EXACT RESULTS ON THE NUMBER OF RESTRICTED EDGE COLORINGS FOR SOME FAMILIES OF LINEAR HYPERGRAPHS

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Abstract. For k-uniform hypergraphs F and H and an integer $r \geq 2$, let $c_{r,F}(H)$ denote the number of r-colorings of the set of hyperedges of H with no monochromatic copy of F and let $c_{r,F}(n) = \max_{H\in\mathcal{H}_n} c_{r,F}(H)$, where the maximum is over the family $\mathcal{H}_n$ of all k-uniform hypergraphs on n vertices. Moreover, let $\mathrm{ex}(n, F)$ be the usual extremal function, i.e., the maximum number of hyperedges of an n-vertex k-uniform hypergraph which contains no copy of F.

Here, we consider the question for determining $c_{r,F}(n)$ for F being the k-uniform expanded, complete 2-graph $H^k_{\ell+1}$ or the k-uniform Fan(k)-hypergraph $F^k_{\ell+1}$ with core of size $(\ell + 1)$, where $\ell \geq k \geq 3$, and we show

$$c_{r,F}(n) = r^{\mathrm{ex}(n,F)}$$

for $r = 2, 3$ and n large enough. Moreover, for $r = 2$ or $r = 3$, for k-uniform hypergraphs H on n vertices the equality $c_{r,F}(H) = r^{\mathrm{ex}(n,F)}$ only holds if H is isomorphic to the ℓ-partite, k-uniform Turán hypergraph on n vertices, once n is large enough.

On the other hand, we show that $c_{r,F}(n)$ is exponentially larger than $r^{\mathrm{ex}(n,F)}$, if $r \geq 4$.

1. Introduction and results

By a k-uniform hypergraph H we mean a pair (V, E) with $V = V(H)$ being its vertex set and $E = E(H) \subseteq [V]^k$ being its set of hyperedges, where $[V]^k$ is the set of all k-element subsets of V. We denote by $e(H)$ its number $|E(H)|$ of hyperedges. For subsets $V' \subseteq V$ and $E' \subseteq E(H)\cap[V']^k$ we call the hypergraph (V', E') a *subhypergraph of* H. For convenience we write $[\ell] := \{1, \ldots, \ell\}$ and $[\ell]^k := [\{1, \ldots, \ell\}]^k$. For a fixed number of colors we consider colorings of the set of hyperedges of k-uniform hypergraphs $H = (V, E)$ without a fixed monochromatic subhypergraph. For k-uniform hypergraphs F and H and an integer r let $c_{r,F}(H)$ denote the number of r-colorings of the set of hyperedges of H with no monochromatic subhypergraph F and let $c_{r,F}(n) = \max_{H\in\mathcal{H}_n} c_{r,F}(H)$, where the maximum runs over all k-uniform hypergraphs on n vertices. A coloring of the set $E(H)$ of hyperedges of H without a monochromatic subhypergraph F is called *F-free coloring.* Moreover, let $\mathrm{ex}(n, F)$ be the usual *extremal* function, i.e., the maximum number of hyperedges of an n-vertex k-uniform hypergraph which contains no subhypergraph F. A hypergraph H with no subhypergraph F is called *F-free.* We say that a hypergraph H on n vertices is *extremal* for F if $e(H) = |E(H)| = \mathrm{ex}(n, F)$ and H is F-free. Moreover,

Date: April 21, 2010.
The second author is supported by GIF grant no. I-889-182.6/2005.

we set

$$\pi_F := \lim_{n \longrightarrow \infty} \mathrm{ex}(n, F)/\binom{n}{k},$$

and call it the *Turán density* for the k-uniform hypergraph F.

Clearly, every coloring of the set of hyperedges of any extremal hypergraph H for F contains no monochromatic subhypergraph F and, consequently,

$$c_{r,F}(n) \geq r^{\mathrm{ex}(n,F)}$$

for all $r \geq 2$. On the other hand, let $\mathrm{forb}_F(n)$ denote the family of all labeled F-free hypergraphs on n vertices. Since every F-free 2-coloring of the set of hyperedges of a hypergraph H gives rise to a member of $\mathrm{forb}_F(n)$, e.g., consider the subhypergraph of H in one of the two colors, we have $c_{2,F}(n) \leq |\mathrm{forb}_F(n)|$.

The family $\mathrm{forb}_F(n)$ has been studied by several people [3, 2, 4, 5, 16, 17, 22, 23, 24]. In particular, in [23] Nagle, Rödl, and Schacht have shown that $|\mathrm{forb}_F(n)| \leq 2^{\mathrm{ex}(n,F)+o(n^k)}$ for every k-uniform hypergraph F, hence,

$$2^{\mathrm{ex}(n,F)} \quad \leq \quad c_{2,F}(n) \leq 2^{\mathrm{ex}(n,F)+o(n^k)}. \tag{1}$$

For F being a graph, Yuster [31] for $\ell = 2$, and Alon, Balogh, Keevash, and Sudakov [1] for arbitrary fixed $\ell \geq 2$ closed the gap between lower and upper bound in (1), namely they proved for the complete graph $K_{\ell+1}$ on $(\ell+1)$ vertices that $c_{2,K_{\ell+1}}(n) = 2^{\mathrm{ex}(n,K_{\ell+1})}$ for n sufficiently large, which has been originally conjectured by Erdős and Rothschild [8]. Moreover, for $r = 3$ colors, Alon, Balogh, Keevash, and Sudakov [1] proved $c_{3,K_{\ell+1}}(n) = 3^{\mathrm{ex}(n,K_{\ell+1})}$, again for n large enough, and in both cases, $r = 2$ or $r = 3$, the equality $c_{r,K_{\ell+1}}(H) = r^{\mathrm{ex}(n,K_{\ell+1})}$ holds for graphs H on n vertices, if and only if H is isomorphic to the ℓ-partite Turán graph on n vertices. On the other hand, Alon et al. showed in [1] that $c_{r,K_{\ell+1}}(n) \gg r^{\mathrm{ex}(n,K_{\ell+1})}$ for every fixed $r \geq 4$.

Recently, Pikhurko and Yilma [26] determined the graphs H on n vertices, which yield $c_{4,F}(H) = c_{4,F}(n)$ for $F = K_3$ and $F = K_4$.

An extension of these results to a hypergraph has been given by Schacht, Rödl and the authors [18]. They showed $c_{r,F}(n) = r^{\mathrm{ex}(n,F)}$ for $r = 2$ or $r = 3$ and n sufficiently large, for F being the 3-uniform hypergraph of the Fano plane, and equality $c_{r,F}(H) = r^{\mathrm{ex}(n,F)}$ for 3-uniform hypergraphs H on n vertices is achieved only if H is the extremal hypergraph for F [11, 14]. Moreover, they showed in [18] that $c_{r,F}(n) \gg r^{\mathrm{ex}(n,F)}$ for any fixed $r \geq 4$ and n large enough.

