such a way that

$$f(\alpha) = \left( \frac{3}{2}\alpha(1-\alpha)^2 \right)^3 - \frac{3}{8}\alpha^2(1-\alpha)^4$$

is close to 0, as this would yield that  $H = H(\alpha)$  contains the “right” number of copies of  $C$ , but clearly  $H$  would not satisfy  $\text{DISC}_{3\alpha(1-\alpha)^2/2}$ . Solving  $f(\alpha) = 0$  is equivalent to solving  $g(\alpha) = \alpha(1-\alpha)^2$  equals  $1/9$ . Since  $g(0) = 0$  and  $g(1/3) = 4/27$ , we infer that there exists an  $\hat{\alpha} \in (0, 1/3]$  such that  $f(\hat{\alpha}) = 0$  (indeed  $\hat{\alpha} \approx 0.16$ ). Hence,  $H(\hat{\alpha})$  has the desired properties. Moreover, we obtain other 3-graphs with the same properties (having the right number of copies of  $C$ , but failing to have  $\text{DISC}_d$ ) for other densities  $d$ , if we consider random sub-hypergraphs of  $H(\hat{\alpha})$ .

**5.6.2. The Pasch-configuration.** Again we will construct a 3-graph  $H$  of density  $d$  which violates  $\text{DISC}_d$ , but has  $\sim d^4 n^6$  labeled copies of the Pasch-configuration  $P$ . For that we first construct a graph  $G$  and then consider its triangles to be the



hyperedges of  $H$ , i.e.,  $H = \mathcal{K}_3(G)$ . Let  $G = G(\alpha)$  be the complete, 5-partite graph with vertex classes  $V_1 \dot{\cup} \dots \dot{\cup} V_5 = V(G)$  and  $|V_1| = |V_2| = |V_3| = |V_4| = (1 - \alpha)n/4$  and  $|V_5| = \alpha n$ . The number of labeled triangles of  $G$  satisfies

$$N_{K_3}(G) = \left( \frac{3}{8}(1 - \alpha)^3 + \frac{9}{4}(1 - \alpha)^2\alpha + o(1) \right) n^3$$

while for the number of labeled  $K_{2,2,2}$  in  $G$  we have

$$N_{K_{2,2,2}}(G) = \left( \frac{(1 - \alpha)^4}{128} (3(1 - \alpha)^2 + 126\alpha^2 + 54\alpha(1 - \alpha)) + o(1) \right) n^6.$$

As above, we are interested in a solution to

$$\left( \frac{3}{8}(1 - \alpha)^3 + \frac{9}{4}(1 - \alpha)^2\alpha \right)^4 = \frac{(1 - \alpha)^4}{128} (3(1 - \alpha)^2 + 126\alpha^2 + 54\alpha(1 - \alpha)),$$

with  $\alpha \in (0, 1/5]$ . Since for  $\alpha = 0$  the left-hand side is smaller than the right-hand side, while for  $\alpha = 1/5$  the inequality switches, there must be an  $\hat{\alpha} \in (0, 1/5]$  such that both sides equal.

Let  $H = H(\hat{\alpha}) = \mathcal{K}_3(G(\hat{\alpha}))$ , i.e.,  $H$  is the 3-graph whose hyperedges correspond to the triangles of  $G(\hat{\alpha})$ . It follows that the number of edges of  $H$  equals the number of triangles in  $G$ , i.e., for  $d_{\hat{\alpha}} = \frac{3}{8}(1 - \hat{\alpha})^3 + \frac{9}{4}(1 - \hat{\alpha})^2\hat{\alpha}$

$$e(H) = (d_{\hat{\alpha}} + o(1)) \binom{n}{3}.$$

On the other hand, every labeled copy of  $K_{2,2,2}$  in  $G$  gives rise to a labeled  $K_{2,2,2}^{(3)}$  in  $H$ , which gives rise to exactly one labeled Pasch-configuration (note, that in fact a copy of  $K_{2,2,2}^{(3)}$  contains exactly two Pasch-configurations, however, those correspond to two different labelings of the same unlabeled copy of  $K_{2,2,2}^{(3)}$ ). Moreover, every labeled copy of the Pasch-configuration  $P$  in  $H$  corresponds to a  $K_{2,2,2}$  in  $G$  and, consequently,

$$N_P(H) = N_{K_{2,2,2}}(G) = (d_{\hat{\alpha}}^4 + o(1))n^6,$$

due to the choice of  $\hat{\alpha}$ . Obviously,  $H = H(\hat{\alpha})$  is 5-partite and does not satisfy  $\text{DISC}_{d_{\hat{\alpha}}}$ , which shows that it has the desired properties.

