On Extremal Hypergraphs for Hamiltonian Cycles

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Abstract

We study sufficient conditions for Hamiltonian cycles in hypergraphs and obtain both Turán- and Dirac-type results. While the Turán-type result gives an exact threshold for the appearance of a Hamiltonian cycle in a hypergraph depending only on the extremal number of a certain path, the Dirac-type result yields just a sufficient condition relying solely on the minimum vertex degree.

Keywords: Extremal numbers for Hamiltonian cycles, Dirac-type conditions, Hamiltonian hypergraphs

1 Introduction and Results

1.1 Turán-type Results

For a fixed graph G we say that the *extremal number* $\mathrm{ex}(n, G)$ of G is the largest integer m such that there exists a graph on n vertices with m edges that does not contain a subgraph isomorphic to G. The corresponding graphs are called *extremal graphs*. Naturally, one can extend this definition to a forbidden spanning structure, e.g. a *Hamiltonian cycle* (a cycle of length n). In [5] Ore proved that a non-Hamiltonian graph on n vertices has at most $\binom{n-1}{2} + 1$ edges, and further, that the unique extremal example is given by an

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$(n-1)$-clique and a vertex of degree one that is adjacent to one vertex of the clique.

A *k-uniform hypergraph* H, or k-graph for short, is a pair (V, E) with a vertex set $V = V(H)$ and an edge set $E = E(H) \subseteq \binom{V}{k}$.

There are several definitions of Hamiltonian cycles in hypergraphs, e.g. Berge Hamiltonian cycles [1]. This note yet follows the definition of Hamiltonian cycles established by Katona and Kierstead [4].

An *l-tight Hamiltonian cycle* in H, $0 \leq l \leq k-1$, $(k-l)|n$, is a spanning sub-k-graph whose vertices can be cyclically ordered in such a way that the edges are segments of that ordering and every two consecutive edges intersect in exactly l vertices. We denote an l-tight Hamiltonian cycle in a k-graph H on n vertices by $C_n^{(k,l)}$, and call the cycle *tight* if it is $(k-1)$-tight.

Working on her thesis [9] in coding theory, Woitas raised the question whether removing $\binom{n-2}{2} - 1$ edges from a complete 3-uniform hypergraph on n vertices leaves a hypergraph containing a 1-tight Hamiltonian cycle. A generalization of this problem is to estimate the extremal numbers of Hamiltonian cycles in k-graphs.

Katona and Kierstead were the first to study sufficient conditions for the appearance of a $C_n^{(k,k-1)}$ in k-graphs. In [4] they showed that for all integers k and n with $k \geq 2$ and $2k-1 \leq n$,

$$\mathrm{ex}\left(n, C_n^{(k,k-1)}\right) \geq \binom{n-1}{k} + \binom{n-2}{k-2}.$$

In [8] Tuza proved the bound

$$\mathrm{ex}\left(n, C_n^{(k,k-1)}\right) \geq \binom{n-1}{k} + p\binom{n-1}{k-2}$$

for all k, n and p such that a partial Steiner system $PS(k-2, 2k-3, n-1)$ of order $n-1$ with $p\binom{n-1}{k-2}/\binom{2k-3}{k-2}$ blocks exists, generalizing the result from [4].

An intuitive approach to forbid Hamiltonian cycles in hypergraphs is to prohibit certain structures in the *link* of one fixed vertex. For a vertex $v \in V$, we define the *link of v in H* to be the $(k-1)$-graph $H(v) = (V \backslash \{v\}, E_v)$ with $\{x_1, \ldots, x_{k-1}\} \in E_v$ iff $\{v, x_1, \ldots, x_{k-1}\} \in E(H)$.

The structure of interest in our case is a generalization of a path for hypergraphs. An *l-tight k-uniform t-path*, denoted by $P_t^{(k,l)}$, is a k-graph on t vertices, $(k-l) \mid (t-l)$, such that there exists an ordering of the vertices, in such a way that the edges are segments of that ordering and every two consecutive edges intersect in exactly l vertices. Observe that a $P_t^{(k,l)}$ has $\frac{t-l}{k-l}$

edges. A k-uniform $(k-1)$-tight path is called *tight*.

For arbitrary k and l we give the exact extremal number and the extremal graphs of l-tight Hamiltonian cycles in this note. The extremal number and the extremal graphs rely on the extremal number of $P(k,l) := P^{(k-1,l-1)}_{\lfloor \frac{k}{k-l} \rfloor (k-l)+l-1}$ and its extremal graphs, respectively.