Quite recently, the present authors together with Schacht [19] studied a general case, where for $r = 2$ or $r = 3$ it has been shown that $c_{r,F}(n) = r^{(\pi_F+o(1))\binom{n}{k}}$ for an arbitrary fixed hypergraph F. Moreover, under a general stability assumption, called s-stability, for $r = 2$ or $r = 3$ it has been shown that hypergraphs H on n vertices and satisfying $c_{r,F}(H) \geq r^{(\pi_F-o(1))\binom{n}{k}}$ must disclose a certain structure. Furthermore, in [19] an exact result for the 3- and 4-uniform generalized triangle has been established.

In this paper we study $c_{r,F}(n)$ for the following two families of linear hypergraphs, where by a *linear* hypergraph we mean a hypergraph whose hyperedges pairwise intersect in at most one vertex. These families are expanded, complete graphs and Fan(k)-hypergraphs which are defined below.

For integers $\ell, k \geq 2$, we define the so-called *expanded, complete graph* $H^k_{\ell+1}$ to be the k-uniform hypergraph obtained as follows. We take $\binom{\ell+1}{2}$ edges of the complete graph $K_{\ell+1}$, called the *core* of $H^k_{\ell+1}$, and we enlarge every edge by a set of $(k-2)$ new vertices. Thus, the vertex set of the hypergraph $H^k_{\ell+1}$ has size $(\ell+1)+\binom{\ell+1}{2}\cdot(k-2)$ and it contains $\binom{\ell+1}{2}$ hyperedges. Clearly, we have the inclusion $H^k_{\ell+1}\supset H^k_{\ell}$.

Similarly, for integers $\ell, k\geq 2, \ell\geq k-1$, we define the *Fan(k)-hypergraph* $F^k_{\ell+1}$ to be the k-uniform hypergraph, which contains $(\ell+1)$ vertices $v_1,\dots,v_{\ell+1}$ called the *core* of $F^k_{\ell+1}$. Moreover, k vertices of this core form a hyperedge, the *core-hyperedge,* say these are the vertices $v_1,\dots,v_k$, and then for each $\{i,j\}\in[\ell+1]^2\setminus[k]^2$ the two-element set $\{v_i,v_j\}$ is enlarged by a set of $(k-2)$ new vertices. Hence, the vertex set of $F^k_{\ell+1}$ has size $(\ell+1)+(\binom{\ell+1}{2}-\binom{k}{2})\cdot(k-2)$ and it contains $1+\binom{\ell+1}{2}-\binom{k}{2}$ hyperedges. Note that F^k_k contains exactly one hyperedge. These families have been studied by Mubayi and Pikhurko [20, 25, 21], where it is shown that for large n and $\ell\geq k$ the unique extremal hypergraph for $H^k_{\ell+1}$ and $F^k_{\ell+1}$ as well is the so-called Turán hypergraph $\mathcal{T}^{(k)}_\ell(n)$, which is a k-uniform hypergraph on n vertices that are partitioned into ℓ classes of cardinality as equal as possible and the hyperedge set consists of all those k-element sets that intersect each class in at most one vertex.

Here we prove the following:

Theorem 1. *Let $F=H^k_{\ell+1}$ be the k-uniform, expanded, complete 2-graph or $F=F^k_{\ell+1}$ the Fan(k)-hypergraph, both with core of size $(\ell+1)$, where $2\leq k\leq\ell$. Let $r=2$ or $r=3$.*

Then, there exists a positive integer $n_r(F)$, such that for every k-uniform hypergraph H on $n\geq n_{r,k}(F)$ vertices it is

$$c_{r,F}(H)\leq r^{\mathrm{ex}(n,F)}.$$

Moreover, for $r=2$ or $r=3$, and n sufficiently large, the only hypergraph H on n vertices with $c_{r,F}(H)=r^{\mathrm{ex}(n,F)}$, is the extremal hypergraph for F, i.e., H is isomorphic to $\mathcal{T}^{(k)}_\ell(n)$, the k-uniform Turán hypergraph on n vertices with ℓ classes.

Note that for $k=2$ we have $H^k_{\ell+1}=F^k_{\ell+1}=K_{\ell+1}$ and $\mathcal{T}^{(k)}_\ell(n)$ is the usual Turán graph with ℓ classes. The case $k=2$ in Thereom 1 is the result of Alon et al.[1].

For proving Theorem 1 we use a structural result obtained in [19]. There, a notion of *s-stability* has been used. However, for the hypergraphs $H^k_{\ell+1}$ and $F^k_{\ell+1}$, it is known that they are 1-stable [25, 21], or simply *stable,* which means that for every $\varepsilon>0$ there exists an $\omega>0$ and an integer n_0 such that if H is $H^k_{\ell+1}$-free or $F^k_{\ell+1}$-free, respectively, on $n\geq n_0$ vertices with $e(H)\geq(\pi_{H^k_{\ell+1}}-\omega)\cdot\binom{n}{k}$ or $e(H)\geq(\pi_{F^k_{\ell+1}}-\omega)\cdot\binom{n}{k}$, respectively, we can delete and/or add to H at most $\varepsilon\cdot n^k$ hyperedges to obtain a hypergraph which is isomorphic to $\mathcal{T}^{(k)}_\ell(n)$. The structural theorem we are going to use is the following.

Theorem 2. *Let $k\in\mathbb{N}$, $k\geq 2$ and $r=2$ or 3. Let F be a stable, k-uniform hypergraph with Turán density $\pi_F>0$. Then, for every $\delta>0$ there exists an $n_0\in\mathbb{N}$ such that for every $n\geq n_0$ the following holds. If two k-uniform hypergraphs H_1,H_2 on $n\geq n_0$ vertices satisfy $c_{r,F}(H_i)\geq r^{\mathrm{ex}(n,F)}$, $i=1,2$, then one can delete or add up to $\delta\cdot n^k$ hyperedges in H_1 to obtain a hypergraph isomorphic to H_2.*

For completenesss, we present the proof for the case when F is a linear hypergraph in the Appendix 7.

In Section 3 we prove Theorem 1, which is the major part of this paper. In Section 4 we give lower bounds $c_{r,F}(n) \gg r^{\mathrm{ex}(n,F)}$ for fixed values $r \geq 4$ and n sufficiently large, where $F = F^k_{\ell+1}$ or $F = H^k_{\ell+1}$, $2 \leq k \leq \ell$. In Section 5 we prove upper bounds on $c_{r,F}(n)$ for any fixed $r \geq 4$, where F is any fixed, linear k-uniform hypergraph with positive Turán density.

