

AN IMPROVED ERROR TERM FOR MINIMUM H -DECOMPOSITIONS OF GRAPHS

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ABSTRACT. We consider partitions of the edge set of a graph G into copies of a fixed graph H and single edges. Let $\phi_H(n)$ denote the minimum number p such that any n -vertex G admits such a partition with at most p parts. We show that $\phi_H(n) = \text{ex}(n, K_r) + \Theta(\text{biex}(n, H))$ for $\chi(H) \geq 3$, where $\text{biex}(n, H)$ is the extremal number of the decomposition family of H . Since $\text{biex}(n, H) = O(n^{2-\gamma})$ for some $\gamma > 0$ this improves on the bound $\phi_H(n) = \text{ex}(n, H) + o(n^2)$ by Pikhurko and Sousa [J. Combin. Theory Ser. B 97 (2007), 1041–1055]. In addition it extends a result of Özkahya and Person [J. Combin. Theory Ser. B, to appear].

1. INTRODUCTION

We study edge decompositions of a graph G into disjoint copies of another graph H and single edges. More formally, an H -decomposition of G is a decomposition $E(G) = \bigcup_{i \in [t]} E(G_i)$ of its edge set, such that for all $i \in [t]$ either $|E(G_i)| = 1$ or G_i is isomorphic to H . Let $\phi_H(G)$ denote the minimum t such there is a decomposition $E(G) = \bigcup_{i \in [t]} E(G_i)$ of this form, and let $\phi_H(n) := \max_{v(G)=n} \phi_H(G)$.

The function $\phi_H(n)$ was first studied in the seventies by Erdős, Goodman and Pósa [3], who showed that the minimal number $k(n)$ such that every n -vertex graph admits an edge decomposition into $k(n)$ cliques equals $\phi_{K_3}(n)$. They also proved that $\phi_{K_3}(n) = \text{ex}(n, K_3)$, where $\text{ex}(n, H)$ is the maximum number of edges in an H -free graph on n vertices. A decade later this result was extended to K_r for arbitrary r by Bollobás [1] who showed that $\phi_{K_r}(n) = \text{ex}(n, K_r)$ for all $n \geq r \geq 3$.

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General graphs H were considered only recently by Pikhurko and Sousa [6], who proved the following upper bound for $\phi_H(n)$.

Theorem 1 (Theorem 1.1 from [6]). *If $\chi(H) = r \geq 3$ then*

$$\phi_H(n) = \text{ex}(n, K_r) + o(n^2).$$

Pikhurko and Sousa also conjectured that if $\chi(H) \geq 3$ and if n is sufficiently large, then the correct value is the extremal number of H .

Conjecture 2. *For any graph H with chromatic number at least 3, there is an $n_0 = n_0(H)$ such that $\phi_H(n) = \text{ex}(n, H)$ for all $n \geq n_0$.*

We remark that the function $\text{ex}(n, H)$ is known precisely only for some graphs H , which renders Conjecture 2 difficult. However, $\text{ex}(n, H)$ is known for the family of *edge-critical* graphs H , that is, graphs with $\chi(H) > \chi(H - e)$ for some edge e . And in fact, after Sousa [9, 7, 8] proved Conjecture 2 for a few special edge-critical graphs, Özkahya and Person [5] verified it for all of them.

Our contribution is an extension of the result of Özkahya and Person to arbitrary graphs H , which also improves on Theorem 1. We need the following definition. Given a graph H with $\chi(H) = r$, the *decomposition family* \mathcal{F}_H of H is the set of bipartite graphs which are obtained from H by deleting $r - 2$ colour classes in some r -colouring of H . Observe that \mathcal{F}_H may contain graphs which are disconnected, or even have isolated vertices. Let \mathcal{F}_H^* be a minimal subfamily of \mathcal{F}_H such that for any $F \in \mathcal{F}_H$, there exists $F' \in \mathcal{F}_H^*$ with $F' \subseteq F$. We define

$$\text{biex}(n, H) := \text{ex}(n, \mathcal{F}_H) = \text{ex}(n, \mathcal{F}_H^*).$$

Our main result states that the $o(n^2)$ error term in Theorem 1 can be replaced by $O(\text{biex}(n, H))$, which is $O(n^{2-\gamma})$ for some $\gamma > 0$ by the result of Kövari, Turán and Sós [4]. Furthermore, we show that our error term is of the correct order of magnitude.