Theorem 1.1 *For any $k \geq 2$, $l \in \{0, \ldots, k-1\}$ there exists an n_0 such that for any $n \geq n_0$ and $(k-l)|n$,*

$$\mathrm{ex}\left(n, C_n^{(k,l)}\right) = \binom{n-1}{k} + \mathrm{ex}\left(n-1, P(k,l)\right)$$

holds. Furthermore, any extremal k-graph on n vertices contains an $(n-1)$-clique and a vertex whose link forms a $P(k,l)$-extremal $(k-1)$-graph.

For example, in the case $k = 3$ and $l = 1$, the theorem above answers the question of Woitas and shows that any $C_n^{(3,1)}$-extremal hypergraph on n vertices consists of a clique $K_{n-1}^{(3)}$ and an isolated vertex. Moreover, in the case $k = 3$ and $l = 2$, the $C_n^{(3,2)}$-extremal hypergraphs consists of a clique $K_{n-1}^{(3)}$ and a vertex whose link graph is extremal for path of length 3. Note that this gives the extremal number for $C_n^{(3,2)}$ to be $\binom{n-1}{3} + n - 1$ if $3|(n-1)$, and $\binom{n-1}{3} + n - 2$ otherwise.

1.2 Dirac-type Results

The problem of finding Hamiltonian cycles and perfect matchings in graphs has been studied very intensively. There are plenty beautiful conditions guaranteeing the existence of such cycles, e.g. Dirac's condition [2].

Over the last couple of years several Dirac-type results in hypergraphs were shown, and along with them, different definitions of *degree* in a k-graph were introduced. They all can be captured by the following definition. The *degree* of $\{x_1, \ldots, x_i\}$, $1 \leq i \leq k-1$, in a k-graph H is the number of edges the set is contained in and is denoted by $\deg(x_1, \ldots, x_i)$. Let

$$\delta_d(H) := \min\{\deg(x_1, \ldots, x_d) | \{x_1, \ldots, x_d\} \subset V(H)\}$$

for $0 \leq d \leq k-1$. If the graph is clear from the context, we omit H and write for short δ_d. Note that $\delta_0 = e(H) := |E(H)|$ and δ_1 is the minimum vertex degree in H.

Following the definitions of Rödl and Ruciński in [6], denote for every d, k, l and n with $0 \leq d \leq k-1$ and $(k-l)|n$ the number $h_d^l(k,n)$ to be the smallest integer h such that every n-vertex k-graph H satisfying $\delta_d(H) \geq h$ contains an l-tight Hamiltonian cycle. Observe that $h_0^l(k,n) = \mathrm{ex}\left(n, C_n^{(k,l)}\right) + 1$.

For further information, an excellent survey of the recent results can be found in [6].

Noting the fact that there are virtually no results on $h_d^l(k,n)$ for $d \leq k-2$, Rödl and Ruciński remarked in [6] that it does not even seem completely trivial to show $h_1^2(3,n) \leq c\binom{n-1}{2}$ for some constant $c < 1$. Further, they mentioned the bound $h_1^2(3,n) \leq \left(\frac{11}{12} + o(1)\right)\binom{n-1}{2}$.

We show the following general upper bound on $h_1^{k-1}(k,n)$.

Theorem 1.2 *For any $k \in \mathbb{N}$ there exists an n_0 such that every k-graph H on $n \geq n_0$ vertices with $\delta_1 \geq \left(1 - \frac{1}{22(1280k^3)^{k-1}}\right)\binom{n-1}{k-1}$ contains a tight Hamiltonian cycle.*

Note that Theorem 1.2 implies $h_d^l(k,n) \leq \left(1 - \frac{1}{22(1280k^3)^{k-1}}\right)\binom{n-d}{k-d}$ for all $l \in \{0,\ldots,k-1\}$ and all $1 \leq d \leq k-1$. This shows that there exists a constant $c < 1$ such that for all l, d it holds that $h_d^l(k,n) \leq c\binom{n-d}{k-d}$, although the constant we show is clearly far from being optimal.

2 Outline of the Proofs

Suppose $H = (V,E)$ is a k-graph on n vertices, n sufficiently large, with at least

$$\binom{n-1}{k} + \mathrm{ex}\left(n-1, P(k,l)\right)$$

edges and no vertex with a $P(k,l)$-free link. Then the vertex set can be partitioned into two sets $V = V' \cup V$" with $|V'| = n'$ and $V'' = \{v_1,\ldots,v_t\}$ such that

$$\delta_1\left(H'\right) \geq (1-\varepsilon)\binom{n'}{k-1} \tag{1}$$

with $H' = H[V']$ and $\varepsilon > 0$ being some carefully chosen constant depending only on k. To obtain V', we iteratively delete vertices $v_1,\ldots,v_t$ of minimum degree from H till the δ_1-condition (1) holds. Counting the non-edges one observes that $t \leq \frac{2}{\varepsilon}$.