2. Notation and Tools

From time to time we ignore divisibilty issues when they (which is always the case here) do not affect our asymptotic considerations. By $(\ell)_k$ we denote the falling factorial $\ell \cdot \ldots \cdot (\ell - k + 1)$. We denote by $h(x) := -x \cdot \log x - (1-x) \cdot \log(1-x)$ for $0 < x < 1$ the entropy function, and we use the inequality

$$\binom{n}{\alpha \cdot n} \leq 2^{h(\alpha)n}$$

for $\alpha \in (0,1)$.

By $\delta(H)$ we denote the minimum vertex degree of H, that is, the minimum number of hyperedges incident to some vertex in H. Further, for the Turán hypergraph $\mathcal{T}^{(k)}_\ell(n)$ defined in the introduction we have the following bounds on its number of hyperedges

$$\binom{\ell}{k} \cdot \left\lfloor \frac{n}{\ell} \right\rfloor^k \quad \leq \quad e(\mathcal{T}^{(k)}_\ell(n)) \leq \binom{\ell}{k} \cdot \left\lceil \frac{n}{\ell} \right\rceil^k, \tag{2}$$

and the following bounds on the minimum degree $\delta(\mathcal{T}^{(k)}_\ell(n))$ of $\mathcal{T}^{(k)}_\ell(n)$ hold:

$$\binom{\ell-1}{k-1} \cdot \left\lceil \frac{n}{\ell} \right\rceil^{k-1} \geq \delta(\mathcal{T}^{(k)}_\ell(n)) \geq \binom{\ell-1}{k-1} \cdot \left\lfloor \frac{n}{\ell} \right\rfloor^{k-1}, \tag{3}$$

as every class $|V_i|$ has size at least $\lfloor n/\ell \rfloor$.

For a partition $\mathcal{P}$ of the vertex set $V(H)$, i.e., $V(H) = V_1 \dot\cup \cdots \dot\cup V_\ell$, a hyperedge e is called *crossing* if e intersects each class V_i, $i \in [\ell]$, in at most one vertex. Let $E_{cross}(\mathcal{P})$ be the set of all crossing hyperedges in the hypergraph H with respect to the partition $\mathcal{P}$. Moreover, let $E_{\mathrm{noncross}}(\mathcal{P}) := E(H) \backslash E_{\mathrm{cross}}(\mathcal{P})$ be the set of all *non-crossing* hyperedges in H, consisting of all hyperedges $e \in E(H)$, which intersect class V_i for some $i \in [\ell]$ in at least 2 vertices, and set $e_{\mathrm{noncross}}(\mathcal{P}) := |E_{\mathrm{noncross}}(\mathcal{P})|$.

For the proof of Theorem 2 (only linear hypergraphs) and for the proof of Theorem 16 in Section 5 we use the weak hypergraph regularity lemma introduced below.

Let H be a k-uniform hypergraph and let $V_1, \ldots, V_k \subseteq V(H)$ be k mutually disjoint subsets (classes) of $V(H)$. We denote by $E_H(V_1, \ldots, V_k)$ the set of all hyperedges in H that intersect every class V_i, $i \in [k]$, in exactly one vertex, and its cardinality is denoted by $e_H(V_1, \ldots, V_k) := |E_H(V_1, \ldots, V_k)|$, i.e.,

$$E_H(V_1, \ldots, V_k) = \{e \in E(H) \colon\ e \subseteq V_1 \cup \ldots \cup V_k, \ \text{ and } \forall i \in [k] \colon |e \cap V_i| \leq 1\}.$$

In the following we use the *weak hypergraph regularity lemma*. This result is an extension of Szemerédi's regularity lemma [30] for graphs. Let H be a k-uniform hypergraph and let $W_1, \ldots, W_k$ be mutually, disjoint, non-empty subsets of the vertex set $V(H)$. We denote by $d_H(W_1, \ldots, W_k)$ the *density* of the k-partite, induced subhypergraph $H[W_1, \ldots, W_k]$ of H, defined by

$$d_H(W_1, \ldots, W_k) = \frac{e_H(W_1, \ldots, W_k)}{|W_1| \cdot \ldots \cdot |W_k|}.$$

For a constant $\varepsilon > 0$, we say that the k-tuple $(V_1, \dots, V_k)$ of mutually disjoint subsets $V_1, \dots, V_k \subseteq V(H)$ is *ε-regular*, if there exists a constant $d \geq 0$, such that

$$|d_H(W_1, \dots, W_k) - d| \leq \varepsilon$$

for all k-tuples $(W_1, \dots, W_k)$ of subsets $W_i \subseteq V_i$, $i \in [k]$, with $|W_1| \cdot \ldots \cdot |W_k| \geq \varepsilon \cdot |V_1| \cdot \ldots \cdot |V_k|$.

A k-tuple $(V_1, \dots, V_k)$ of mutually disjoint subsets $V_1, \dots, V_k \subseteq V(H)$, which is not ε-regular, is called *ε-irregular*.

An *ε-regular partition* $V(H) = V_1 \dot\cup \cdots \dot\cup V_t$ of a vertex set $V(H)$ has the following properties:

(i) $||V_i| - |V_j|| \leq 1$ for all i, j, and

(ii) for all but at most $\varepsilon \cdot \binom{t}{k}$ many k-element subsets $\{i_1, \dots, i_k\} \subseteq [t]$ the k-tuple $(V_{i_1}, \dots, V_{i_k})$ is ε-regular.

The colored version of the weak regularity lemma (see e.g. [6, 9, 29]) states the following.

Theorem 3. *Let $k \geq 2$, and $r \geq 1$, and $t_0 \geq 1$ be fixed integers. For every $\varepsilon > 0$, there exist $T_0 = T_0(r, t_0, \varepsilon)$ and $N_0 = N_0(r, t_0, \varepsilon)$ such that for every k-uniform hypergraph H on $n \geq N_0$ vertices, whose hyperedges are r-colored. i.e., $E(H) = E_1 \dot\cup \cdots \dot\cup E_r$, there exists a partition $V(H) = V_1 \dot\cup \cdots \dot\cup V_t$, with $t_0 \leq t \leq T_0$, which is ε-regular simultaneously with respect to each subhypergraph $H_i = (V, E_i)$, $i \in [r]$.* □

For a hypergraph H and a regular partition of its vertex set we use the concept of a *cluster-hypergraph.*

Definition 4. *For a hypergraph H, an ε-regular partition $V(H) = V_1 \dot\cup \cdots \dot\cup V_t$ of its vertex set, and a number $\gamma > 0$ let $H(\gamma)$ be the cluster-hypergraph with vertex set $V(H(\gamma)) = [t]$ and the set $E(H(\gamma))$ of hyperedges, where for $1 \leq i_1 < \cdots < i_k \leq t$ it is $\{i_1, \dots, i_k\} \in E(H(\gamma))$ if and only if the k-tuple $(V_{i_1}, \dots, V_{i_k})$ is ε-regular and its density satisfies $d_H(V_{i_1}, \dots, V_{i_k}) \geq \gamma$.*

In [15], in the context of the weak hypergraph regularity lemma, a counting lemma for *linear* hypergraphs was proved, from which one obtains the following.