Moreover, we remark that the graph  $G = G(\hat{\alpha})$  from above has the properties

$$N_{K_3}(G) = (d_{\hat{\alpha}} + o(1))n^3 \quad \text{and} \quad N_{K_{2,2,2}}(G) = (d_{\hat{\alpha}}^4 + o(1))n^6$$

while it obviously fails to satisfy  $\text{DISC}_{d_{\hat{\alpha}}}$  for graphs. This answers a question of Shapira and Yuster from [25].

## REFERENCES

1. N. Alon, R. A. Duke, H. Lefmann, V. Rödl, and R. Yuster, *The algorithmic aspects of the regularity lemma*, J. Algorithms **16** (1994), no. 1, 80–109. [5.5](#)
2. F. R. K. Chung, *Quasi-random hypergraphs revisited*, Corrigendum: Quasi-random classes of hypergraphs, Random Structures and Algorithms **1** (1990), 363–382. [1](#), [5.2](#)
3. ———, *Quasi-random classes of hypergraphs*, Random Structures Algorithms **1** (1990), no. 4, 363–382. [1](#), [5.2](#)
4. ———, *Regularity lemmas for hypergraphs and quasi-randomness*, Random Structures Algorithms **2** (1991), no. 2, 241–252. [1](#), [3.2.2](#), [3.2.2](#), [5.2](#)
5. F. R. K. Chung and R. L. Graham, *Quasi-random hypergraphs*, Random Structures Algorithms **1** (1990), no. 1, 105–124. [1](#), [5.2](#)
6. ———, *Quasi-random set systems*, J. Amer. Math. Soc. **4** (1991), no. 1, 151–196. [1](#), [1.1.3](#)
7. ———, *Cohomological aspects of hypergraphs*, Trans. Amer. Math. Soc. **334** (1992), no. 1, 365–388. [1](#)

