

# HIGHLY CONNECTED COLOURED SUBGRAPHS VIA THE REGULARITY LEMMA

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ABSTRACT. For integers  $n, r, s, k \in \mathbb{N}$ ,  $n \geq k$  and  $r \geq s$ , let  $m(n, r, s, k)$  be the largest (in order)  $k$ -connected component with at most  $s$  colours one can find in *any*  $r$ -colouring of the edges of the complete graph  $K_n$  on  $n$  vertices. Bollobás asked for the determination of  $m(n, r, s, k)$ .

Here, bounds are obtained in the cases  $s = 1, 2$  and  $k = o(n)$ , which extend results of Liu, Morris and Prince. Our techniques use Szemerédi's Regularity Lemma for many colours.

We shall also study a similar question for bipartite graphs.

## 1. INTRODUCTION AND RESULTS

For basic definitions from graph theory we refer the reader to [2]. Let  $n, r, s, k \in \mathbb{N}$  with  $s \leq r$  and  $k \leq n$ . Given a graph  $G = (V, E)$  with  $|V(G)| = n$ , and an  $r$ -colouring of its edges,  $f : E(G) \rightarrow [r]$ , define

$$M(f, G, r, s, k) := \max\{|V(H)| : H \subseteq G, |f(E(H))| \leq s \text{ and } H \text{ is } k\text{-connected}\}.$$

That is,  $M(f, G, r, s, k)$  is the order of the largest  $k$ -connected subgraph  $H$  in  $G$  whose edges are coloured with at most  $s$  different colours in the  $r$ -colouring  $f$ . Let

$$m(G, r, s, k) := \min_f \{M(f, G, r, s, k)\}.$$

In the case  $G = K_n$ , we write  $M(f, n, r, s, k)$  and  $m(n, r, s, k)$  respectively.

The question of determining  $m(n, r, s, k)$  (in its full generality) was first posed by Bollobás. In particular, Bollobás and Gyárfás [3] conjectured that  $m(n, 2, 1, k) = n - 2k + 2$  for  $n > 4(k - 1)$ . They proved this for  $k = 2$ , and they also showed that  $m(n, 2, 1, k) \geq n - 6(2k - 3)$  for  $n \geq 16k - 22$  and  $k \geq 2$ . Further partial results of the conjecture were subsequently proved by Liu, Morris and Prince. They proved the conjecture for  $k = 3$  [6], and that  $m(n, 2, 1, k) = n - 2k + 2$  for  $n \geq 13k - 15$  [7]. Also in [7], they studied  $m(n, r, 1, k)$  for general  $r$ . They conjectured that, given  $r$  and  $k$  with  $r \geq 3$ , there exists  $n_0 = n_0(r, k)$  such that  $m(n, r, 1, k)$  is approximately equal to  $\frac{n-k+1}{r-1}$ , if  $r - 1$  is a prime power, for every  $n \geq n_0$ . They proved the case  $r = 3$  with  $n_0 = 480k$ .

The question becomes much more harder to study when one looks for multicoloured  $k$ -connected components, i.e.  $s \geq 2$ . In a subsequent paper [8], Liu, Morris and Prince conducted further research, determining more values and bounds for  $m(n, r, s, k)$  with  $s \geq 2$ .

In the two papers [7, 8], Liu et al. proved (among other facts) the following lower bounds for  $n, r, k \in \mathbb{N}$  and  $r \geq 3$ :

- $m(n, r, 1, k) \geq \frac{n}{r-1} - 11(k^2 - k)r$ ,
- $m(n, r, 2, k) \geq \frac{4n}{r+1} - 17kr(r + 2k + 1)$ .

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Moreover, in the first case, when  $r - 1$  is a prime power, they showed an upper bound on  $m(n, r, 1, k)$  is  $\frac{n-k+1}{r-1} + r$ , and in the second case, when  $r + 1$  is a power of 2, an upper bound on  $m(n, r, 2, k)$  is  $\frac{4n}{r+1} + 4$ . In the proofs of the above lower bounds, Liu et al. used Mader's Theorem [9]. But, these lower bounds are good only for  $k = o(\sqrt{n})$ .

We will study asymptotic lower bounds on  $m(n, r, s, k)$  when  $n$  is large. Using ideas from [7, 8] and the many-colours version of Szemerédi's Regularity Lemma [11], we extend the above results to the sublinear case: when  $k = o(n)$  (and hence, more consistent with the two aforementioned conjectures), showing that "asymptotically", connectedness should play a lesser role. Namely, that multicoloured components will contain nearly spanning highly connected subgraphs (of connected coloured graphs). More precisely, we shall prove the following main results.

**Theorem 1.** *For every  $\gamma \in (0, \frac{1}{4})$ ,  $n, r \in \mathbb{N}$  with  $r \geq 3$ , there exist integers  $N_0 = N_0(\gamma, r)$  and  $T_0 = T_0(\gamma, r)$  such that, for all  $n \geq N_0$ ,*

$$m\left(n, r, 1, \frac{1-4\gamma}{rT_0}n\right) \geq \frac{n}{r-1} - \frac{6\gamma n}{r-1}.$$

*In particular, for fixed  $r$ , and  $k = o(n)$ ,  $m(n, r, 1, k) \geq \frac{n}{r-1} - o(n)$ , with equality if  $r - 1$  is a prime power.*

**Theorem 2.** *For every  $\gamma \in (0, \frac{1}{4})$ ,  $n, r \in \mathbb{N}$  with  $r \geq 3$ , there exist integers  $N_0 = N_0(\gamma, r)$  and  $T_0 = T_0(\gamma, r)$  such that, for all  $n \geq N_0$ ,*

$$m\left(n, r, 2, \frac{1-4\gamma}{rT_0}n\right) \geq \frac{4n}{r+1} - \frac{8\gamma n}{r+1}.$$

*In particular, for fixed  $r$ , and  $k = o(n)$ ,  $m(n, r, 2, k) \geq \frac{4n}{r+1} - o(n)$ , with equality if  $r + 1$  is a power of 2.*

Bollobás and Gyárfás [3] noted that every 2-coloured complete graph on  $n$  vertices contains an  $\lceil \frac{n}{16} \rceil$ -connected subgraph on at least  $\frac{n}{4}$  vertices. Thus, the above theorems may also be seen as a step in this direction.

With the same techniques, we are also able to partially prove another conjecture of Liu et al.: Conjecture 2 of [7]. For this, it is more convenient if we define the analogous function to  $m(n, r, s, k)$  for bipartite graphs. For  $n, n', r, s, k \in \mathbb{N}$  with  $s \leq r$  and  $k \leq n \leq n'$ , and an  $r$ -colouring  $f : E(K_{n, n'}) \rightarrow [r]$ , define

$$\begin{aligned} M_{\text{bip}}(f, n, n', r, s, k) &:= \max\{|V(H)| : H \subseteq K_{n, n'}, |f(E(H))| \leq s \text{ and } H \text{ is } k\text{-connected}\}, \\ m_{\text{bip}}(n, n', r, s, k) &:= \min_f \{M_{\text{bip}}(f, n, n', r, s, k)\}. \end{aligned}$$

The conjecture then states that, provided  $n, n' \geq rk$ , we have  $m_{\text{bip}}(n, n', r, 1, k) \geq \frac{n+n'}{r}$  (and so independent of  $k$ ). We shall prove the following partial result.

**Theorem 3.** *For every  $\gamma \in (0, \frac{1}{34})$ ,  $n, n', r \in \mathbb{N}$ , with  $r \geq 2$  and  $n' \geq n$ , there exist integers  $N_0 = N_0(\gamma, r)$  and  $T_0 = T_0(\gamma, r)$  such that, for all  $n \geq N_0$ ,*

$$\frac{n+n'}{r} - \frac{3\gamma(n+n')}{r} \leq m_{\text{bip}}\left(n, n', r, 1, \frac{1-4\gamma}{rT_0}n\right) \leq \frac{n+n'}{r} + 2.$$

*In particular, for fixed  $r$ , and  $k = o(n)$ , we have  $m_{\text{bip}}(n, n', r, 1, k) = \frac{n+n'}{r} - o(n)$ .*

## 2. TOOLS: REGULARITY LEMMAS

In this section, we shall discuss the concept of  $\varepsilon$ -regularity for graphs and the celebrated Szemerédi's Regularity Lemma [11]. For further details, see the excellent survey of Komlós and Simonovits [5].

For a graph  $G = (V, E)$  and two disjoint subsets  $A, B$  of  $V$ , we write  $E(A, B)$  to denote the set of the edges from  $E$  that intersect both  $A$  and  $B$ . We set  $e(A, B) := |E(A, B)|$ . We write  $G[A, B]$  for the bipartite subgraph of  $G$  induced by  $A$  and  $B$ , ie:  $G[A, B]$  has vertex classes  $A$  and  $B$ , and edge set  $E(A, B)$ . We often call  $(A, B)$  a *pair* without writing  $G$  explicitly when it is clear from the context or if it is not important.

Let  $\varepsilon \in (0, 1)$ . We define a pair  $(V_1, V_2)$  (a bipartite graph with vertex classes  $V_1$  and  $V_2$ ) to be  $\varepsilon$ -regular, if for every  $U_i \subseteq V_i$  with  $|U_i| \geq \varepsilon|V_i|$ ,  $i = 1, 2$ , the following inequality holds:

$$|d(V_1, V_2) - d(U_1, U_2)| < \varepsilon,$$

where  $d(X, Y) := \frac{e(X, Y)}{|X||Y|}$  is the (*edge density*) of the pair  $(X, Y)$ . Note that if  $(V_1, V_2)$  is  $\varepsilon$ -regular, then it is also  $\varepsilon'$ -regular for any  $\varepsilon' \in (\varepsilon, 1)$ . The pair  $(V_1, V_2)$  is said to be  $(\varepsilon, d)$ -regular, if it is  $\varepsilon$ -regular with  $d(V_1, V_2) \geq d$ .