Lemma 5. *Let F be a fixed k-uniform, linear hypergraph.*

For each $\gamma > 0$ there exists $\varepsilon = \varepsilon(\gamma) > 0$ and an integer $m_0 = m_0(\gamma)$ such that for every positive integer t the following holds.

Let H be a k-uniform hypergraph with an ε-regular partition $V(H) = V_1 \dot\cup \cdots \dot\cup V_t$, where $|V_i| \geq m_0$ for every $i \in [t]$. If the cluster-hypergraph $H(\gamma)$ contains a subhypergraph F, then the hypergraph H contains a subhypergraph F too. □

3. Expanded, Complete 2-Graph and Fan(k)-Hypergraph

As already discussed in the introduction, we apply Theorem 2 for some linear stable hypergraph F. In this paper F is either $H^k_{\ell+1}$ or $F^k_{\ell+1}$ and from the stability for F it immediately follows for a hypergraph H_1 from Theorem 2 that one can add or delete up to $\delta \cdot n^k$ hyperedges to obtain a hypergraph which is isomorphic to $\mathcal{T}^{(k)}_\ell(n)$. Due to the structure of $\mathcal{T}^{(k)}_\ell(n)$, this implies that there exists a partition $\mathcal{P}$ of the vertex set $V(H_1) = V_1 \dot\cup \ldots \dot\cup V_\ell$ such that

$$e_{\text{noncross}}(\mathcal{P}) < \delta \cdot n^k. \tag{4}$$

Recall, that stability for the expanded, complete 2-graph $H^k_{\ell+1}$ was shown by Pikhurko [25] and for the Fan(k)-hypergraph $F^k_{\ell+1}$ by Mubayi and Pikhurko [21].

Proof of Theorem 1. Here we only prove the case $r = 3$, as the arguments for $r = 2$ are very similar. Let $2 \le k \le \ell$ and let $F = H^k_{\ell+1}$ or $F = F^k_{\ell+1}$, unless otherwise specified.

Let n_0 be given by Theorem 2 (applied with δ, which will be specified later) and let $n_r(F) = n_3(F) \ge n_0$ be sufficiently large.

The proof proceeds by contradiction as follows. Assume that we are given a hypergraph H on $n > n_3$ vertices with $c_{3,F}(H) \ge 3^{\mathrm{ex}(n,F)+m}$ for some $m \ge 0$. We show the following lemma, which is central in our considerations.

Lemma 6. *Let $F = F^k_{\ell+1}$ or $F = H^k_{\ell+1}$.*

If $c_{3,F}(H) \ge 3^{\mathrm{ex}(n,F)+m}$ for some $m \ge 0$ and H is not isomorphic to the Turán hypergraph $\mathcal{T}^{(k)}_\ell(n)$, then there exists an induced subhypergraph H' of H on n' vertices with $n' \ge n-2$ and

$$c_{3,F}(H') \ge 3^{\mathrm{ex}(n',F)+m+1}. \tag{5}$$

Inductively, for n sufficiently large, we arrive at some subhypergraph H_0 of H on at most n_0 vertices that admits at least $3^{\mathrm{ex}(n_0,F)+\binom{n_0}{k}+1}$ many F-free 3-colorings of its set of hyperedges, which is impossible and yields a contradiction. Thus, it is left to verify Lemma 6, however, this is the major part of the proof. □

Before proving Lemma 6 we introduce some further notation.

We often will be interested in how hyperedges in the link of a vertex intersect a particular partition. For a k-uniform hypergraph $H = (V, E)$, a partition $\mathcal{P}$ of its vertex set with $V(H) = V_1 \dot\cup \cdots \dot\cup V_\ell$ into ℓ mutually disjoint classes, and any vertex $v \in V$, we distinguish between three different types of hyperedges incident to v. Let $v \in V_j$ be a vertex for some $j \in [\ell]$. We refer to those hyperedges $e \in E(H)$ incident to vertex v and intersecting every class $V_i, i \in [\ell]$, in at most one vertex as *crossing* hyperedges. Moreover, hyperedges incident to vertex v, that intersect class V_j in exactly one further vertex different from v and else intersect any other class $V_i, i \in [\ell] \setminus \{j\}$, in at most one vertex are referred to as *defective* hyperedges. Finally, all remaining hyperedges incident to vertex v are called *bad* hyperedges. More formally, crossing hyperedges incident to vertex v form the following subset of $E(H)$:

$$E_{\mathrm{cross}}(v) := \{e \in E(H)\colon\ v \in e \text{ and } \forall i \in [\ell] : |e \cap V_i| \le 1\},$$

while the set of defective hyperedges incident to vertex $v \in V_j$ is

$$E_{\mathrm{defect}}(v) := \{e \in E(H)\colon\ v \in e \text{ and } |e \cap V_j| = 2 \text{ and } \forall i \in [\ell] \setminus j : |e \cap V_i| \le 1\,\}.$$

Let $E_{\mathrm{bad}}(v) = \{e \in E(H)\colon\ v \in e\} \setminus (E_{\mathrm{cross}}(v) \dot\cup E_{\mathrm{defect}}(v))$ be the set of bad hyperedges, or, equivalently, for $v \in V_j$:

$$E_{\mathrm{bad}}(v) := \{e \in E(H)\colon\ v \in e \text{ and, } |e \cap V_j| \ge 3 \text{ or } \exists i \in [\ell] \setminus \{j\} \text{ with } |e \cap V_i| \ge 2\}.$$

Let $\tau\colon [\ell] \longrightarrow \{0, 1, \ldots, k\}$ be a function such that $\sum_{i=1}^{\ell} \tau(i) = k$. Then, for a k-element subset (hyperedge) e of V we say that e is of type τ, if $|e \cap V_i| = \tau(i)$ for each $i \in [\ell]$. We thus may formulate different types of hyperedges by specifying the types of these hyperedges. Therefore, for example, a crossing hyperedge has type τ, where k elements of $[\ell]$ are mapped to 1 and the remaining $(\ell - k)$ are mapped to 0. Note that there are $\binom{k+\ell-1}{\ell-1}$ distinct types of hyperedges with respect to the partition $V(H) = V_1 \dot\cup \cdots \dot\cup V_\ell$, provided $V_i \neq \varnothing$ for each $i \in [\ell]$. We also occasionally say a type τ intersects the class V_i if $\tau(i) \geq 1$.

For a vertex $v \in V(H)$ and a type τ associated with v we write $\deg^\tau(v) := |E^\tau(v)|$, where $E^\tau(v)$ denotes the set of all those hyperedges of type τ in H, which are incident to vertex v.