Theorem 3. *For every integer $r \geq 3$ and every graph H with $\chi(H) = r$ there are constants $c = c(H) > 0$ and $C = C(H)$ and an integer n_0 such that for all $n \geq n_0$ we have*

$$\text{ex}(n, K_r) + c \cdot \text{biex}(n, H) \leq \phi_H(n) \leq \text{ex}(n, K_r) + C \cdot \text{biex}(n, H).$$

Since for every edge-critical H and every n we have $\text{biex}(n, H) = 0$, this is indeed an extension of the result of Özkahya and Person.

2. OUTLINE OF THE PROOF AND AUXILIARY LEMMAS

The lower bound of Theorem 3 is obtained as follows. We let F be an n -vertex \mathcal{F}_H^* -free graph with $\text{biex}(n, H)$ edges, and let $c = (r - 1)^{-2}$. There is an $n/(r - 1)$ -vertex subgraph F' of F with at least $c \cdot e(F)$ edges. We let G be obtained from the complete balanced $(r - 1)$ -partite graph on n vertices by inserting F' into the largest part. Clearly, we

have $e(G) \geq \text{ex}(n, K_r) + c \cdot \text{biex}(n, H)$, and by definition of \mathcal{F}_H^* , the graph G is H -free, and therefore satisfies $\phi_H(G) = e(G) \geq \text{ex}(n, K_r) + c \cdot \text{biex}(n, H)$.

The upper bound of Theorem 3 is an immediate consequence of the following result.

Theorem 4. *For every integer $r \geq 3$ and every graph H with $\chi(H) = r$ there is a constant $C = C(H)$ and an integer n_0 such that the following holds. Every graph G on $n \geq n_0$ vertices and with*

$$e(G) \geq \text{ex}(n, K_r) + C \cdot \text{biex}(n, H)$$

satisfies $\phi_H(G) \leq \text{ex}(n, K_r)$.

The proof of this theorem (see Section 3) uses the auxiliary lemmas collected in this section and roughly proceeds as follows. We start with a graph $G = (V, E)$ on n vertices with $e(G) \geq \text{ex}(n, K_r) + C \cdot \text{biex}(n, H)$. For contradiction we assume that $\phi_H(G) > \text{ex}(n, K_r)$. This allows us to use a stability-type result (Lemma 5), which supplies us with a partition $V = V_1, \dots, V_{r-1}$ with parts of roughly the same size and with few edges inside each part. Since $e(G) \geq \text{ex}(n, K_r) + C \cdot \text{biex}(n, H)$ we also know that between two parts only few edges are missing. Next, in each part V_i we identify the (small) set X_i of those vertices with many edges to V_i and set $V'_i := V_i \setminus X_i$ and $X := \bigcup X_i$.

Then we consider the graph $G[V \setminus X]$, and identify a copy of some F in the decomposition family of H in any $G[V'_i]$, which we then complete to a copy of H using the classes V'_j with $j \neq i$ (see Lemma 6). We delete this copy of H from G and repeat this process. We shall show that this is possible until the number of edges in all $V_i \setminus X_i$ drops below $\text{biex}(n, H)$, and thus this gives many edge-disjoint copies of H in G .

Finally, we find edge-disjoint copies of H each of which has one of its colour classes in X and the other $(r - 1)$ colour classes in V'_1, \dots, V'_{r-1} (see Lemma 7). It is possible to find many copies of H in this way, because every vertex in X has many neighbours in *every* V_i .

In total these steps will allow us to find enough H -copies to obtain a contradiction.

2.1. Notation. Let G be a graph and $V(G) = V_1 \dot{\cup} \dots \dot{\cup} V_s$ a partition of its vertex set. We write $e(V_i)$ for the number of edges of G with both ends in V_i and $e(V_i, V_j)$ for the number of edges of G with one end in V_i and one end in V_j . Moreover, for $v \in V$ we let $\deg_G(v, V_i) = \deg(v, V_i)$ denote the number of neighbours of v in V_i . An edge of G is called *crossing* (for $V_1 \dot{\cup} \dots \dot{\cup} V_s$) if its ends lie in different classes of this partition. A subgraph H is called *crossing* if all of its edges are crossing, and *non-crossing* if none of its edges is crossing. The *chromatic excess* $\sigma(H)$ of H denotes the smallest size of a colour class in a proper $\chi(H)$ -colouring of H .