The following well-known lemmas make  $\varepsilon$ -regular pairs very useful in applications.

**Lemma 4** (Facts 1.3, 1.4 of [5]). *Given  $0 < 2\varepsilon < \eta < 1$ , let  $(V_1, V_2)$  be an  $\varepsilon$ -regular pair with density  $\eta$ . Then,*

$$\begin{aligned} |\{x \in V_1 : |\Gamma(x)| \leq (\eta - \varepsilon)|V_2|\}| &\leq \varepsilon|V_1|, \\ |\{x \in V_1 : |\Gamma(x)| \geq (\eta + \varepsilon)|V_2|\}| &\leq \varepsilon|V_1|, \\ |\{(x, y) : x, y \in V_1, |\Gamma(x) \cap \Gamma(y)| \leq (\eta - \varepsilon)^2|V_2|\}| &\leq 2\varepsilon|V_1|^2, \\ |\{(x, y) : x, y \in V_1, |\Gamma(x) \cap \Gamma(y)| \geq (\eta + \varepsilon)^2|V_2|\}| &\leq 2\varepsilon|V_1|^2. \end{aligned}$$

*Note that  $(x, y)$  is an ordered pair.*

**Lemma 5** (Slicing Lemma; Fact 1.5 of [5]). *Let  $\varepsilon, \alpha \in (0, 1)$  with  $\varepsilon < \alpha$ , and  $(V_1, V_2)$  be an  $\varepsilon$ -regular pair. If  $U_i \subseteq V_i$  and  $|U_i| \geq \alpha|V_i|$  for  $i = 1, 2$ , then  $(U_1, U_2)$  is  $\varepsilon'$ -regular, where  $\varepsilon' = \max(\frac{\varepsilon}{\alpha}, 2\varepsilon)$ .*

For a given graph  $G$ , the partition  $V(G) = V_0 \dot{\cup} V_1 \dot{\cup} V_2 \dot{\cup} \dots \dot{\cup} V_t$  of its vertex set is said to be  $\varepsilon$ -regular if the following conditions hold:

- $|V_0| \leq \varepsilon|V(G)|$ ,
- $|V_i| = |V_j|$ , for every  $1 \leq i, j \leq t$ , and
- all but at most  $\varepsilon t^2$  pairs  $(V_i, V_j)$  are  $\varepsilon$ -regular,  $1 \leq i < j \leq t$ .

The classes of the partition are called *clusters*, and  $V_0$  is the *exceptional set*.

Szemerédi's Regularity Lemma [11] then says that, given  $\varepsilon \in (0, 1)$  and an integer  $t_0$ , we can find integers  $N_0 = N_0(\varepsilon, t_0)$  and  $T_0 = T_0(\varepsilon, t_0)$  such that, for every graph  $G$  on at least  $N_0$  vertices,  $V(G)$  admits a partition into  $t + 1$  classes, for some  $t_0 \leq t \leq T_0$ , which is  $\varepsilon$ -regular. Roughly speaking, this says that any graph of sufficiently large order can be approximated by a multipartite graph with a bounded number of equal classes, where the distribution of the edges between most pairs of classes is, in some sense, as in a random graph.

Here, we shall utilise a straightforward generalisation of the original proof of Szemerédi: the *many-colours regularity lemma*.

**Theorem 6** (Many-colours regularity lemma; Theorem 1.18 of [5]). *For every  $\varepsilon \in (0, 1)$  and  $r, t_0 \in \mathbb{N}$ , there exist integers  $N_0 = N_0(\varepsilon, r, t_0)$  and  $T_0 = T_0(\varepsilon, r, t_0)$  such that the following holds. Every graph  $G = (V, E)$  with  $|V| \geq N_0$ , whose edges are  $r$ -coloured:  $E = E_1 \dot{\cup} \dots \dot{\cup} E_r$ , admits a partition of its vertex set:  $V = V_0 \dot{\cup} V_1 \dot{\cup} \dots \dot{\cup} V_t$ , for some  $t_0 \leq t \leq T_0$ , which is  $\varepsilon$ -regular simultaneously with respect to every subgraph  $G_i = (V, E_i)$ ,  $i \in [r]$ .*

Finally, we will need the notion of a *cluster graph* of a graph  $G$ . Let  $G$  be a graph on  $n$  vertices, and  $V(G) = V_1 \dot{\cup} \cdots \dot{\cup} V_t$  be a partition of its vertex set. For  $0 < \varepsilon < \eta < 1$  (we think of  $\eta$  as being much larger than  $\varepsilon$ , but much smaller than 1), define a new graph  $R(\eta)$ , the *cluster graph* (or *reduced graph*), where  $V(R(\eta)) = [t]$ , and  $ij \in E(R(\eta))$  if  $(V_i, V_j)$  is  $(\varepsilon, \eta)$ -regular. One also sometimes writes  $\{V_1, \dots, V_t\}$  for  $V(R(\eta))$ .

The “remaining underlying graph”  $G'$  is then the subgraph of  $G$ , where  $V(G') = V(G)$ , and  $xy \in E(G')$  if  $xy \in E(G)$  and  $x \in V_i, y \in V_j$ , where  $ij \in E(R(\eta))$ . That is, we keep an edge for  $G'$  if the pair that it belongs to is  $(\varepsilon, \eta)$ -regular.

If in the original regularity lemma, we let  $t_0 = \lceil \frac{1}{\varepsilon} \rceil$  and  $|V(G)|$  be sufficiently large, then  $G$  admits an  $\varepsilon$ -regular partition. If  $\eta$  is as above, we can obtain the graph  $R(\eta)$  from  $G$  (ignoring the exceptional set) on  $t \geq \frac{1}{\varepsilon}$  vertices, and the graph  $G'$ . Then, in  $G'$ , we have disregarded at most

$$t\varepsilon n \binom{n}{t} + \binom{\lfloor \varepsilon n \rfloor}{2} + \varepsilon t^2 \binom{n}{t}^2 + \eta \binom{n}{t}^2 \binom{t}{2} + t \binom{\lfloor n/t \rfloor}{2} < \frac{7}{2} \eta n^2$$

edges from  $G$ . Thus, for small  $\eta$ , the graph  $G'$  is “almost” our original graph  $G$ . We can similarly apply this for the many-colours version of the regularity lemma, and get that for every colour  $i \in [r]$ , the corresponding graph  $G'_i$  is “close” to the graph  $G_i$ .

It turns out that, if we can find a component in  $R(\eta)$  of order  $c \geq 2$ , the underlying subgraph  $G'$  of  $G$  contains a subgraph  $H$  on roughly at least  $c \cdot \frac{n}{t}$  vertices, which becomes  $o(n)$ -connected after deleting roughly at most  $\varepsilon c \cdot \frac{n}{t}$  vertices from it.

**Lemma 7** (Tree decomposition lemma). *Let  $\varepsilon \in (0, 1)$ , and suppose that the graph  $G$  has a partition:  $V(G) = V_1 \dot{\cup} \cdots \dot{\cup} V_t$ , where  $|V_i| = m$  for every  $i$ . For  $3\varepsilon < \eta < 1$ , let  $R(\eta)$  be the cluster graph of this partition. If  $R(\eta)$  contains a component of order  $c \geq 2$ , then  $G$  contains an  $(\eta - 3\varepsilon)m$ -connected component on at least  $(1 - \varepsilon)cm$  vertices.*

*Proof.* Let  $C$  be a component in  $R(\eta)$  of order  $c \geq 2$ . Fix any spanning tree  $T$  of  $C$ , and assume without loss of generality that  $V(T) = V(C) = \{1, \dots, c\}$ . The underlying subgraph  $H$  of  $G$  that corresponds to  $T$  has  $V(H) = V_1 \dot{\cup} \cdots \dot{\cup} V_c$ , and  $xy \in E(H)$  if  $xy \in E(G)$  and  $x \in V_i, y \in V_j$ , where  $ij \in E(T)$ .

We shall show that, by deleting at most  $\varepsilon m$  vertices from each  $V_i, i \in [c]$ , we will get a subgraph  $H'$  of  $H$  which is  $(\eta - 3\varepsilon)m$ -connected. We proceed as follows. Let  $L_1$  be the leaves of  $T$ . For  $j > 1$ , let  $L_j$  be the leaves of  $T - \bigcup_{i=1}^{j-1} L_i$  (defined inductively). We have a partition  $V(T) = L_1 \dot{\cup} \cdots \dot{\cup} L_p$ . Note that  $|L_p| = 1$  or  $2$ , and if  $p = 1$ , then  $|L_p| = 2$ . Now, run the following algorithm.

Step 1. If  $p = 1$ , then proceed to Step 2. Otherwise,  $p > 1$ . In this case, take a vertex of  $L_1$ , say  $V_i$ . It has exactly one neighbour in  $T - L_1$ , say  $V_j$ . Since  $(V_i, V_j)$  is  $(\varepsilon, \eta)$ -regular, by Lemma 4, we may delete all vertices from  $V_i$  with at most  $(\eta - \varepsilon)m$  neighbours in  $V_j$ , and obtain  $V'_i \subseteq V_i$  with  $|V'_i| \geq (1 - \varepsilon)m$ . Now, disregard the vertex  $V_i$  from  $T$ , and repeat this procedure on every vertex of  $L_1$ . Then, repeat the whole procedure successively on  $L_2, L_3, \dots, L_{p-1}$ .