We also speak about crossing, defective or bad types, when we consider types of crossing, defective or bad hyperedges, respectively. Note however, a hyperedge might be of bad or defective type depending on the vertex under consideration incident to it.

Recall that the set $E_{\text{cross}}(\mathcal{P})$ contains all crossing hyperedges in the hypergraph H with respect to a partition $\mathcal{P}$, and $E_{\text{noncross}}(\mathcal{P}) := E(H) \setminus E_{\text{cross}}(\mathcal{P})$ is the set of all *non-crossing* hyperedges in H, with $e_{\text{noncross}}(\mathcal{P}) := |E_{\text{noncross}}(\mathcal{P})|$.

For $t \in [k]$, and t pairwise distinct vertices $v_1, \ldots, v_t$ let $L_H(v_1, \ldots, v_t)$ be the set of all $(k-t)$-element subsets $S \subseteq V(H)$, such that $v_1, \ldots, v_t$ together with S form a hyperedge in the k-uniform hypergraph H, i.e.,

$$L_H(v_1, \ldots, v_t) = \{e \setminus \{v_1, \ldots, v_t\} : e \in E(H) \text{ and } v_1, \ldots, v_t \in e\}.$$

We call $L_H(v_1, \ldots, v_t)$ the $(k-t)$-uniform *common link hypergraph* of the vertices $v_1, \ldots, v_t$, or *common link graph* if $k - t = 2$.

Proof of Lemma 6. Let $F = F^k_{\ell+1}$ or $F = H^k_{\ell+1}$, $2 \leq k \leq \ell$. Let H be a k-uniform hypergraph on n vertices, $H \neq \mathcal{T}^{(k)}_\ell(n)$ and let $c_{3,F}(H) \geq 3^{\text{ex}(n,F)+m}$ with $m \geq 0$, which implies $e(H) \geq e(\mathcal{T}^{(k)}_\ell(n))$.

Without loss of generality we may assume that the minimum degrees of the hypergraph H and the Turán hypergraph $\mathcal{T}^{(k)}_\ell(n)$ satisfy

$$\delta(H) \geq \delta(\mathcal{T}^{(k)}_\ell(n)). \tag{6}$$

Otherwise, let v be a vertex of minimum degree in H and consider the sub-hypergraph $H' := H - \{v\}$. Since $e(\mathcal{T}(n-1)) = e(\mathcal{T}^{(k)}_\ell(n)) - \delta(\mathcal{T}^{(k)}_\ell(n)) \leq e(\mathcal{T}^{(k)}_\ell(n)) - (\delta(H)+1)$ we infer

$$c_{3,F}(H') \geq \frac{c_{3,F}(H)}{3^{\delta(H)}} = 3^{e(\mathcal{T}^{(k)}_\ell(n)) - \delta(H) + m} \geq 3^{e(\mathcal{T}^{(k)}_\ell(n-1)) + m + 1}, \tag{7}$$

that is, by deleting vertex v from H and all hyperedges incident to it, by a simple averaging argument we obtain (7), and hence (5).

Consider a partition $\mathcal{P}$ with $V(H) = V_1 \dot\cup \cdots \dot\cup V_\ell$, that maximizes the number of crossing hyperedges in H. Let $e_{\text{cross}}(\mathcal{P})$ be this maximum number of crossing hyperedges in H. Therefore, this partition $\mathcal{P}$ minimizes the total number of bad and defective hyperedges. Moreover, since $c_{3,F}(H) \geq 3^{\text{ex}(n,F)}$ by assumption, by Theorem 2 we know, by our choice of $\delta > 0$ that

$$e(H) - e_{\text{cross}}(\mathcal{P}) < \delta \cdot n^k, \tag{8}$$

which gives an upper bound on the number of hyperedges in H. Since $e(H) \geq e(\mathcal{T}^{(k)}_\ell(n))$, we have $e_{cross}(\mathcal{P}) > e(\mathcal{T}^{(k)}_\ell(n)) - \delta \cdot n^k$, i.e., less than $\delta \cdot n^k$ crossing hyperedges are missing in H. With $e(H) \geq e(\mathcal{T}^{(k)}_\ell(n))$ and (8), we obtain the following lower and upper bounds on the sizes of the classes V_i:

Lemma 7. *For each $i \in [\ell]$ it is*

$$n/\ell - (\ell-1)\cdot\delta^{1/k}\cdot n \le |V_i| \le n/\ell + \ell^2\cdot\delta^{1/k}\cdot n. \tag{9}$$

Proof. Assume that $|V_i| = n/\ell - (\ell-1)\cdot x\cdot n/\ell$ for some $i \in [\ell]$ and $x \ge 0$. As a product with a fixed number of terms with given sum is maximal if all are equal, with Pascal's identity, i.e., $\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}$, we must have (neglecting roundings) for the number of crossing hyperedges in H:

$$\begin{aligned}
&\binom{\ell-1}{k-1}\cdot\left(\frac{n}{\ell} - \frac{(\ell-1)\cdot x\cdot n}{\ell}\right)\cdot\left(\frac{n}{\ell}+\frac{x\cdot n}{\ell}\right)^{k-1} + \binom{\ell-1}{k}\cdot\left(\frac{n}{\ell}+\frac{x\cdot n}{\ell}\right)^{k}\\
&\ge \binom{\ell}{k}\cdot\left(\frac{n}{\ell}\right)^k - \delta\cdot n^k\\
&\iff \binom{\ell-1}{k-1}\cdot(1-(\ell-1)\cdot x)\cdot(1+x)^{k-1} + \binom{\ell-1}{k}\cdot(1+x)^k \ge \binom{\ell}{k} - \delta\cdot\ell^k\\
&\iff \binom{\ell-1}{k-1}\cdot\left(1+\sum_{i=1}^{k-1}\left(\binom{k-1}{i} - (\ell-1)\cdot\binom{k-1}{i-1}\right)\cdot x^i - (\ell-1)\cdot x^k\right) +\\
&+\binom{\ell-1}{k}\cdot\left(1+\sum_{i=1}^{k-1}\binom{k}{i}\cdot x^i + x^k\right) \ge \binom{\ell}{k} - \delta\cdot\ell^k\\
&\iff \binom{\ell-1}{k-1} + \binom{\ell-1}{k} - \left((\ell-1)\cdot\binom{\ell-1}{k-1} - \binom{\ell-1}{k}\right)\cdot x^k +\\
&+\sum_{i=1}^{k-1}\binom{k-1}{i-1}\cdot\left(\binom{\ell-1}{k}\cdot\frac{k}{i} + \binom{\ell-1}{k-1}\cdot\frac{k-i}{i} - (\ell-1)\cdot\binom{\ell-1}{k-1}\right)\cdot x^i\\
&\ge \binom{\ell}{k} - \delta\cdot\ell^k\\
&\iff \sum_{i=1}^{k-1}\binom{\ell-1}{k-1}\cdot\binom{k-1}{i-1}\cdot\left(\frac{\ell}{i}-\ell\right)\cdot x^i \ge \binom{\ell}{k}\cdot(k-1)\cdot x^k - \delta\cdot\ell^k.
\end{aligned} \tag{10}$$

With $\ell \ge k \ge 2$ and $(\ell/i - \ell) \le 0$ for $i \ge 1$, we conclude from (10) that $\delta\cdot\ell^k \ge x^k$, thus $x \le \ell\cdot\delta^{1/k}$, i.e., $|V_i| \ge n/\ell - (\ell-1)\cdot\delta^{1/k}\cdot n$ for all $i \in [\ell]$.