2.2. Auxiliary lemmas. The proof of Theorem 4 relies on the following three lemmas. Firstly, we use a stability-type result which was observed in [5].

Lemma 5 (stability lemma [5]). *For every $\gamma > 0$, every integer $r \geq 3$, and every graph $H \neq K_r$ with $\chi(H) = r$ there is an integer n_0 such that the following holds. If $G = (V, E)$ has $n \geq n_0$ vertices and satisfies $\phi_H(G) \geq \text{ex}(n, K_r)$, then there is a partition $V = V_1 \dot{\cup} \cdots \dot{\cup} V_{r-1}$ such that*

- (a) $\deg(v, V_i) \leq \deg(v, V_j)$ for all $v \in V_i$ and all $i, j \in [r-1]$,
- (b) $\sum_i e(V_i) < \gamma n^2$, and
- (c) $\frac{n}{r-1} - 2\sqrt{\gamma}n \leq |V_i| \leq \frac{n}{r-1} + 2r\sqrt{\gamma}n$. □

We remark that this lemma is stated in [5] only with assertion (b). However, we can certainly assume that the partition obtained is a maximal $(r-1)$ -cut, which implies (a), and for (c) see Claim 8 in [5].

The following lemma allows us to find many H -copies in a graph G with a partition such that each vertex has few neighbours inside its own partition class.

Lemma 6. *For every integer $r \geq 3$, every graph H with $\chi(H) = r$ and every positive $\beta \leq 1/(100e(H)^4)$ there is an integer n_0 such that the following holds. Let $G = (V, E)$ be a graph on $n \geq n_0$ vertices, with a partition $V = V_1 \dot{\cup} \cdots \dot{\cup} V_{r-1}$ such that for all $i, j \in [r-1]$ with $i \neq j$*

- (i) $\deg(v, V_j) \geq \left(\frac{1}{r-1} - \beta\right)n$ for every $v \in V_i$,
- (ii) $\sum_{i=1}^{r-1} e(V_i) \leq \beta^2 n^2 / e(H)$ and $\Delta(V_i) \leq 2\beta n$.

Then we can consecutively delete edge-disjoint copies of H from G , until $e(V_i) \leq \text{biex}(n, H)$ for all $i \in [r-1]$. Moreover, these H -copies can be chosen such that each of them contains a non-crossing $F \in \mathcal{F}_H^$ and all edges in $E(H) \setminus E(F)$ are crossing.*

Proof. Let $G = (V, E)$ be a graph and $V = V_1 \dot{\cup} \cdots \dot{\cup} V_{r-1}$ be a partition satisfying the conditions of the lemma. We proceed by selecting copies of H in G and deleting them, one at a time, in the following way. First we find a copy of some $F \in \mathcal{F}_H^*$ in $G[V_i]$ for some partition class V_i . Then we extend this F to a copy of H , using only vertices v of G for $H \setminus F$ which have at least $\left(\frac{1}{r-1} - 2\beta\right)n$ neighbours in every partition class other than their own. We say that such vertices v are β -active.

We need to show that this deletion process can be performed until $e(V_i) \leq \text{biex}(n, H)$ for all $i \in [r-1]$. Clearly, while $e(V_i) > \text{biex}(n, H)$ for some i , we find some $F \in \mathcal{F}_H^*$ in $G[V_i]$. Let such a copy of F be fixed in the following and assume without loss of generality that $V(F) \subseteq V_{r-1}$. It remains to show that F can be extended to a copy of H .