Step 2. We are left with  $L_p$ . If  $|L_p| = 2$ , let  $L_p = \{V_k, V_\ell\}$ . By Lemma 4, we may delete all vertices from  $V_k$  with at most  $(\eta - \varepsilon)m$  neighbours in  $V_\ell$ , and all vertices from  $V_\ell$  with at most  $(\eta - \varepsilon)m$  neighbours in  $V_k$ . We obtain  $V'_k \subseteq V_k$  and  $V'_\ell \subseteq V_\ell$  with  $|V'_k|, |V'_\ell| \geq (1 - \varepsilon)m$ . If  $|L_p| = 1$ , let  $L_p = \{V_k\}$ . Take an arbitrary, fixed neighbour of  $V_k$  in  $T$ , say  $V_\ell$ , and (similarly) delete the vertices from  $V_k$  with at most  $(\eta - \varepsilon)m$  neighbours in  $V_\ell$ . We obtain  $V'_k \subseteq V_k$  with  $|V'_k| \geq (1 - \varepsilon)m$ .

For every  $i \in [c]$ , we have now deleted at most  $\varepsilon m$  vertices from  $V_i$ , obtaining  $V'_i \subseteq V_i$ . Let  $H'$  be the remaining subgraph of  $H$ . Then,  $|V(H')| \geq (1 - \varepsilon)cm$ . We claim that  $H'$  is the required  $(\eta - 3\varepsilon)m$ -connected subgraph. We shall prove a stronger assertion: deleting *any*  $(\eta - 3\varepsilon)m$  vertices

from every  $V'_i$  does not disconnect  $H'$ . So, delete such a set of vertices, let  $V''_i \subseteq V'_i$  be the remaining subsets,  $i \in [c]$ , and let  $H''$  be the remaining subgraph of  $H'$ . We want to show that  $H''$  is connected.

We first show that the pair  $(V''_k, V''_\ell)$  is connected. It suffices to show that, for every  $x \in V''_k$  and  $y \in V''_\ell$ ,  $x$  is connected to  $y$ . Observe that the minimum degree of the pair  $(V'_k, V'_\ell)$  is at least  $(\eta - 2\varepsilon)m$ . So, if  $X = \Gamma_H(x) \cap V''_\ell$  and  $Y = \Gamma_H(y) \cap V''_k$ , then  $|X|, |Y| \geq \varepsilon m$ . It now suffices to show that  $E(X, Y) \neq \emptyset$ . Since  $(V_k, V_\ell)$  is  $(\varepsilon, \eta)$ -regular, we have  $d(X, Y) > d(V_k, V_\ell) - \varepsilon \geq \eta - \varepsilon > 0$ , which implies  $E(X, Y) \neq \emptyset$ .

Now, any  $V_q \in V(T) \setminus \{V_k\}$  is connected to  $V_k$  by a unique path in  $T$ , say,  $V_{q_1} \cdots V_{q_r}$ , where  $q_1 = q$  and  $q_r = k$ . It is easy to see from the algorithm that, for every  $1 \leq s < r$ , every vertex of  $V''_{q_s}$  has at least  $(\eta - 2\varepsilon)m - (\eta - 3\varepsilon)m = \varepsilon m$  neighbours in  $V''_{q_{s+1}}$ . Hence, for any  $V_r, V_{r'} \in V(T)$  (which may be the same) and any  $u \in V''_r, v \in V''_{r'}$ ,  $u$  and  $v$  are connected to some  $u', v' \in V''_k$  by  $H''$ , respectively. The lemma follows, since  $u'$  and  $v'$  are connected by  $(V''_k, V''_\ell)$ , so that  $u$  and  $v$  are connected by  $H''$ .  $\square$

We now have all the  $\varepsilon$ -regularity results that we will need for tackling Theorems 1 and 2. For Theorem 3, we will need the following version of the regularity lemma for bipartite graphs for many colours.

**Theorem 8.** *For every  $\varepsilon \in (0, 1)$  and  $r, t_0 \in \mathbb{N}$ , there exist integers  $N_0 = N_0(\varepsilon, r, t_0)$  and  $T_0 = T_0(\varepsilon, r, t_0)$  such that the following holds. Every bipartite graph  $G = (U \dot{\cup} V, E)$  with  $|U| = n$ ,  $|V| = n'$ , and  $n' \geq n \geq N_0$ , whose edges are  $r$ -coloured:  $E = E_1 \dot{\cup} \cdots \dot{\cup} E_r$ , admits a partition of its vertex set:  $U = U_0 \dot{\cup} U_1 \dot{\cup} \cdots \dot{\cup} U_t$  and  $V = V_0 \dot{\cup} V_1 \dot{\cup} \cdots \dot{\cup} V_t$ , for some  $t_0 \leq t \leq T_0$ , so that*

- $|U_0| \leq \varepsilon n$  and  $|V_0| \leq \varepsilon n'$ ,
- $|U_i| = |U_j|$  and  $|V_{i'}| = |V_{j'}|$ , for every  $1 \leq i, j, i', j' \leq t$ , and
- in every subgraph  $G_k = (U \dot{\cup} V, E_k)$ ,  $k \in [r]$ , all but at most  $\varepsilon t^2$  pairs of  $\{(U_i, V_j)\}_{i, j \geq 1}$  are  $\varepsilon$ -regular.

The proof of Theorem 8 is a straightforward generalisation (see for example [5] for details) of the proof of the regularity lemma for bipartite graphs. For the proof of the latter see [11], see also [10] for a weaker form and [12] for a more recent proof as well.

Unfortunately, if we attempt to use Theorem 8 directly to tackle Theorem 3, we will run into a major difficulty. Given a large bipartite graph  $G$  with part-sizes  $n$  and  $n'$ , where  $n' \geq n$ , it turns out that, when we partition  $V(G)$ , we would like all the cluster sizes to be roughly the same. If we use the partition given by Theorem 8, this is certainly far from being true if  $n' \gg n$ . So, our next aim is to suitably refine the partition of  $V(G)$  as given by Theorem 8 while, in some sense, “preserving” a large proportion of  $\varepsilon$ -regular pairs.

We shall roughly divide our consideration into the cases  $n' \sim n$  and  $n' \gg n$ . More precisely, given  $\varepsilon > 0$ , we consider the cases  $n' \leq 33\varepsilon^{-5}n$  and  $n' > 33\varepsilon^{-5}n$ .

**Theorem 9** (Many-colours regularity lemma for bipartite graphs). *For every  $\varepsilon \in (0, \frac{1}{34})$  and  $r, t_0 \in \mathbb{N}$ , there exist integers  $N_0 = N_0(\varepsilon, r, t_0)$  and  $T_0 = T_0(\varepsilon, r, t_0)$ , such that the following holds. Let  $G = (U \dot{\cup} V, E)$  be a bipartite graph, with  $|U| = n$ ,  $|V| = n'$ , and  $n' \geq n \geq N_0$ . Then, whenever the edges of  $G$  are  $r$ -coloured:  $E = E_1 \dot{\cup} \cdots \dot{\cup} E_r$ ,  $G$  admits a partition of its vertex set:  $U = U_0 \dot{\cup} U_1 \dot{\cup} \cdots \dot{\cup} U_t$  and  $V = V_0 \dot{\cup} V_1 \dot{\cup} \cdots \dot{\cup} V_{t'}$ , for some  $t_0 \leq t \leq T_0$ , so that*

- $|U_0| \leq \varepsilon n$  and  $|V_0| \leq \varepsilon n'$ ,
- $|U_i| = |V_j|$  for all  $1 \leq i \leq t, 1 \leq j \leq t'$ , and
- – either, in every  $G_k = (U \dot{\cup} V, E_k)$ ,  $k \in [r]$ , all but at most  $\varepsilon t t'$  pairs of  $\{(U_i, V_j)\}_{i, j \geq 1}$  are  $\varepsilon$ -regular, if  $n' \leq 33\varepsilon^{-5}n$ ,
- or, all but at most  $\varepsilon t t'$  pairs of  $\{(U_i, V_j)\}_{i, j \geq 1}$  are  $(\varepsilon, \frac{1}{r} - \frac{1}{4}\varepsilon^5)$ -regular, each one with respect to some colour, if  $n' > 33\varepsilon^{-5}n$ ,  $r < 4\varepsilon^{-5}$ , and  $G = K_{n, n'}$ .

The case  $n' > 33\varepsilon^{-5}n$  will be the trickier case. We shall derive a key lemma to help us to prove this case. To do this, we first recall a lemma of Alon et al. [1], which has a sufficient condition for a bipartite graph to be  $\varepsilon$ -regular.

For a pair  $(V_1, V_2)$  with  $|V_1| = |V_2| = n$ , density  $d$ , and  $Y \subseteq V_1$ , define the *deviation* of  $Y$  by

$$\sigma(Y) = \frac{1}{|Y|^2} \sum_{y_1, y_2 \in Y} \sigma(y_1, y_2),$$

where for distinct  $y_1, y_2 \in V_1$

$$\sigma(y_1, y_2) = |\Gamma(y_1) \cap \Gamma(y_2)| - d^2n$$

is the *neighbourhood deviation* of  $y_1$  and  $y_2$ .

Here then, is the lemma of Alon et al.

**Lemma 10** (Regularity criterion; Lemma 3.2 of [1]). *Let  $G$  be a bipartite graph with classes  $V_1$  and  $V_2$ , where  $|V_1| = |V_2| = n$ , and density  $d$ . Let  $2n^{-1/4} < \varepsilon < \frac{1}{16}$ . Assume that*

$$|\{x \in V_1 : |\deg(x) - dn| \geq \varepsilon^4 n\}| \leq \frac{1}{8}\varepsilon^4 n,$$

and that for every  $Y \subseteq V_1$  with  $|Y| \geq \varepsilon n$ , we have  $\sigma(Y) < \frac{1}{2}\varepsilon^3 n$ . Then,  $G$  is  $\varepsilon$ -regular.