Now, assume that for some $i \in [\ell]$ we have $|V_i| \ge n/\ell + (\ell-1)^2\cdot x\cdot n/\ell$ for some $x \ge 0$. Then, there must exist some class V_j, $j \in [\ell]$, with $|V_j| \le n/\ell - (\ell-1)\cdot x\cdot n/\ell$, hence the considerations from above apply, and we conclude $x \le \ell\cdot\delta^{1/k}$, i.e., $|V_i| \le n/\ell + \ell^2\cdot\delta^{1/k}$, which gives (9). □

In the following our argument splits into three cases depending on the link hypergraph of a vertex. First we assume that there exists a vertex v incident to at least $\beta\cdot n^{k-1}$ *bad* hyperedges with respect to the partition $\mathcal{P}$ (Case 1). If this is not the case, then we assume that there exists a vertex v, which is incident to at least $\beta\cdot n^{k-1}$ *defective* hyperedges with respect to the partition $\mathcal{P}$ (Case 2). Finally, if neither Case 1 nor Case 2 holds, we deal with Case 3, where every vertex is adjacent to at most $2\cdot\beta\cdot n^{k-1}$ many *defective* or *bad* hyperedges with respect to the partition $\mathcal{P}$. Thus, in Case 3 by the assumption (6) on the high minimum degree of the hypergraph H, with $0 < \beta \ll 1$ we know that every vertex is adjacent mostly to *crossing* hyperedges with respect to the partition $\mathcal{P}$.

We will omit the explicit setting of the small parameters δ, β and also ε as well (which appears at a later stage of the proof). It is sufficient to keep in mind that

$$0 < \delta \ll \beta \ll \varepsilon \ll 1. \tag{11}$$

Obviously, for the partition $\mathcal{P}$ and any vertex v there are exactly

- $c_\ell := \binom{\ell-1}{k-1}$ types of crossing hyperedges incident to v, and
- $d_\ell := \binom{\ell-1}{k-2}$ types of defective hyperedges incident to v, and
- $b_\ell := \binom{k+\ell-2}{k-1} - \binom{\ell}{k-1}$ types of bad hyperedges incident to v.

The further organization of the proof is, that every case is presented in its own subsection.

3.0.1. *Case 1: H satisfies $\exists i \in [\ell]$ and $\exists v \in V_i$: $|E_{\text{bad}}(v)| \geq \beta \cdot n^{k-1}$.* Recall, that $e(H) \geq \text{ex}(n, F)$ and that by assumption the hypergraph H is not isomorphic to the Turán hypergraph $\mathcal{T}_\ell^{(k)}(n)$, thus H contains at least one subhypergraph F. Indeed, by the assumption $c_{3,F}(H) \geq 3^{\text{ex}(n,F)+m}$ for some $m \geq 0$, it turns out that Case 1 never holds for appropriately chosen small $\beta > 0$.

Assume without loss of generality that $i = 1$, and let $v \in V_1$ be a vertex such that $|E_{\text{bad}}(v)| \geq \beta \cdot n^{k-1}$. There are at most b_ℓ types of *bad* hyperedges incident to vertex v. Thus, for at least one type τ we know that $|E^\tau(v)| \geq \beta \cdot n^{k-1}/b_\ell$. By an averaging argument, there exist $(k-3)$ distinct vertices $w_1, \ldots, w_{k-3}$, all distinct from vertex v, such that the common link graph

$$L(v, w_1, \ldots, w_{k-3}) = \{e \setminus \{v, w_1, \ldots, w_{k-3}\} \colon\ e \in E^\tau_{bad}(v) \text{ and } w_1, \ldots, w_{k-3} \in e\}$$

contains at least $\beta \cdot n^2/b_\ell$ edges, which all are contained in some class V_j, $j \in [\ell]$, that is, together with any edge from $L(v, w_1, \ldots, w_{k-3})$, the vertices v and $w_1, \ldots, w_{k-3}$ form a hyperedge of type τ in H. Then, greedily we can find a matching $M \subseteq L(v, w_1, \ldots, w_{k-3})$, hence $M \subseteq [V_j]^2$, of size $m \geq \beta \cdot n/(2 \cdot b_\ell)$. Let

$$M := \{\{a_1, b_1\}, \ldots, \{a_m, b_m\}\}$$

and $e_s := \{a_s, b_s\}$, $s \in [m]$, where without loss of generality $m \leq n/(5 \cdot \ell)$, otherwise we delete some edges from M.

First we deal with the case when $F = H^k_{\ell+1}$. We know by (9) that, for $0 < \delta \leq (1/(5 \cdot \ell^2))^k$ and for n sufficiently large, every class V_i, $i \in [\ell]$, has size at least $4 \cdot n/(5 \cdot \ell)$, thus we may select from every class V_i, $i \neq j$, two disjoint subsets A_i and B_i each of size $n/(3 \cdot \ell)$, where both A_i and B_i are disjoint from the set $\{v, w_1, \ldots, w_{k-3}\}$. Moreover, we define for the class V_j the sets $A_j := \{a_1, \ldots, a_m\}$ and $A^*_j := \{b_1, \ldots, b_m\}$ and a subset $B_j \subset V_j$ of size $n/(3 \cdot \ell)$, which is disjoint from $A_j \cup A^*_j \cup \{v, w_1, \ldots, w_{k-3}\}$.