By an *end* of a path $P_t^{(k,l)}$ we mean the tuple consisting of its first $k-1$ vertices, $(x_1,\ldots,x_{k-1})$, or the tuple consisting of its last $k-1$ vertices in reverse order, $(x_t,\ldots,x_{t-k+2})$, considering the ordered vertices. For an i-tuple $(x_1,\ldots,x_i)$ in H we write $\boldsymbol{x_i}$, $1 \leq i \leq n$. We call $\boldsymbol{x_{k-1}}$ *good* if all x_is are pairwise distinct and for all $i \in \{1, \dots, k-1\}$ it holds that

$$\deg(x_1, \dots, x_i) \geq \left(1 - \varrho^{k-i}\right) \binom{n-i}{k-i}, \tag{2}$$

where ϱ again depends on ε and k only. A path is called *good* if both of its ends are good.

In the following we give a brief overview over the structure of the proofs of Theorems 1.1 and 1.2, along with some more definitions. The main idea is to utilize the notion of *absorbers* and *connectors* (the so-called *absorbing technique*) that was originally developed by Rödl, Ruciński and Szemerédi [7].

i. At first, we prove the existence of a set $\mathcal{S}$ of one l-tight good path or several vertex-disjoint good tight paths containing the vertices from V''. (Actually one does not need this step in the proof of Theorem 1.2.)

ii. We say that a tuple $\boldsymbol{x_{2k-2}}$ *absorbs* a vertex $v \in V$ if both $\boldsymbol{x_{2k-2}}$ and $(x_1, \dots, x_{k-1}, v, x_k, \dots, x_{2k-2})$ induce good paths in H, meaning that the corresponding ordering of the paths is $\boldsymbol{x_{2k-2}}$ or $(x_1, \dots, x_{k-1}, v, x_k, \dots, x_{2k-2})$, respectively, and the ends are good. We prove the existence of a set $\mathcal{A}$ of linear size, providing that any remaining vertex from V' can be absorbed by linearly many tuples of $\mathcal{A}$. We call an element of $\mathcal{A}$ an *absorber*.

iii. For $\boldsymbol{x_i}, \boldsymbol{y_j} \in V^{k-1}$ we define $\boldsymbol{x_i} \times \boldsymbol{y_j} = (x_1, \dots, x_i, y_1, \dots, y_j)$. Let $\boldsymbol{x_{k-1}}$ and $\boldsymbol{y_{k-1}}$ be good. We say that a tuple $\boldsymbol{z_{k-1}}$ *connects* $\boldsymbol{x_{k-1}}$ with $\boldsymbol{y_{k-1}}$ if $(x_{k-1}, \dots, x_1) \times \boldsymbol{z_{k-1}} \times \boldsymbol{y_{k-1}}$ induces a path in H with respect to the order. Notice that the connecting-operation is not symmetric. We prove the existence of a set $\mathcal{C}$ such that any pair of good $(k-1)$-tuples in H' can be connected by linearly many elements of $\mathcal{C}$. We call the elements of $\mathcal{C}$ *connectors*.

iv. We modify $\mathcal{A}$ and $\mathcal{C}$ such that $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{S}$ are pairwise vertex-disjoint.

v. We create a good tight path that contains all elements of the modified $\mathcal{A}$, respecting their ordering.

vi. We extend the path from Step **v** until it covers almost all of the remaining vertices that do not participate in the modified $\mathcal{C}$ and $\mathcal{S}$.

vii. Using connectors, we create a cycle containing the path(s) from $\mathcal{S}$ and the good path from Step **vi**.

viii. In the final step all remaining vertices are absorbed by the absorbers in the cycle.

For the full details we refer the reader to our paper [3].

3 Concluding Remarks

In general, we conjecture that an extremal graph of any bounded spanning structure consists of an $(n-1)$-clique and a further extremal graph.

Conjecture 3.1 *For any $k \in \mathbb{N}$ there exists an n_0 such that for every k-graph H on $n \geq n_0$ vertices without a spanning subgraph isomorphic to a forbidden hypergraph F of bounded maximum vertex degree,*

$$|e(H)| \leq \binom{n-1}{k} + \operatorname{ex}\left(n-1, \{F(v) : v \in V\}\right)$$

holds, and the bound is tight.

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