Now, here is the lemma that we will require.

**Lemma 11.** *Let  $\varepsilon \in (0, \frac{1}{16})$ ,  $k \in \mathbb{N}$  and  $(X_1, X), \dots, (X_k, X)$  be  $(\frac{1}{16}\varepsilon^5)$ -regular pairs, with  $d_i := d(X_i, X) \in (\frac{1}{4}\varepsilon^5, 1)$  for every  $i$ , and moreover,  $|X_i| = m$  for every  $i$ ,  $|X| = m'$ ,  $m' \geq m > 16\varepsilon^{-4}$ , and  $\ell m = m'$  for some  $\ell \in \mathbb{N}$ . Then, there exists a partition  $X = U_1 \dot{\cup} \dots \dot{\cup} U_\ell$ , where  $|U_j| = m$  for every  $j$ , such that at least*

$$\left[1 - 2m^3 \exp\left(-\frac{3}{16^4}\varepsilon^{20}m\right)\right]k\ell$$

of the pairs  $(X_i, U_j)$  are  $\varepsilon$ -regular, and  $d(X_i, U_j) \in (d_i - \frac{1}{4}\varepsilon^5, d_i + \frac{1}{4}\varepsilon^5)$ .

*Proof.* We shall prove that, by taking a random partition of  $X$  into parts of size  $m$ , the conclusion holds with probability at least  $1 - \frac{1}{m} > 0$ . To do this, we shall apply Lemma 10. We show that for every  $i$ , most vertices from  $X_i$  and most pairs of vertices from  $X_i$  have roughly the expected degrees and co-degrees in a randomly chosen subset of  $X$  of size  $m$ .

Fix  $X_i$ , and let  $\{v_1, \dots, v_m\}$  be its vertex set. For every  $1 \leq r \leq m$ , let  $Z_r$  be the random variable that counts the neighbours of the vertex  $v_r \in X_i$  in a subset  $U \subset X$  of size  $m$ , chosen uniformly at random.  $Z_r$  has a hypergeometric distribution  $\text{Hg}(m, m', \deg(v_r))$ . Let  $\tilde{\varepsilon} = \frac{1}{16}\varepsilon^5$ . By Lemma 4, all but at most  $2\tilde{\varepsilon}m$  vertices of  $X_i$  have degrees in  $X$  lying in the interval  $(d_i m' - \tilde{\varepsilon}m', d_i m' + \tilde{\varepsilon}m')$ . Let  $X'_i$  be these vertices, and  $v_r \in X'_i$ . By Chernoff's inequality (see, for example, Theorem 2.10 in [4]), we have

$$\begin{aligned} \mathbb{P}(Z_r \leq d_i m - 2\tilde{\varepsilon}m) &\leq \mathbb{P}\left(Z_r \leq (\deg(v_r) - \tilde{\varepsilon}m')\frac{m}{m'}\right) \leq \mathbb{P}\left(Z_r \leq (1 - \tilde{\varepsilon})\frac{\deg(v_r)m}{m'}\right) \leq \\ &\leq \exp\left(-\frac{\tilde{\varepsilon}^2 \deg(v_r)m}{3m'}\right) \leq \exp\left(-\frac{1}{3}\tilde{\varepsilon}^2(d_i - \tilde{\varepsilon})m\right), \end{aligned}$$

and similarly,  $\mathbb{P}(Z_r \geq d_i m + 2\tilde{\varepsilon}m) \leq \exp(-\frac{1}{3}\tilde{\varepsilon}^2(d_i - \tilde{\varepsilon})m)$ . Applying this to every such vertex in  $\bigcup_{i=1}^k X'_i$ , we have

$$\begin{aligned} \mathbb{P}\left(|\Gamma(v) \cap U| \in (d_i m - 2\tilde{\varepsilon}m, d_i m + 2\tilde{\varepsilon}m) \text{ whenever } v \in \bigcup_{i=1}^k X'_i, v \in X'_i\right) &\geq \\ &\geq 1 - \sum_{i=1}^k 2m \exp\left(-\frac{1}{3}\tilde{\varepsilon}^2(d_i - \tilde{\varepsilon})m\right) \geq 1 - 2km \exp\left(-\frac{1}{16^3}\varepsilon^{15}m\right). \end{aligned}$$

Next, for fixed  $X_i$  and for every ordered pair  $(v_s, v_t)$  of vertices from  $X_i$ , let the random variable  $Z_{st}$  be the number of common neighbours of  $v_s$  and  $v_t$  in a randomly chosen subset  $U$  of  $X$  of size  $m$ . We have  $Z_{st} \sim \text{Hg}(m, m', |\Gamma(v_s) \cap \Gamma(v_t)|)$ . Let  $X''_i \subseteq X_i \times X_i$  be those ordered pairs with common degree in  $X$  lying in  $((d_i - \tilde{\varepsilon})^2 m', (d_i + \tilde{\varepsilon})^2 m')$ . By Lemma 4,  $X''_i$  consists of all but at most  $4\tilde{\varepsilon}m^2$  pairs from  $X_i$ . Again, by Chernoff's inequality, for  $(v_s, v_t) \in X''_i$ ,

$$\mathbb{P}(Z_{st} \geq (d_i + \tilde{\varepsilon})^2 m + \tilde{\varepsilon}m) \leq \exp\left(-\frac{1}{3}\tilde{\varepsilon}^2(d_i - \tilde{\varepsilon})^2 m\right),$$

by a similar calculation. Applying this to all such ordered pairs in  $\bigcup_{i=1}^k X''_i$ , we have

$$\begin{aligned} \mathbb{P}\left(|\Gamma(v_s) \cap \Gamma(v_t) \cap U| < (d_i + \tilde{\varepsilon})^2 m + \tilde{\varepsilon}m \text{ whenever } (v_s, v_t) \in \bigcup_{i=1}^k X''_i, (v_s, v_t) \in X''_i\right) &\geq \\ &\geq 1 - \sum_{i=1}^k m^2 \exp\left(-\frac{1}{3}\tilde{\varepsilon}^2(d_i - \tilde{\varepsilon})^2 m\right) \geq 1 - km^2 \exp\left(-\frac{3}{16^4}\varepsilon^{20}m\right). \end{aligned}$$

So now, choose a set  $U \subset X$  with  $|U| = m$ , uniformly at random. For each  $1 \leq i \leq k$ , let  $G_i = (X_i, U)$  and  $d'_i = d(X_i, U)$ . With probability at least  $1 - 2km^2 \exp(-\frac{3}{16^4}\varepsilon^{20}m)$ , we have, for every  $i$ ,

$$d'_i \geq \frac{(d_i m - 2\tilde{\varepsilon}m)(m - 2\tilde{\varepsilon}m)}{m^2} \geq d_i - 4\tilde{\varepsilon} = d_i - \frac{1}{4}\varepsilon^5,$$

and,

$$d'_i \leq \frac{(d_i m + 2\tilde{\varepsilon}m)(m - 2\tilde{\varepsilon}m) + 2\tilde{\varepsilon}m \cdot m}{m^2} \leq d_i + 4\tilde{\varepsilon} = d_i + \frac{1}{4}\varepsilon^5.$$

With these, it is now easy to show that for every  $i$ ,

$$|\{x \in X_i : |\deg_{G_i}(x) - d'_i m| \geq \varepsilon^4 m\}| \leq |\{x \in X_i : |\deg_{G_i}(x) - d_i m| \geq 2\tilde{\varepsilon}m\}| \leq 2\tilde{\varepsilon}m \leq \frac{1}{8}\varepsilon^4 m.$$

Next, fix  $i$ , and let  $Y \subseteq X_i$  with  $|Y| \geq \varepsilon m$ . Let  $Y_1$  be those ordered pairs of  $Y$  which are in  $X''_i$ , and  $Y_2$  be those which are not. So,  $|Y_2| \leq 4\tilde{\varepsilon}m^2$ . We have

$$\begin{aligned} \sigma(Y) &= \frac{1}{|Y|^2} \left( \sum_{(y_1, y_2) \in Y_1} \sigma(y_1, y_2) + \sum_{(y_1, y_2) \in Y_2} \sigma(y_1, y_2) \right) \\ &\leq \frac{1}{|Y|^2} \left[ ((d_i + \tilde{\varepsilon})^2 m + \tilde{\varepsilon}m - (d_i - 4\tilde{\varepsilon})^2 m)(|Y|^2 - |Y_2|) + (m - (d_i - 4\tilde{\varepsilon})^2 m)|Y_2| \right] \\ &= \frac{1}{|Y|^2} \left[ ((d_i + \tilde{\varepsilon})^2 + \tilde{\varepsilon} - (d_i - 4\tilde{\varepsilon})^2)m|Y|^2 + (1 - (d_i + \tilde{\varepsilon}^2) - \tilde{\varepsilon})m|Y_2| \right] \\ &< 11\tilde{\varepsilon}m + \frac{4\tilde{\varepsilon}m}{\varepsilon^2} = \frac{11}{16}\varepsilon^5 m + \frac{1}{4}\varepsilon^3 m < \frac{1}{2}\varepsilon^3 m. \end{aligned}$$

Since  $2m^{-1/4} < \varepsilon < \frac{1}{16}$ , Lemma 10 implies that  $(X_i, U)$  is  $\varepsilon$ -regular. This is then true for every  $i$ , with probability at least  $1 - 2km^2 \exp(-\frac{3}{16^4}\varepsilon^{20}m)$ .