By condition (i), at the beginning of the deletion process every vertex is β -active, and every vertex which gets inactive has lost at least

βn neighbours in some partition class other than its own. Further, by condition (ii) we can find at most $\beta^2 n^2 / e(H)$ copies of H in this way. Hence we conclude that even after the very last deletion step, the number of vertices which are not β -active is at most

$$\frac{\beta^2 n^2}{e(H)} e(H) \cdot \frac{1}{\beta n} = \beta n.$$

In addition, by condition (ii) we have $\Delta(V_{r-1}) \leq 2\beta n$ at the beginning of the deletion process. Recall moreover that, in this process, we use inactive vertices only in copies of some graph in \mathcal{F}_H^* (and not to complete such a copy to an H -copy). Hence, throughout the process, we have for all $j \in [r-1]$ and all $v \in V \setminus V_j$ that

$$(1) \quad \deg(v, V_j) \geq \left(\frac{1}{r-1} - 2\beta\right)n - e(H) - e(H) \cdot 2\beta n \geq \frac{n}{r-1} - 5e(H)\beta n.$$

By condition (i) each partition class V_j has size at least $(\frac{1}{r-1} - 2\beta)n$, and thus size at most $\frac{n}{r-1} + 2r\beta n$. Moreover, by (1) each vertex $v \in V_j$ has at most

$$\frac{n}{r-1} + 2r\beta n - \left(\frac{n}{r-1} - 6e(H)\beta n\right) \leq 8e(H)\beta n$$

non-neighbours in each $V_{j'}$ with $j \neq j'$. Hence, any set $S \subseteq V \setminus V_j$ with $|S| \leq r \cdot v(H)$ has at least $|V_j| - 8r \cdot v(H)e(H)\beta n$ common neighbours in V_j . In particular, S has at least

$$\left(\frac{1}{r-1} - 2\beta\right)n - \beta n - 8r \cdot v(H)e(H)\beta n \geq \frac{n}{r-1} - 11e(H)^3 \beta n > \beta n \geq v(H)$$

common neighbours in V_j which are β -active, where we used the condition $\beta \leq 1/(100e(H)^4)$ in the second inequality, and in the last inequality that n is sufficiently large.

When F gets selected in the deletion process, we use the above observation to construct within the β -active common neighbours of F a copy of the complete $(r-2)$ -partite graph with $v(H)$ vertices in each part, as follows. We inductively find sets $S_i \subseteq V_i$ of size $v(H)$ which form the parts of this complete $(r-2)$ -partite graph. For each $1 \leq i \leq r-2$ in turn, we note that $v(F) + (i-1)v(H) \leq r \cdot v(H)$, and therefore the set $v(F) \cup S_1 \cup \dots \cup S_{i-1}$ has at least $\beta n \geq v(H)$ common neighbours in V_i which are β -active. We let S_i be any set of $v(H)$ of these β -active common neighbours. Thus we can extend F to a copy of H in G . \square

With the help of the next lemma we will find H -copies using those vertices which have many neighbours in their own partition class.

Lemma 7. *For every integer $r \geq 3$, every graph H with $\chi(H) = r$, and every positive $\beta \leq 1/(2e(H)^2)$ there are integers K and n_0 such that the following holds. Let $G = (V, E)$ be a graph on $n \geq n_0$ vertices, with a partition $V = X \dot{\cup} V'_1 \dot{\cup} \dots \dot{\cup} V'_{r-1}$ such that*

- (i) $e(V'_i, V'_j) > |V'_i||V'_j| - \beta^6 n^2$ for each $i, j \in [r-1]$ with $i \neq j$,
- (ii) $|X| \leq \beta^6 n$.

Then we can consecutively delete edge-disjoint copies of H from G , until for all but at most $K(\sigma(H) - 1)$ vertices $x \in X$ there is an $i \in [r - 1]$ such that $\deg(x, V'_i) \leq \beta^2 n$. Moreover, these H -copies can be chosen such that they are crossing for the partition $X \dot{\cup} V'_1 \dot{\cup} \dots \dot{\cup} V'_{r-1}$ and each of them uses exactly $\sigma(H)$ vertices of X .

Proof. Without loss of generality we assume that there are only crossing edges in G (otherwise delete the non-crossing edges). We proceed as follows. In the beginning we set $X' := X$. Then we identify $\sigma(H)$ vertices in X' which are completely joined to a complete $(r - 1)$ -partite graph $K_{r-1}(v(H))$ in $V \setminus X$, with $v(H)$ vertices in each part. The subgraph of G identified in this way clearly contains a copy of H with the desired properties, whose edges we delete from G . Next we delete those vertices x from X' with $\deg(x, V'_i) \leq \beta^2 n$ for some $i \in [r - 1]$. Then we continue with the next copy of H .