We want to find $\Theta(n^k)$ copies of $H^k_{\ell+1}$ (these need not be subhypergraphs $H^k_{\ell+1}$ in H, as some hyperedges might be missing), such that, on average, only $\Theta(n^{k-1})$ of the copies share some hyperedge (which contains some matching edge from M), and moreover these copies are "almost" hyperedge-disjoint from other copies. More precisely, we show:

Lemma 8. *There exists a family $\mathcal{F}$ of subhypergraphs $H^k_{\ell+1}$ in H with the following properties:*

- $\mathcal{F} = \mathcal{F}_1 \dot\cup \cdots \dot\cup \mathcal{F}_m$ *with* $|M| = m \geq \beta \cdot n/(2 \cdot b_\ell)$, *and*
- $|\mathcal{F}| \geq \frac{\beta}{3^{k+\ell-3}\ell^{\ell+3k-5} \cdot 4 \cdot b_\ell} \cdot n^k$, *and*

- *for all $F_1 \in \mathcal{F}_s$ and $F_2 \in \mathcal{F}_t$ with $s \neq t$ it is $E(F_1) \cap E(F_2) = \varnothing$, i.e., any two subhypergraphs $H^k_{\ell+1}$ from different subfamilies do not have any hyperedges in common, and*
- *for all $F_1, F_2 \in \mathcal{F}_s$, $s \in [m]$, it is $E(F_1) \cap E(F_2) = e = \{v, w_1, \ldots, w_{k-3}\} \dot\cup e_s$ with $e_s \in M$ and $e \in E$, and we call the hyperedge e the* common *hyperedge of the subfamily $\mathcal{F}_s$.*

Proof. We use in our arguments the following simple claim:

Claim 9. *Let G be the complete, r-partite, r-uniform hypergraph with classes of sizes $c_i \cdot N$ for constants $0 < c_i \leq 1$, $i \in [r]$.*

Then, there exists a linear subhypergraph $\mathcal{G}$ of G with at least $(N^2/r^2) \cdot \prod_{i=1}^r c_i$ hyperedges.

Proof. Given the complete, r-partite, r-uniform hypergraph as specified in the assumption, we start by picking any hyperedge f from G and delete all hyperedges from G that intersect f in at least 2 vertices. Then, we repeat this procedure with the resulting subhypergraph until the remaining hyperedges pairwise meet in at most one vertex. In each step we delete at most $\binom{r}{2} \cdot N^{r-2}$ hyperedges. This way, we clearly find a linear subhypergraph $\mathcal{G}$ with at least $(N^2/r^2) \cdot \prod_{i=1}^r c_i$ hyperedges. □

For $r = \ell$, we apply Claim 9 to the complete, ℓ-partite, ℓ-uniform hypergraph with vertex classes $A_1, \ldots, A_\ell$, where $|A_i| = n/(3 \cdot \ell)$ for each $i \neq j$ and $|A_j| = m \geq \beta \cdot n/(2 \cdot b_\ell)$, and we obtain a linear family $\mathcal{G}$ on $A_1 \dot\cup \cdots \dot\cup A_\ell$ with

$$|\mathcal{G}| \geq \frac{\beta}{2 \cdot \ell^2 \cdot b_\ell} \cdot \left(\frac{1}{3 \cdot \ell}\right)^{\ell-1} \cdot n^2. \tag{12}$$

Let $B = B_1 \dot\cup \cdots \dot\cup B_\ell$, i.e., $|B| = n/3$ since $|B_i| = n/(3 \cdot \ell)$, $i \in [\ell]$, and note that by the choice of the sets A_i, B_i, $i \in [\ell]$, we have $B \cap (A_1 \cup \cdots \cup A_\ell) = \varnothing$. Partition the set B into $(\binom{\ell+1}{2} - 1)$ mutually disjoint subsets $B_{(x,y)}$, $1 \leq x < y \leq \ell + 1$ but $(x, y) \neq (j, j + 1)$, each of size

$$|B_{(x,y)}| \geq \frac{n}{3 \cdot \ell^2} \quad \text{such that} \quad |B_{(x,y)} \cap B_i| \geq \frac{n}{3 \cdot \ell^3} \text{ for all } i \in [\ell],$$

and set

$$T := \left(\frac{n}{3 \cdot \ell^3}\right)^{k-2}. \tag{13}$$

For each pair (x, y), $1 \leq x < y \leq \ell + 1$ but $(x, y) \neq (j, j + 1)$, choose T pairwise distinct $(k - 2)$-element subsets from the set $B_{(x,y)}$, crossing with respect to the partition $\mathcal{P}$, and enumerate these as $u_{(x,y)}(1), \ldots, u_{(x,y)}(T)$.

With any hyperedge $e \in \mathcal{G}$ we associate an ℓ-tuple $\hat{e}$, such that for $\hat{e} = (v_1, \ldots, v_\ell)$, we have $e = \{v_1, \ldots, v_\ell\} \in \mathcal{G}$, where $v_i \in A_i$ for $i \in [\ell]$ (and therefore $v_j = a_s$ for some $s \in [m]$). We enlarge $\hat{e}$ by the vertex b_s to $\hat{e}^* = (v_1, \ldots, v_{j-1}, a_s, b_s, v_{j+1}, \ldots, v_\ell)$, where $\{a_s, b_s\}$ is an edge from the matching M. Let $(\hat{e}^*)_i$ denote the entry of $\hat{e}^*$ in coordinate i, $i \in [\ell + 1]$. Fix some integer $p \in [T]$. For every pair (x, y) of integers, $1 \leq x < y \leq \ell + 1$ but $(x, y) \neq (j, j + 1)$, we enlarge the 2-element set $\{(\hat{e}^*)_x, (\hat{e}^*)_y\}$ by the $(k - 2)$-element set $u_{(x,y)}(p)$ to a k-element set. Moreover, we extend the 2-element set $\{a_s, b_s\}$ by the $(k-2)$-element set $\{v, w_1, \ldots, w_{k-3}\}$, which is a hyperedge in H. These $\binom{\ell+1}{2}$ many $(k - 2)$-element sets $u_{(x,y)}(p)$ and $\{v, w_1, \ldots, w_{k-3}\}$ are pairwise disjoint by construction, and we obtain a copy $H(\hat{e}, p)$ of $H^k_{\ell+1}$ with core $\hat{e}^*$ in the complete ℓ-partite k-uniform hypergraph $K[V_1, \ldots, V_\ell]$ (on the same vertex set as H).

We construct this way such copies $H(\hat{e}, p)$ of $H^k_{\ell+1}$ for every $e \in \mathcal{G}$ and every $p \in [T]$. For $s \in [m]$, define the families $\mathcal{F}_s := \{H(\hat{e}, p) : e \in \mathcal{G}, p \in [T], (\hat{e})_j = a_s\}$.

We claim that distinct copies $H(\hat{e}, p)$ and $H(\hat{e}', p')$ of $H^k_{\ell+1}$ from the same subfamily $\mathcal{F}_s$ intersect in the k-element set $\{a_s, b_s, v, w_1, \ldots, w_{k-3}\} \in E$ only, while copies $H(\hat{e}, p)$ and $H(\hat{e}', p')$ of $H^k_{\ell+1}$ from distinct subfamilies $\mathcal{F}_s$ and $\mathcal{F}_t$, $s \neq t$, respectively, do not have any k-element set in common.

Namely, if $e, e' \in \mathcal{G}$, where $(\hat{e})_j \neq (\hat{e}')_j$, then by construction $|e \cap e'| \leq 1$, and thus for any $p, p' \in [1, T]$ the copies $H(\hat{e}, p)$ and $H(\hat{e}', p')$ of $H^k_{\ell+1}$ do not have any k-element set in common.