Finally, for a random partition  $X = U_1 \dot{\cup} \dots \dot{\cup} U_\ell$ , where  $|U_j| = m$  for every  $j$ , let the random variable  $N$  be the number of pairs  $(X_i, U_j)$  which are not both  $\varepsilon$ -regular and with  $d(X_i, U_j) \in (d_i - \frac{1}{4}\varepsilon^5, d_i + \frac{1}{4}\varepsilon^5)$ . Then,  $\mathbb{E}(N) \leq k\ell \cdot 2m^2 \exp(-\frac{3}{16^4}\varepsilon^{20}m)$ . Thus, by Markov's inequality, we have

$$\mathbb{P}\left(N \geq k\ell \cdot 2m^3 \exp\left(-\frac{3}{16^4}\varepsilon^{20}m\right)\right) \leq \frac{1}{m},$$

which implies the lemma.  $\square$

We are now ready to prove Theorem 9.

*Proof of Theorem 9.* Let such  $\varepsilon, r$  and  $t_0$  be given. Let  $G = (U \dot{\cup} V, E)$  be a bipartite graph with  $|U| = n$ ,  $|V| = n'$  and  $n' \geq n$ , and whose edges are  $r$ -coloured:  $E = E_1 \dot{\cup} \dots \dot{\cup} E_r$ . We consider two cases.

*Case 1* ( $n' \leq 33\varepsilon^{-5}n$ ). Let  $N'_0$  and  $T'_0$  be the integers obtained from Theorem 8, using  $\varepsilon' = \varepsilon^8$ . Choose  $N_0 = N'_0$  and  $T_0 \geq T'_0\varepsilon^{-7}$ . If  $n \geq N_0$ , then Theorem 8 applies for  $G$ . So, we have a partition  $U = U'_0 \dot{\cup} U'_1 \dot{\cup} \dots \dot{\cup} U'_s$  and  $V = V'_0 \dot{\cup} V'_1 \dot{\cup} \dots \dot{\cup} V'_s$ , for some  $t_0 \leq s \leq T'_0$ , so that

- $|U'_0| \leq \varepsilon^8 n$  and  $|V'_0| \leq \varepsilon^8 n'$ ,
- $|U'_i| = m$  and  $|V'_j| = m'$  for all  $1 \leq i, j \leq s$  and some  $m, m'$ , and
- in every  $G_k = (U \dot{\cup} V, E_k)$ ,  $k \in [r]$ , all but at most  $\varepsilon^8 s^2$  pairs of  $\{(U'_i, V'_j)\}_{i,j \geq 1}$  are  $\varepsilon^8$ -regular.

For each  $U'_i$ , divide it into subsets of size  $\lceil \varepsilon^7 m' \rceil$ , leaving a remaining set of size less than  $\lceil \varepsilon^7 m' \rceil$ . Unite these remaining sets with  $U'_0$  and let  $U_0$  be the union. Then,  $|U_0| \leq \varepsilon^8 n + \varepsilon^7 n \cdot \frac{m'}{m} \leq \varepsilon^8 n + \frac{33\varepsilon^2}{1-\varepsilon^8} n \leq \varepsilon n$  if  $\varepsilon < \frac{1}{34}$ . Repeat this with each  $V'_j$ , again dividing into sets of size  $\lceil \varepsilon^7 m' \rceil$ , and let  $V_0$  be the analogous union of  $V'_0$  with the remaining sets. Then,  $|V_0| \leq \varepsilon^8 n' + \varepsilon^7 n' \leq \varepsilon n'$ .

Now, let  $U_1, \dots, U_t \subset U$  and  $V_1, \dots, V_{t'} \subset V$  be the subsets of size  $\lceil \varepsilon^7 m' \rceil$  in the above partition. If  $U_p \subset U'_i$  and  $V_q \subset V'_j$ , and  $(U'_i, V'_j)$  is  $\varepsilon^8$ -regular in every colour, then applying Lemma 5 with  $\alpha = \varepsilon^7$  gives that  $(U_p, V_q)$  is  $\varepsilon$ -regular in every colour. So, there are at least

$$(1 - \varepsilon^8)s^2 \frac{m}{\lceil \varepsilon^7 m' \rceil} \cdot \frac{m'}{\lceil \varepsilon^7 m' \rceil} = (1 - \varepsilon^8)tt' \geq (1 - \varepsilon)tt'$$

such pairs  $(U_p, V_q)$ . Finally,  $t' = \frac{m'}{\lceil \varepsilon^7 m' \rceil} s \leq T_0$ , so  $t_0 \leq s \leq t \leq t' \leq T_0$ . This proves Case 1.

*Case 2* ( $n' > 33\varepsilon^{-5}n$ ,  $r < 4\varepsilon^{-5}$  and  $G = K_{n,n'}$ ). Let  $N'_0$  and  $T'_0$  be the integers obtained from Theorem 8, using  $\varepsilon' = \frac{1}{33}\varepsilon^5$ . Choose  $N_0 > \max\left(\frac{16\varepsilon^{-4}T'_0}{1-\varepsilon}, \frac{\theta T'_0}{1-\varepsilon}, N'_0\right)$  and  $T_0 = T'_0$ , where  $\theta = \theta(\varepsilon)$  is the largest solution to  $2\theta^3 \exp\left(-\frac{3}{16^4}\varepsilon^{20}\theta\right) = \frac{1}{2}\varepsilon$ . If  $n \geq N_0$ , then Theorem 8 applies for  $K_{n,n'}$ . So, we have a partition  $U = U_0 \dot{\cup} U_1 \dot{\cup} \dots \dot{\cup} U_t$  and  $V = W'_0 \dot{\cup} W'_1 \dot{\cup} \dots \dot{\cup} W'_t$ , where  $t_0 \leq t \leq T_0$ , such that

- $|U_0| \leq \frac{1}{33}\varepsilon^5 n$  and  $|W'_0| \leq \frac{1}{33}\varepsilon^5 n'$ ,
- $|U_i| = m$  and  $|W'_j| = m''$ , for every  $1 \leq i, j \leq t$  and some  $m, m''$ , and
- in every  $G_k = (U \dot{\cup} V, E_k)$ ,  $k \in [r]$ , all but at most  $\frac{1}{33}\varepsilon^5 t^2$  pairs of  $\{(U_i, W'_j)\}_{i,j \geq 1}$  are  $(\frac{1}{33}\varepsilon^5)$ -regular.

We have  $|U_0| \leq \varepsilon n$ . Now, let  $m'' = \ell m + \ell'$ , where  $\ell, \ell' \in \mathbb{Z}$  and  $0 \leq \ell' < m$ . For each  $W'_j$ , remove a set of size  $\ell'$  and unite these sets with  $W'_0$ , forming a new set  $V_0$ . Then,  $|V_0| \leq \frac{1}{33}\varepsilon^5 n' + \frac{1}{32}\varepsilon^5 (n' - |W'_0|) \leq \varepsilon n'$ . Also, one can show that  $\ell > 33\varepsilon^{-5} - 2$ , so that  $\ell' < \frac{m''}{\ell+1} < \frac{1}{32}\varepsilon^5 m''$ . Let  $W_j$  be the remaining set from  $W'_j$ . Now,  $\frac{|W_j|}{|W'_j|} = 1 - \frac{\ell'}{m''} > 1 - \frac{1}{32}\varepsilon^5$ . If  $(U_i, W'_j)$  is  $(\frac{1}{33}\varepsilon^5)$ -regular in every colour, then applying Lemma 5 with  $\alpha = 1 - \frac{1}{32}\varepsilon^5$  gives that  $(U_i, W_j)$  is  $(\frac{1}{16}\varepsilon^5)$ -regular in every colour.



Now, consider the subgraph  $H \subseteq K_{n,n'}$  as follows.  $V(H) = V(K_{n,n'}) \setminus (U_0 \cup V_0)$ . If  $(U_i, W_j)$  is  $(\frac{1}{16}\varepsilon^5)$ -regular in every colour, choose a colour whose density is at least  $\frac{1}{r}$ , and keep only these edges for  $H$ . Finally, disregard the colours. Now, in  $H$ , for fixed  $W_j$ , let the pairs  $(U_1^j, W_j), \dots, (U_{k_j}^j, W_j)$  be precisely the  $(\frac{1}{16}\varepsilon^5)$ -regular pairs, where  $U_1^j, \dots, U_{k_j}^j \in \{U_1, \dots, U_t\}$ , and with densities at least  $\frac{1}{r}$ , so that  $d(U_1^j, W_j), \dots, d(U_{k_j}^j, W_j) \in (\frac{1}{4}\varepsilon^5, 1)$ . By the choices of  $N_0$  and  $T_0$ , Lemma 11 applies to the pairs  $(U_1^j, W_j), \dots, (U_{k_j}^j, W_j)$ . So, there is a partition  $W_j = V_1^j \dot{\cup} \dots \dot{\cup} V_{\ell}^j$  into sets of size  $m$ , such that at least  $[1 - 2m^3 \exp(-\frac{3}{16^4}\varepsilon^{20}m)]k_j\ell$  of the pairs  $(U_p^j, V_q^j)$  are  $\varepsilon$ -regular, with densities greater than  $\frac{1}{r} - \frac{1}{4}\varepsilon^5$ .

Now, taking the union over  $W_j$  and noting that  $m > \theta$ , at least

$$\sum_{j=1}^t \left[1 - 2m^3 \exp\left(-\frac{3}{16^4}\varepsilon^{20}m\right)\right]k_j\ell \geq \left(1 - \frac{1}{2}\varepsilon\right)\left(1 - \frac{1}{33}\varepsilon^5\right)t^2\ell \geq (1 - \varepsilon)tt'$$

pairs  $(U_p, V_q)$  are  $\varepsilon$ -regular, with densities greater than  $\frac{1}{r} - \frac{1}{4}\varepsilon^5$ , where  $\{V_1, \dots, V_{t'}\}$  are all the  $V_{t'}^j$ . Case 2 follows after we reinstate the colours of the edges from  $H$ .