We need to show that this process can be repeated until $X' \leq K(\sigma(H) - 1)$. Indeed, assume that we still have $X' > K(\sigma(H) - 1)$. Observe that since $\sum_{x \in X} \deg(x) < |X|n$ we can find less than $|X|n \leq \beta^6 n^2$ copies of H with the desired properties in total, where we used condition (ii). Hence, throughout the process at most $e(H)\beta^6 n^2$ edges are deleted from G . In addition, for each $x \in X'$ we have by definition $\deg(x, V'_i) > \beta^2 n$ for all $i \in [r - 1]$. Hence we can choose for each i a set $S_i \subseteq N_{V'_i}(x)$ of size $\beta^2 n$. By condition (i) the graph $G[\dot{\cup} S_i]$ has density at least

$$\frac{\binom{r-1}{2}((\beta^2 n)^2 - \beta^6 n^2 - e(H)\beta^6 n^2)}{\binom{(r-1)\beta^2 n}{2}} \geq \frac{r-2}{r-1}(1 - (e(H)+1)\beta^2) > \frac{r-3}{r-2},$$

where we used $\beta \leq 1/(2e(H)^2)$ in the last inequality. Thus, since n is sufficiently large, we can apply the supersaturation theorem of Erdős and Simonovits [2], to conclude that the graph $G[\dot{\cup} S_i]$ contains at least $\delta n^{(r-1)v(H)}$ copies of $K_{r-1}(v(H))$, where $\delta > 0$ depends only on β and $e(H)$. Choosing $K := 1/\delta$, we can then use the pigeonhole principle and the fact that $|X'| > K(\sigma(H) - 1)$ to infer that there are $\sigma(H)$ vertices in X' which are all adjacent to the vertices of one specific copy of $K_{r-1}(v(H))$ in $G[\dot{\cup} S_i]$ as desired. \square

In addition we shall use the following easy fact about $\text{biex}(n, H)$.

Fact 8. *Let H be an r -chromatic graph, $r \geq 3$. If $\text{biex}(n, H) < n - 1$ then $\sigma(H) = 1$.*

Proof. If $\sigma(H) \geq 2$, then each F from \mathcal{F}_H^* contains a matching of size 2. Thus $\text{biex}(n, H) \geq n - 1$ since the star $K_{1, n-1}$ does not contain two disjoint edges. \square

3. PROOF OF THEOREM 4

In this section we show how Lemmas 5, 6 and 7 imply Theorem 4.

Proof of Theorem 4. Let r and H with $\chi(H) = r \geq 3$ be given. If $H = K_r$, then the result of Bollobás [1] applies, hence we can assume that $H \neq K_r$. We choose

$$(2) \quad \beta := \frac{1}{100e(H)^4} \quad \text{and} \quad \gamma := \frac{\beta^{12}}{1000e(H)^4}.$$

Let K be the constant from Lemma 7 and choose

$$(3) \quad C := K \cdot v(H)\beta^{-1}.$$

Finally let n_0 be sufficiently large for Lemmas 5, 6 and 7.

Now let G be a graph with $n \geq n_0$ vertices and

$$(4) \quad e(G) \geq \text{ex}(n, K_r) + C \cdot \text{biex}(n, H),$$

and assume for contradiction that

$$(5) \quad \phi_H(G) \geq \text{ex}(n, K_r).$$

Observe first that we may assume without loss of generality that

$$(6) \quad \delta(G) \geq \delta(T_{r-1}(n)).$$

Indeed, if this is not the case, we can consecutively delete vertices of minimum degree until we arrive at a graph G_{n^*} on n^* vertices with $\delta(G_{n^*}) \geq \delta(T_{r-1}(n^*))$. Denote the sequence of graphs obtained in this way by $G_n := G, G_{n-1}, \dots, G_{n^*}$. We have

$$\text{ex}(n, K_r) \leq \phi_H(G) \leq \phi_H(G_{n-1}) + \delta(T_{r-1}(n)) - 1$$

and thus $\phi_H(G_{n-1}) \geq \text{ex}(n-1, K_r) + 1$. Similarly $\phi_H(G_{n-i}) \geq \text{ex}(n-i, K_r) + i$. Since n is sufficiently large there is an i^* such that $n - i^* \geq n_0$ and $i^* \geq \binom{n-i^*}{2} + 1$. Hence $n^* > n - i^* \geq n_0$, since otherwise $\phi_H(G_{n^*}) \geq \text{ex}(n^*, K_r) + \binom{n^*}{2} + 1$, a contradiction. Thus we may assume (6).