8. F. R. K. Chung, R. L. Graham, and R. M. Wilson, *Quasi-random graphs*, *Combinatorica* **9** (1989), no. 4, 345–362. [1](#), [1.1.4](#), [1.2](#), [1.3](#)
9. A. Czygrinow and V. Rödl, *An algorithmic regularity lemma for hypergraphs*, *SIAM J. Comput.* **30** (2000), no. 4, 1041–1066 (electronic). [5.5](#)
10. D. Dellamonica, Jr. and V. Rödl, *Hereditary quasi-random properties of hypergraphs*, submitted. [1.3](#), [5.4](#)
11. P. Frankl and V. Rödl, *The uniformity lemma for hypergraphs*, *Graphs Combin.* **8** (1992), no. 4, 309–312. [1](#), [3.2.2](#), [5.2](#)
12. ———, *Extremal problems on set systems*, *Random Structures Algorithms* **20** (2002), no. 2, 131–164. [1](#), [5.2](#)
13. D. H. Gottlieb, *A certain class of incidence matrices*, *Proc. Amer. Math. Soc.* **17** (1966), 1233–1237. [3.2](#), [3.2.1](#)
14. W. T. Gowers, *Quasirandomness, counting and regularity for 3-uniform hypergraphs*, *Combin. Probab. Comput.* **15** (2006), no. 1-2, 143–184. [1](#), [2.5](#), [5.2](#)
15. ———, *Hypergraph regularity and the multidimensional Szemerédi theorem*, *Ann. of Math.* (2) **166** (2007), no. 3, 897–946. [1](#), [5.2](#)
16. H. Hatami, *Graph norms and Sidorenko’s conjecture*, *Israel J. Math.*, to appear. [1.2](#)
17. J. Hiob, *Über eine Erweiterung des Satzes von Chung, Graham und Wilson*, Bachelor thesis, Fachbereich Mathematik, Universität Hamburg, 2010. [5.3](#)
18. Y. Kohayakawa, B. Nagle, V. Rödl, and M. Schacht, *Weak hypergraph regularity and linear hypergraphs*, submitted. [1](#), [1.1.1](#), [1.1.2](#), [1.1.4](#), [2](#), [3.2.2](#)
19. Y. Kohayakawa, V. Rödl, and J. Skokan, *Hypergraphs, quasi-randomness, and conditions for regularity*, *J. Combin. Theory Ser. A* **97** (2002), no. 2, 307–352. [1](#), [5.2](#)
20. B. Nagle, A. Poerschke, V. Rödl, and M. Schacht, *Hypergraph regularity and quasi-randomness*, *Proceedings of the Twentieth Annual ACM-SIAM Symposium on Discrete Algorithms* (C. Mathieu, ed.), ACM, 2009, pp. 227–235. [1](#), [5.2](#)
21. V. Rödl, *On a packing and covering problem*, *European J. Combin.* **6** (1985), no. 1, 69–78. [3.2.3](#), [3.2.3](#)
22. V. Rödl and J. Skokan, *Regularity lemma for  $k$ -uniform hypergraphs*, *Random Structures Algorithms* **25** (2004), no. 1, 1–42. [1](#), [5.2](#)
23. A. Shapira, *Quasi-randomness and the distribution of copies of a fixed graph*, *Combinatorica*, to appear. [1](#), [3.2.1](#)
24. A. Shapira and R. Yuster, *The effect of induced subgraphs on quasi-randomness*, *Proceedings of the 19th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, ACM Press (2008), 789–798. [1](#), [1.2](#), [1.3](#), [3.2](#)
25. ———, *On the density of a graph and its blowup*, submitted. [5.6.2](#)
26. M. Simonovits and V. T. Sós, *Szemerédi’s partition and quasirandomness*, *Random Structures Algorithms* **2** (1991), no. 1, 1–10. [1](#), [3.2.2](#)
27. ———, *Hereditarily extended properties, quasi-random graphs and not necessarily induced subgraphs*, *Combinatorica* **17** (1997), no. 4, 577–596. [1](#), [1.3](#)
28. ———, *Hereditary extended properties, quasi-random graphs and induced subgraphs*, *Combin. Probab. Comput.* **12** (2003), no. 3, 319–344, *Combinatorics, probability and computing* (Oberwolfach, 2001). [1](#), [1.3](#)
29. J. Skokan and L. Thoma, *Bipartite subgraphs and quasi-randomness*, *Graphs Combin.* **20** (2004), no. 2, 255–262. [1](#), [1.2](#), [2.2](#)
30. A. Steger, *Die Kleitman–Rothschild Methode*, Ph.D. thesis, Forschungsinstitut für Diskrete Mathematik, Rheinische Friedrichs–Wilhelms–Universität Bonn, March 1990. [1](#), [3.2.2](#)
31. E. Szemerédi, *Regular partitions of graphs*, *Problèmes combinatoires et théorie des graphes* (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976), Colloq. Internat. CNRS, vol. 260, CNRS, Paris, 1978, pp. 399–401. [1](#), [3.2.2](#)
32. A. Thomason, *Pseudorandom graphs*, *Random graphs ’85* (Poznań, 1985), North-Holland Math. Stud., vol. 144, North-Holland, Amsterdam, 1987, pp. 307–331. [1](#)
33. ———, *Random graphs, strongly regular graphs and pseudorandom graphs*, *Surveys in combinatorics 1987* (New Cross, 1987), London Math. Soc. Lecture Note Ser., vol. 123, Cambridge Univ. Press, Cambridge, 1987, pp. 173–195. [1](#)
34. R. Yuster, *Quasi-randomness is determined by the distribution of copies of a fixed graph in equicardinal large sets*, *Proceedings of the 12th International Workshop on Randomization and Computation (RANDOM)*, Springer Verlag (2008), 596–601. [1](#), [1.1.2](#), [5.1](#)

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