This completes the proof of Theorem 9. □

### 3. PROOFS OF THEOREMS 1, 2 AND 3

To obtain Theorems 1 and 2, we will generalise the following results.

**Theorem 12** (Theorem 11 of [7]). *Let  $n, r \in \mathbb{N}$ , with  $r \geq 2$ . Then*

$$m(n, r, 1, 1) \geq \frac{n}{r-1}.$$

**Theorem 13** (Theorem 16 of [8]). *Let  $n, r \in \mathbb{N}$ , with  $r \geq 3$ . Then*

$$m(n, r, 2, 1) \geq \frac{4n}{r+1}.$$

Our main goal will be to “relax” the term  $m(n, r, s, 1)$  to  $m(G, r, s, 1)$ , for  $s = 1, 2$ , and  $r \geq 3$  in both cases. Here,  $G$  will be an almost complete graph - it should miss at most  $\gamma n^2$  edges, where  $n$  is the order of  $G$  and  $\gamma > 0$  is small.

A key lemma of [7] that we will need is the following.

**Lemma 14** (Lemma 9 of [7]). *Let  $m, n \in \mathbb{N}$  and  $c \in [0, 1]$ . If  $G$  is a bipartite graph with part-sizes  $m$  and  $n$ , and  $e(G) \geq cmn$ , then  $G$  has a component of order at least  $c(m+n)$ .*

We are now ready to prove the new versions of Theorems 12 and 13.

**Theorem 15.** *Let  $\gamma \in (0, 1)$  and  $n, r \in \mathbb{N}$ ,  $r \geq 3$ . Let  $G$  be a graph on  $n$  vertices with  $e(G) \geq \binom{n}{2} - \gamma n^2$ . Then  $m(G, r, 1, 1) \geq \frac{n}{r-1} - \frac{9\gamma n}{2(r-1)}$ .*

*Proof.* Fix an  $r$ -colouring of  $E(G)$ . We first construct a bipartition  $V(G) = V_1 \dot{\cup} V_2$  such that  $|V_1|, |V_2| \geq \frac{n}{3}$  and  $E(V_1, V_2)$  has no edges of colour 1. Consider all the components of  $G$  in colour 1 (including isolated vertices, if any). Let these have vertex sets  $C_1, \dots, C_p$  with  $|C_1| \geq |C_2| \geq \dots \geq |C_p|$ , and  $p \geq 2$  (we are clearly done if  $p = 1$ ). If  $|C_1| \geq \frac{n}{r-1}$ , then the theorem holds, so assume that  $|C_1| < \frac{n}{r-1} \leq \frac{n}{2}$ . Now, take  $V_1 = \bigcup_{i=1}^t C_i$ ,  $V_2 = \bigcup_{i=t+1}^p C_i$ , where  $t \in \{1, \dots, p-2\}$  is the unique integer such that  $\sum_{i=1}^{t-1} |C_i| < \frac{n}{3}$  and  $\sum_{i=1}^t |C_i| \geq \frac{n}{3}$ . One can then easily check that  $|V_1|, |V_2| \geq \frac{n}{3}$ .

Now,  $e(V_1, V_2) \geq |V_1||V_2| - \gamma n^2$ , and there exist at least  $\frac{e(V_1, V_2)}{r-1}$  edges in  $E(V_1, V_2)$  of the same colour. But then Lemma 14 asserts the existence of a monochromatic component of order at least

$$\frac{|V_1||V_2| - \gamma n^2}{(r-1)|V_1||V_2|} \cdot n \geq \left( \frac{1}{r-1} - \frac{9\gamma}{2(r-1)} \right) n.$$

□

We remark that Theorem 15 breaks down for  $r = 2$ , as the following example of a 2-colouring on a graph  $G$  on  $n$  vertices, and missing  $\gamma n^2$  edges, shows. Take three disjoint vertex sets  $V_1, V_2, V_3$  for  $V(G)$ , where  $|V_1| = |V_2| = \sqrt{\gamma}n$  and  $|V_3| = (1 - 2\sqrt{\gamma})n$ . Colour the edges within  $V_1$  and between  $V_1$  and  $V_3$  blue, those within  $V_2$  and between  $V_2$  and  $V_3$  red, those within  $V_3$  arbitrarily, while between  $V_1$  and  $V_2$  are non-edges of  $G$ . Then, it is easy to see that this gives  $m(G, 2, 1, 1) \leq (1 - \sqrt{\gamma})n$ , showing that Theorem 15 does not hold even if we replace the constant  $\frac{9}{2}$  by any other constant.

A generalisation of Theorem 13 we will need is as follows. Its proof is similar to the proof of Theorem 13. It is, however, more technically involved, and uses some additional ideas.

**Theorem 16.** *For every  $\gamma \in (0, 1)$  and  $n, r \in \mathbb{N}$ ,  $r \geq 3$ , there exists a  $\delta = \delta(\gamma, r)$  with  $0 < \delta < \gamma$  such that, if  $G$  is a graph on  $n$  vertices with  $e(G) \geq \binom{n}{2} - \delta n^2$ , then  $m(G, r, 2, 1) \geq \frac{4n}{r+1} - \frac{4\gamma n}{r+1}$ .*

*Proof.* Let  $\gamma \in (0, 1)$  and integer  $r \geq 3$  be given. We will show how to choose  $0 < \delta = \delta(\gamma, r) < \gamma$  to satisfy the theorem as we proceed through the proof.

Fix an  $r$ -colouring of  $E(G)$  and let  $H$  be a largest connected monochromatic subgraph of  $G$ . Assume that  $H$  is of colour 1. We set  $A = V(H)$  and  $|A| = \frac{cn(1-\gamma)}{r+1}$ . If  $c \geq 4$  then the theorem holds. So assume now that  $c < 4$ .

Let  $B = V(G) \setminus A$ . Then, no edge in  $G[A, B]$  has colour 1. Thus, there exists a colour, say 2, which occurs at least  $\frac{|A||B| - \delta n^2}{r-1}$  times in  $G[A, B]$ . Let  $B_2 \subset B$  be the vertices of  $B$  which send an edge of colour 2 to  $A$ . If  $|B_2| \geq \frac{(4-c)(1-\gamma)n}{r+1}$ , then the set  $A \cup B_2$  is connected by colours 1 and 2, and  $|A \cup B_2| \geq \frac{4(1-\gamma)n}{r+1}$ , so we are done. Thus, we may assume that  $|B_2| < \frac{(4-c)(1-\gamma)n}{r+1}$ , and we can choose a set  $B'_2$  such that  $B_2 \subset B'_2 \subset B$  and  $|B'_2| = \frac{(4-c)(1-\gamma)n}{r+1}$ .

Now consider the bipartite subgraph  $H_2$  of  $G[A, B'_2]$  whose edges are of colour 2. A straightforward but tedious calculation yields

$$e(H_2) \geq \frac{|A||B| - \delta n^2}{r-1} = \left( \frac{r+1-c(1-\gamma)}{1-\gamma} - \frac{(r+1)^2\delta}{c(1-\gamma)^2} \right) \frac{1}{(4-c)(r-1)} |A||B'_2|. \quad (1)$$

Provided that  $\delta \leq \frac{1}{9}$ , Theorem 15 gives  $\frac{cn(1-\gamma)}{r+1} \geq \frac{n}{r-1} - \frac{9\delta n}{2(r-1)} \geq \frac{n}{2(r-1)}$ . Then, if  $\delta \leq \frac{\gamma}{2r-2}$  as well, (1) gives

$$e(H_2) \geq \frac{r+1-c}{(4-c)(r-1)} |A||B'_2|.$$

Now, applying Lemma 14 on  $H_2$ , we have a connected monochromatic subgraph on at least

$$\frac{r+1-c}{(4-c)(r-1)} (|A| + |B'_2|) = \frac{r+1-c}{(4-c)(r-1)} \cdot \frac{4(1-\gamma)n}{r+1}$$

vertices. Because  $H$  was chosen to be a largest monochromatic subgraph in  $G$ , we obtain

$$\frac{r+1-c}{(4-c)(r-1)} \cdot \frac{4(1-\gamma)n}{r+1} \leq \frac{c(1-\gamma)n}{r+1},$$

and if one solves this inequality for  $c$  (see the proof of Theorem 13 in [8]), again a straightforward calculation yields

$$2 \leq c \leq \frac{2(r+1)}{r-1}.$$

We will only need  $c \geq 2$  for later.