Next, by (5), we can apply Lemma 5, which provides us with a partition $V_1 \dot{\cup} \dots \dot{\cup} V_{r-1}$ of $V(G)$ such that assertions (a), (b) and (c) in Lemma 5 are satisfied. Let $m := \sum_{i=1}^{r-1} e(V_i)$. Equation (4) and Lemma 5(b) imply

$$(7) \quad C \cdot \text{biex}(n, H) \leq m \leq \gamma n^2 \stackrel{(2)}{\leq} \beta^2 n^2 / e(H).$$

Further, by the definition of m we clearly have $e(G) \leq \text{ex}(n, K_r) + m$. Hence it will suffice to find $\frac{m}{e(H)-1} + 1$ edge-disjoint copies of H in G , since this would imply

$$\phi_H(G) \leq \text{ex}(n, K_r) + m - \left(\frac{m}{e(H)-1} + 1 \right) (e(H) - 1) < \text{ex}(n, K_r),$$

contradicting (5). So this will be our goal in the following, which we shall achieve by first applying Lemma 6 and then Lemma 7.

We prepare these applications by identifying for every $i \in [r-1]$ the set X_i of vertices in V_i with high degree to its own class, that is,

$$X_i := \left\{ v \in V_i : \deg(v, V_i) \geq \frac{1}{2}\beta n \right\}.$$

Let $X := \dot{\cup}_{i \in [r-1]} X_i$. This implies

$$(8) \quad |X| \leq \frac{2m}{\frac{1}{2}\beta n} \stackrel{(7)}{\leq} \frac{2\gamma n^2}{\frac{1}{2}\beta n} \stackrel{(2)}{\leq} \sqrt{\gamma} n \stackrel{(2)}{\leq} \beta^6 n.$$

In addition we set $V'_i := V_i \setminus X_i$ for all $i \in [r-1]$, $n' := |V \setminus X|$, $m' := \sum_{i=1}^{r-1} e(V_i \setminus X_i)$ and $m_X := m - m' = e(X) + \sum_{i=1}^{r-1} e(X_i, V'_i)$.

Step 1. We want to apply Lemma 6 to the graph $G[V \setminus X]$ and the partition $V'_1 \dot{\cup} \dots \dot{\cup} V'_{r-1}$. We first need to check that the conditions are satisfied. By Lemma 5(c) and (8) we have for each $i, j \in [r-1]$ with $i \neq j$ that $|V \setminus (V_i \cup V'_j)| \leq (\frac{r-3}{r-1} + 6\sqrt{\gamma})n$. Moreover, by (8) we clearly have $n' \geq n/2$. Hence, by the definition of X , for each $v \in V'_i$ we have

$$(9) \quad \begin{aligned} \deg(v, V'_j) &\stackrel{(6)}{\geq} \delta(T_{r-1}(n)) - |V \setminus (V_i \cup V'_j)| - \frac{1}{2}\beta n \\ &\geq \left(\frac{1}{r-1} - 7\sqrt{\gamma} - \frac{1}{2}\beta\right)n \stackrel{(2)}{\geq} \left(\frac{1}{r-1} - \beta\right)n' \end{aligned}$$

and thus condition (i) of Lemma 6 is satisfied. Condition (ii) of Lemma 6 holds by (7) and the definition of X . Therefore we can apply Lemma 6.

This lemma asserts that we can consecutively delete copies of H from $G[V \setminus X]$, each containing a non-crossing $F \in \mathcal{F}_H^*$ and crossing edges otherwise, until $e(V'_i) \leq \text{biex}(n', H)$. Denote the graph obtained after these deletions by G_1 .