Next, we aim to show that a large number of vertices of  $B$  send edges of at least two different colours to  $A$ . Suppose that we have  $\beta n$  vertices of  $B$  which send edges of exactly one colour to  $A$ . Then, at least  $\frac{\beta n}{r-1}$  vertices of  $B$  send edges of one particular colour to  $A$ . Let  $B_3$  be such a set of vertices, with  $|B_3| \geq \frac{\beta n}{r-1}$ . Then, again by Lemma 14, we have a monochromatic component of order at least

$$\frac{|A||B_3| - \delta n^2}{|A||B_3|} (|A| + |B_3|) \geq \left( 1 + \frac{\beta(r+1)}{(r-1)c(1-\gamma)} - \frac{\delta(r^2-1)}{\beta c(1-\gamma)} - \frac{\delta(r+1)^2}{c^2(1-\gamma)^2} \right) |A|.$$

We will have a contradiction against the maximality of  $H$  if

$$\frac{\beta(r+1)}{(r-1)c(1-\gamma)} - \frac{\delta(r^2-1)}{\beta c(1-\gamma)} - \frac{\delta(r+1)^2}{c^2(1-\gamma)^2} > 0,$$

and solving this as a quadratic in  $\beta$ , we get  $\beta > \frac{\delta(r^2-1)}{2c(1-\gamma)} + \frac{r-1}{2} \sqrt{\frac{\delta^2(r+1)^2}{c^2(1-\gamma)^2} + 4\delta}$ . So, we have

$$\beta \leq \frac{\delta(r^2-1)}{2c(1-\gamma)} + \frac{r-1}{2} \sqrt{\frac{\delta^2(r+1)^2}{c^2(1-\gamma)^2} + 4\delta}. \quad (2)$$

Also, there are at most

$$\frac{\delta n^2}{|A|} = \frac{\delta(r+1)n}{c(1-\gamma)} \quad (3)$$

vertices in  $B$  which have no neighbours in  $A$ . So, from (2) and (3), the number of vertices in  $B$  sending edges of at least two different colours to  $A$  is at least

$$|B| - \frac{\delta n^2}{|A|} - \beta n \geq \left( 1 - \frac{c(1-\gamma)}{r+1} - \frac{\delta(r+1)}{c(1-\gamma)} - \frac{\delta(r^2-1)}{2c(1-\gamma)} - \frac{r-1}{2} \sqrt{\frac{\delta^2(r+1)^2}{c^2(1-\gamma)^2} + 4\delta} \right) n. \quad (4)$$

We would like the quantity on the right of (4) to be at least  $\frac{(r+1-c)(1-\gamma)n}{r+1}$ . This is true if and only if (after some calculation)

$$\gamma \geq \frac{\delta(r+1)^2}{2c(1-\gamma)} + \frac{r-1}{2} \sqrt{\frac{\delta^2(r+1)^2}{c^2(1-\gamma)^2} + 4\delta}.$$

This holds if both

$$\frac{\delta(r+1)^2}{2c(1-\gamma)} \leq \frac{\gamma}{3} \quad \text{and} \quad \frac{r-1}{2} \sqrt{\frac{\delta^2(r+1)^2}{c^2(1-\gamma)^2} + 4\delta} \leq \frac{\gamma}{3}$$

are satisfied. So sufficiently (recalling that  $c \geq 2$ ), we can let

$$\delta \leq \frac{4\gamma(1-\gamma)}{3(r+1)^2} \quad \text{and} \quad \delta \leq \frac{16\gamma^2(1-\gamma)^2}{9(r-1)^2((r+1)^2 + 16(1-\gamma)^2)}. \quad (5)$$

It follows that, taking  $\delta$  which satisfies (5) (as well as  $\delta \leq \frac{1}{9}$  and  $\delta \leq \frac{\gamma}{2r-2}$ ) and using the pigeonhole principle,  $B$  contains at least

$$\frac{2}{r-1} \cdot \frac{(r+1-c)(1-\gamma)n}{r+1}$$

vertices, each sending an edge of some colour, say  $j$ , to  $A$ . Let  $D \subset B$  be these vertices. Now, recalling that  $c \geq 2$  and  $r \geq 3$ , we have a connected subgraph using colours 1 and  $j$  on at least

$$|A \dot{\cup} D| \geq \frac{c(1-\gamma)n}{r+1} + \frac{2(r+1-c)(1-\gamma)n}{(r-1)(r+1)} \geq \frac{4(1-\gamma)n}{r+1}$$

vertices.

So altogether, the theorem holds if we initially chose  $\delta = \delta(\gamma, r)$  such that

$$0 < \delta \leq \min \left\{ \frac{1}{9}, \frac{\gamma}{2r-2}, \frac{4\gamma(1-\gamma)}{3(r+1)^2}, \frac{16\gamma^2(1-\gamma)^2}{9(r-1)^2((r+1)^2 + 16(1-\gamma)^2)} \right\}.$$

□

Likewise, we have the following analogous statement for bipartite graphs, which is an easy corollary from Lemma 14.

**Lemma 17.** *Let  $\gamma \in (0, 1)$ , and  $m, n, r \in \mathbb{N}$ . Let  $G$  be a bipartite graph with part-sizes  $m$  and  $n$ , and  $e(G) \geq mn - \gamma mn$ . Then, if we have an  $r$ -colouring of  $E(G)$ , there is a monochromatic connected subgraph on at least  $\frac{m+n}{r} - \frac{\gamma(m+n)}{r}$  vertices.*

Now, Theorems 15 and 16, together with Theorem 6 and Lemma 7, imply immediately Theorems 1 and 2.

*Proof of Theorem 1.* Given  $\gamma \in (0, \frac{1}{4})$  and integer  $r \geq 3$ , let  $\varepsilon = \frac{\gamma}{r}$  and  $t_0 = \lceil \frac{1}{\varepsilon} \rceil$ . Obtain  $N_0 = N_0(\varepsilon, r, t_0) = N_0(\gamma, r)$  and  $T_0 = T_0(\varepsilon, r, t_0) = T_0(\gamma, r)$  from Theorem 6. Now, given  $K_n$  with  $n \geq N_0$ , and an  $r$ -colouring  $f : E(K_n) \rightarrow [r]$ , let  $G_i$  be the graph on  $V(K_n)$  with the edges of colour  $i$ , for  $i \in [r]$ . Then, there exists a partition  $V(K_n) = V_0 \dot{\cup} V_1 \dot{\cup} \dots \dot{\cup} V_t$ , for some  $\frac{1}{\varepsilon} \leq t \leq T_0$ , which is simultaneously  $\varepsilon$ -regular with respect to every  $G_i$ . We have  $|V_0| \leq \varepsilon n$ , and  $|V_i| = |V_j|$  for all  $i, j \geq 1$ . Now, for each  $G_i$ , we obtain the cluster graph  $R_i(\frac{1}{r})$  of the graph  $G_i - V_0$ , so that  $|V(R_i(\frac{1}{r}))| = t$ . Let the graph  $R(\frac{1}{r}) = \bigcup_{i=1}^r R_i(\frac{1}{r})$ , and let  $g : E(R(\frac{1}{r})) \rightarrow [r]$  be an  $r$ -colouring satisfying  $g(uv) \in \{i \in [r] : uv \in E(R_i(\frac{1}{r}))\}$ , for every  $uv \in E(R(\frac{1}{r}))$ . Now, note that  $e(R(\frac{1}{r})) \geq \binom{t}{2} - \gamma t^2$ , since there are at least  $\binom{t}{2} - r\varepsilon t^2 = \binom{t}{2} - \gamma t^2$  pairs  $(V_p, V_q)$  which are  $\varepsilon$ -regular in every  $G_i$ , and for such a pair  $(V_p, V_q)$ , we have  $d(V_p, V_q) \geq \frac{1}{r}$  in some  $G_i$ . So, Theorem 15 implies that we can find a monochromatic connected subgraph of  $R(\frac{1}{r})$  (in the colouring  $g$ ), say of colour  $\ell$ , on at least  $\frac{t}{r-1} - \frac{9\gamma t}{2(r-1)}$  vertices. This is a connected subgraph of  $R_\ell(\frac{1}{r})$ , so Lemma 7 implies that  $G_\ell$  has a  $[(\frac{1}{r} - \frac{3\gamma}{r})\frac{n-|V_0|}{t}]$ -connected subgraph of order at least

$$\left(1 - \frac{\gamma}{r}\right) \left(\frac{t}{r-1} - \frac{9\gamma t}{2(r-1)}\right) \left(\frac{n-|V_0|}{t}\right) \geq \frac{n}{r-1} - \frac{6\gamma n}{r-1}.$$

The first part follows, since  $(\frac{1}{r} - \frac{3\gamma}{r})\frac{(n-|V_0|)}{t} \geq \frac{1-4\gamma}{rT_0}n$ .

To see the second part, simply let  $\gamma \rightarrow 0$  and  $n \rightarrow \infty$ . Then,  $\frac{1-4\gamma}{rT_0}n = o(n)$  and  $\frac{6\gamma n}{r-1} = o(n)$ . The “equality” part comes from the fact (Lemma 8 of [7]) that  $m(n, r, 1, k) \leq \frac{n-k+1}{r-1} + r$  if  $r-1$  is a prime power. □

*Proof of Theorem 2.* This is essentially the same as the previous proof. We will use Theorem 16 instead of Theorem 15. Given  $\gamma \in (0, \frac{1}{4})$  and integer  $r \geq 3$ , we first obtain  $\delta = \delta(\gamma, r)$  as given by Theorem 16, and then set  $\varepsilon = \frac{\delta}{r}$  and  $t_0 = \lceil \frac{1}{\varepsilon} \rceil$ . Now, apply Theorems 6, 16, and Lemma 7 as before. We obtain the corresponding  $N_0$  and  $T_0$ , and then taking an  $r$ -colouring of  $E(K_n)$ , where  $n \geq N_0$ , we obtain the corresponding  $t$ . This time, we get a 2-coloured,  $(\frac{1-4\delta}{rT_0}n)$ -connected subgraph of  $K_n$  on at least

$$\left(1 - \frac{\delta}{r}\right) \left(\frac{4t}{r+1} - \frac{4\delta t}{r+1}\right) \left(\frac{n-\varepsilon n}{t}\right) \geq \frac{4n}{r+1} - \frac{8\delta n}{r+1} \geq \frac{4n}{r+1} - \frac{8\gamma n}{r+1}$$

vertices. We are done, since this subgraph is also  $(\frac{1-4\gamma}{rT_0}n)$ -connected.

The second part is now trivial, again by letting  $\gamma \rightarrow 0$  and  $n \rightarrow \infty$ . The “equality” part comes from the fact (Lemma 13 of [8]) that  $m(n, r, 2, k) \leq \frac{4n}{r+1} + 4$  if  $r+1$  is a power of 2. □

A similar idea would enable us to prove Theorem 3: we shall use Theorem 9, and Lemmas 17 and 7.

*Proof of Theorem 3.* We first prove the lower bound. Let  $\gamma \in (0, \frac{1}{34})$  and  $n, n', r \in \mathbb{N}$ , where  $r \geq 2$  and  $n' \geq n$ . Let  $\varepsilon = \frac{\gamma}{r}$  and  $t_0 = \lceil \frac{1}{\varepsilon} \rceil$ . Obtain  $N_0$  and  $T_0$  from Theorem 9. Now, given  $K_{n, n'} = (U \dot{\cup} V, E)$  with  $|U| = n \geq N_0$  and  $|V| = n'$ , and an  $r$ -colouring of it, we have a partition  $U = U_0 \dot{\cup} U_1 \dot{\cup} \dots \dot{\cup} U_t$  and  $V = V_0 \dot{\cup} V_1 \dot{\cup} \dots \dot{\cup} V_{t'}$  such that  $|U_0| \leq \varepsilon n$ ,  $|V_0| \leq \varepsilon n'$ ,  $|U_i| = |V_j| = m$  for all  $i, j \geq 1$  and some  $m$ , and  $\frac{1}{\varepsilon} \leq t \leq T_0$ . Now, we consider two cases, defining our cluster graphs slightly differently in each case.

*Case 1* ( $n' \leq 33\varepsilon^{-5}n$ ). In this case, at least  $(1 - \varepsilon)tt'$  pairs of  $\{(U_i, V_j)\}_{i, j \geq 1}$  are  $\varepsilon$ -regular in every colour. By considering cluster graphs with  $\eta = \frac{1}{r}$  of the coloured subgraphs on  $K_{n, n'} - (U_0 \cup V_0)$  as before, it follows that, by Lemmas 17 and 7, we have a  $(\frac{1-4\gamma}{rT_0}n)$ -connected, monochromatic subgraph on at least

$$\left(1 - \frac{\gamma}{r}\right) \left(\frac{t+t'}{r} - \frac{\gamma(t+t')}{r}\right) \left(\frac{n+n' - \varepsilon n - \varepsilon n'}{t+t'}\right) \geq \frac{n+n'}{r} - \frac{3\gamma(n+n')}{r}$$

vertices, since  $m = \frac{n-|U_0|}{t} = \frac{n'-|V_0|}{t'}$ , so that  $m = \frac{n+n'-|U_0|-|V_0|}{t+t'} \geq \frac{n+n'-\varepsilon n-\varepsilon n'}{t+t'}$ .

*Case 2* ( $n' > 33\varepsilon^{-5}n$ ). In this case, at least  $(1 - \varepsilon)tt'$  pairs of  $\{(U_i, V_j)\}_{i, j \geq 1}$  are  $(\varepsilon, \frac{1}{r} - \frac{1}{4}\varepsilon^5)$ -regular, each one in some colour. For each  $i \in [r]$ , let  $G_i$  be the subgraph of  $K_{n, n'}$ , with the edges of colour  $i$ . Let  $R_i(\frac{1}{r} - \frac{1}{4}\varepsilon^5)$  be the cluster graph derived from  $G_i - (U_0 \cup V_0)$ , and  $R(\frac{1}{r} - \frac{1}{4}\varepsilon^5) = \bigcup_{i=1}^r R_i(\frac{1}{r} - \frac{1}{4}\varepsilon^5)$ . Then,  $e(R(\frac{1}{r} - \frac{1}{4}\varepsilon^5)) \geq (1 - \varepsilon)tt' \geq (1 - \gamma)tt'$ . As before, Lemmas 17 and 7 imply that some  $G_\ell$  has a  $[(\frac{1}{r} - \frac{1}{4}\varepsilon^5 - 3\varepsilon)m]$ -connected subgraph on at least  $\frac{n+n'}{r} - \frac{3\gamma(n+n')}{r}$  vertices, by the same calculation as in Case 1. Case 2 follows, since  $(\frac{1}{r} - \frac{1}{4}\varepsilon^5 - 3\varepsilon)m \geq \frac{1-4\gamma}{rT_0}n$  if  $r \geq 2$ .

Now, we show the upper bound. For  $k \in \mathbb{N}$ , we shall describe an  $r$ -colouring of  $E(K_{n, n'})$ , where  $n' \geq n \geq rk$ , such that the largest monochromatic  $k$ -connected subgraph has order at most  $\frac{n+n'}{r} + 2$ . So, take such a  $K_{n, n'} = (U \dot{\cup} V, E)$ , where  $|U| = n$  and  $|V| = n'$ . Partition each of  $U$  and  $V$  into  $r$  parts:  $U = U_1 \dot{\cup} \dots \dot{\cup} U_r$ , and  $V = V_1 \dot{\cup} \dots \dot{\cup} V_r$ , each as equally as possible. Note that  $|U_i|, |V_j| \geq k$  for every  $i, j$ . Now, colour the edges of  $E(U_i, V_j)$  with colour  $i - j \pmod{r}$ . Then, the largest monochromatic  $k$ -connected subgraph has order  $\lceil \frac{n}{r} \rceil + \lceil \frac{n'}{r} \rceil \leq \frac{n+n'}{r} + 2$ .

The final part is now trivial, again by letting  $\gamma \rightarrow 0$  and  $n \rightarrow \infty$ , and using a similar argument as before.  $\square$

#### 4. OPEN PROBLEMS

Having now improved the lower bounds for  $m(n, r, 1, k)$  and  $m(n, r, 2, k)$  for large  $n$ , an obvious question to ask would be what happens for  $s \geq 3$ . Our main problem is that we do not have analogous results to Theorems 15 and 16 for  $s \geq 3$ . Note that (for some special values of  $r$ ) Theorem 15 says that  $m(G, r, 1, 1)$  is ‘‘close’’ to  $m(n, r, 1, 1)$  if  $G$  is ‘‘close’’ to  $K_n$ , and Theorem 16 says a similar thing for  $s = 2$ . So, we pose the following conjecture, which asks for an analogous statement for  $s \geq 3$ .

**Conjecture 18.** *Let  $\gamma \in (0, 1)$ , and  $n, r, s \in \mathbb{N}$  with  $r \geq s \geq 3$ . Then there exist  $\delta = \delta(\gamma, r, s)$  with  $0 < \delta < \gamma$  such that the following holds. If  $G$  is a graph on  $n$  vertices with  $e(G) \geq \binom{n}{2} - \delta n^2$ , then  $m(G, r, s, 1) \geq (1 - \gamma)m(n, r, s, 1)$ .*

If some form of Conjecture 18 holds, then we can conceivably apply the same techniques, and get  $m(n, r, s, k) \geq m(n, r, s, 1) - o(n)$  for  $r, s$  fixed,  $r \geq s \geq 3$ , and  $k = o(n)$ . Thus, we may only have to worry about 1-connectedness if we were to tackle Bollobás’ question in this direction.

Another problem is that we have not said much about  $m_{\text{bip}}(n, n', r, s, k)$  for  $s \geq 2$ . It is easy to adapt the proof of Theorem 3 and get  $m_{\text{bip}}(n, n', r, s, k) \geq \frac{s(n+n')}{r} - o(n)$ . But this could be weak except for  $s = 2$ , where we may slightly adjust the upper bound construction in the proof of Theorem 3, and get that  $m_{\text{bip}}(n, n', r, 2, k) \leq \frac{2(n+n')}{r} + 4$  if  $r$  is a power of 2 (this is similar to the construction given in Lemma 13 of [8]). So, we finish by stating the problem itself.

**Problem 19.** Determine  $m_{\text{bip}}(n, n', r, s, k)$  for  $s \geq 2$ .

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## REFERENCES

- [1] N. Alon, R. A. Duke, H. Lefmann, V. Rödl and R. Yuster, The algorithmic aspects of the regularity lemma, *J. Algorithms*, **16(1)** (1994), 80-109.
- [2] B. Bollobás, “Modern Graph Theory”, Springer-Verlag, New York, 1998, xiii+394pp.
- [3] B. Bollobás and A. Gyárfás, Highly connected monochromatic subgraphs, *Discrete Math.*, **308(9)** (2008), 1722-1725.
- [4] S. Janson, T. Łuczak and A. Ruciński, “Random graphs”, Wiley-Interscience Series in Discrete Mathematics and Optimization, New York, 2000, xii+333pp.
- [5] J. Komlós and M. Simonovits, Szemerédi’s regularity lemma and its applications in graph theory, Paul Erdős is eighty, Proceedings of Colloquia of the Bolyai Mathematical Society 2 (Keszthely, 1993), 295-352.
- [6] H. Liu, R. Morris and N. Prince, Highly connected monochromatic subgraphs: addendum, manuscript.
- [7] H. Liu, R. Morris and N. Prince, Highly connected monochromatic subgraphs of multicoloured graphs, to appear in *J. Graph Theory*.
- [8] H. Liu, R. Morris and N. Prince, Highly connected multicoloured subgraphs of multicoloured graphs, *Discrete Math.*, **308(22)** (2008), 5096-5121.
- [9] W. Mader, Existenz  $n$ -fach zusammenhängender Teilgraphen in Graphen genügend grosser Kantendichte, *Abh. Math. Sem. Univ. Hamburg*, **37** (1972), 86-97.
- [10] E. Szemerédi, On sets of integers containing no  $k$  elements in arithmetic progression, *Acta Arithmetica* **27**(1975), 199-245.
- [11] E. Szemerédi, Regular partitions of graphs, Problèmes combinatoires et théorie des graphes, *Proceedings du Colloque International CNRS*, **260**, CNRS, Paris, (1978), 399-401.
- [12] T. Tao, Szemerédi’s regularity lemma revisited, *Contrib. Discrete Math.*, **1** (2006), 8-28.

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