We have $\max_{F \in \mathcal{F}_H^*} e(F) \leq e(H) - 2$, since $\chi(H) \geq 3$. Hence each copy of H deleted in this way uses at most $e(H) - 2$ non-crossing edges, and so this gives at least

$$(10) \quad \frac{m' - (r-1) \text{biex}(n, H)}{e(H) - 2} \geq \frac{m' - r \text{biex}(n, H)}{e(H) - 2}$$

edge-disjoint copies of H in $G[V \setminus X]$.

By assertion (b) of Lemma 5 and the assumption $e(G) \geq \text{ex}(n, K_r)$, we have that $e_G(V_i, V_j) \geq |V_i||V_j| - \gamma n^2$, and thus $e_G(V'_i, V'_j) \geq |V'_i||V'_j| - \gamma n^2$. Again by assertion (b) of Lemma 5, in obtaining G_1 as described above we delete at most γn^2 copies of H , so we have

$$(11) \quad \begin{aligned} e_{G_1}(V'_i, V'_j) &\geq |V'_i||V'_j| - \gamma n^2 - (e(H) - 1)\gamma n^2 \\ &= |V'_i||V'_j| - e(H)\gamma n^2 \stackrel{(2)}{\geq} |V'_i||V'_j| - \beta^6 n^2. \end{aligned}$$

Step 2. Next we want to apply Lemma 7 to G_1 and the partition $X \dot{\cup} V'_1 \dot{\cup} \dots \dot{\cup} V'_{r-1}$. Note that condition (i) of Lemma 7 is satisfied by (11) and condition (ii) by (8). Hence Lemma 7 allows us to delete crossing copies of H from G until all vertices x of a subset $X_0 \subseteq X$ with $|X| - |X_0| \leq K(\sigma(H) - 1) =: K'$ have $\deg(x, V'_{i(x)}) \leq \beta^2 n$ for some $i(x) \in [r-1]$. Denote the graph obtained after these deletions by G_2 .

Now let $x \in X_j$ for some $j \in [r - 1]$ be arbitrary. We set $m_x := \deg_G(x, V_j \setminus X)$. Since no edges adjacent to x were deleted in step 1, if $x \in X_0$ then the number of edges adjacent to x deleted in step 2 is at least $\deg_G(x, V_{i(x)} \setminus X) - \beta^2 n \geq m_x - 2\beta^2 n$, where we used assertion (a) of Lemma 5 and (8) in the inequality. Hence, since $m_X = \sum_{x \in X} m_x + e(X)$, in total at least

$$\begin{aligned} m_X - K'n - |X_0|\beta^2 n - e(X) &\geq m_X - K'n - 2\beta^2 n|X| \\ &\stackrel{(8)}{\geq} m_X - K'n - 8\beta m \end{aligned}$$

edges adjacent to X were deleted in step 2. By Fact 8 we have $K' = K(\sigma(H) - 1) = 0$ if $\text{biex}(n, H) < n - 1$. If $\text{biex} \geq n - 1 \geq n/2$ on the other hand, then $m \geq Cn/2$ by (7) and thus $K'n \leq 2K'm/C$. Observe moreover that, because $H \neq K_3$, each H -copy deleted in this step uses at least 2 edges which are not adjacent to X . We conclude that at least

$$(12) \quad \frac{m_X - \frac{2K(\sigma(H)-1)}{C}m - 8\beta m}{e(H) - 2} \stackrel{(3)}{\geq} \frac{m_X - 9\beta m}{e(H) - 2}$$

edge-disjoint copies of H were deleted from G_1 in step 2.

Combining (10) and (12) reveals that G contains

$$\begin{aligned} \frac{m' - r \text{biex}(n, H)}{e(H) - 2} + \frac{m_X - 9\beta m}{e(H) - 2} &\stackrel{(7)}{\geq} \frac{m - \frac{r}{C}m - 9\beta m}{e(H) - 2} \\ &\stackrel{(3)}{\geq} \frac{m - 10\beta m}{e(H) - 2} \stackrel{(2)}{\geq} \frac{m}{e(H) - 1} + 1 \end{aligned}$$

edge-disjoint copies of H , which gives the desired contradiction. \square

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