Now let $e, e' \in \mathcal{G}$ with $(\hat{e})_j = (\hat{e}')_j = a_s$. If $e = e'$, and $p \neq p'$, then the copies $H(\hat{e}, p)$ and $H(\hat{e}, p')$ of $H^k_{\ell+1}$ intersect in the k-element set $\{a_s, b_s, v, w_1, \ldots, w_{k-3}\}$ only, as $u_{(x,y)}(p) \neq u_{(x,y)}(p')$ for all (x, y), $1 \leq x < y \leq \ell + 1$ and $(x, y) \neq (j, j + 1)$, and as the sets $B_{(x,y)}$ are pairwise disjoint. If $e \neq e'$ and $(\hat{e})_j = (\hat{e}')_j = a_s$, then for any $p, p' \in [T]$, with $|e \cap e'| = 1$ again we infer that the copies $H(\hat{e}, p)$ and $H(\hat{e}, p')$ of $H^k_{\ell+1}$ only intersect in the k-element set $\{a_s, b_s, v, w_1, \ldots, w_{k-3}\}$, which is a hyperedge in H.

Thus, using (12) and (13) we have found at least

$$T \cdot |\mathcal{G}| \geq \left(\frac{n}{3 \cdot \ell^3}\right)^{k-2} \cdot \left(\frac{1}{3 \cdot \ell}\right)^{\ell-1} \cdot \frac{\beta}{2 \cdot \ell^2 \cdot b_\ell} \cdot n^2 = \frac{\beta}{3^{k+\ell-3} \cdot \ell^{\ell+3k-5} \cdot 2 \cdot b_\ell} \cdot n^k$$

copies of $H^k_{\ell+1}$ in $K[V_1, \ldots, V_\ell]$. However, not all of these copies of $H^k_{\ell+1}$ might be present in H as subhypergraphs, as by (8) some but at most $\delta \cdot n^k$ crossing hyperedges are missing in H. But as all common hyperedges $\{a_s, b_s, v, w_1, \ldots, w_{k-3}\}$, $s \in [m]$, are present in H, we obtain for $0 < \delta \ll \beta$ at least

$$\frac{\beta}{3^{k+\ell-3} \cdot \ell^{\ell+3k-5} \cdot 2 \cdot b_\ell} \cdot n^k - \delta \cdot n^k \overset{(11)}{\geq} \frac{\beta}{3^{k+\ell-3} \cdot \ell^{\ell+3k-5} \cdot 4 \cdot b_\ell} \cdot n^k$$

subhypergraphs $H^k_{\ell+1}$ in H with the desired properties, as claimed in Lemma 8. $\square$

Next we consider the case $F = F^k_{\ell+1}$.

Lemma 10. *There exists a family $\mathcal{F}$ of subhypergraphs $F^k_{\ell+1}$ in the hypergraph H with the following properties:*

- *$\mathcal{F} = \mathcal{F}_1 \dot\cup \cdots \dot\cup \mathcal{F}_m$ with $|M| = m = \beta \cdot n/(2 \cdot b_\ell)$, and*
- *$|\mathcal{F}| \geq \frac{1}{53 \cdot \binom{\ell}{k} \cdot (2 \cdot b_\ell)^k} \cdot \beta^k \cdot n^k$, and*
- *for all $F_1 \in \mathcal{F}_s$ and $F_2 \in \mathcal{F}_t$ with $s \neq t$ it is $E(F_1) \cap E(F_2) = \varnothing$, i.e., any two subhypergraphs $F^k_{\ell+1}$ from different subfamilies do not have any hyperedges in common, and*
- *for all $F_1, F_2 \in \mathcal{F}_s$, $s \in [m]$, it is $E(F_1) \cap E(F_2) = e = \{v, w_1, \ldots, w_{k-3}\} \dot\cup e_s$ with $e_s \in M$ and $e \in E$, and we call the hyperedge e the* common *hyperedge of the subfamily $\mathcal{F}_s$.*

Proof. For the proof we use the following claim

Claim 11. *Let $r \geq k \geq 3$ be integers. Let $0 < c < 1/(12 \cdot \binom{r}{k})$ be a constant. Let G be the complete, r-partite, r-uniform hypergraph with classes each of size N.*

Then, there exists a subhypergraph $\mathcal{G}$ of G with at least $c \cdot N^k/4$ hyperedges such that

- *distinct hyperedges $e, e' \in E(\mathcal{G})$ do not have any k-element subset in common, and*
- *for each two-element set $\{v, w\}$ of vertices in $V(\mathcal{G})$ there are at most $2 \cdot c \cdot N^{k-2}$ hyperedges $e \in E(\mathcal{G})$, which contain $\{v, w\}$.*

Proof. We show the existence of the subhypergraph $\mathcal{G}$ by a probabilitic argument.

With probability $p = c/N^{r-k}$ for some constant $c > 0$ we pick uniformly at random and independently of each other hyperedges from G. Let S be the random variable counting the number of chosen hyperedges. Then, the expected number $E[S]$ satisfies

$$\mathbb{E}[S] = p \cdot N^r = c \cdot N^k.$$

The random variable S is binomially distributed hence by Chernoff's inequality, see e.g. [12, Theorem 2.1], $\mathbb{P}(\mathbb{E}[S] - S > \alpha \cdot \mathbb{E}[S]) \leq e^{-\alpha^2 \mathbb{E}[S]/2}$ for $0 < \alpha < 1$, we infer for N large enough

$$\mathbb{P}(S < c \cdot N^k/2) < e^{-cN^k/8} < \frac{1}{3}. \tag{14}$$

Let P count the number of pairs of chosen r-element subsets which have a k-element set in common. Then we infer for the expectation $\mathbb{E}[P]$

$$\mathbb{E}[P] \leq \binom{r}{k} \cdot N^k \cdot (N^{r-k})^2 \cdot p^2 = c^2 \cdot \binom{r}{k} \cdot N^k,$$

and by Markov's inequality we have

$$\mathbb{P}(P > 3 \cdot c^2 \cdot \binom{r}{k} \cdot N^k) < \frac{1}{3}. \tag{15}$$

Now, for a fixed pair $\{v, w\}$ of vertices from different classes, let $R_{v,w}$ be the random variable counting the number of r-element sets, which contain both vertices v and w, hence

$$\mathbb{E}[R_{v,w}] = p \cdot N^{r-2} = c \cdot N^{k-2}.$$

The random variable $R_{v,w}$ is binomially distributed, hence by Chernoff's inequality, we infer

$$\mathbb{P}(R_{v,w} > 2 \cdot \mathbb{E}[R_{v,w}]) < e^{-(c/3)N^{k-2}}.$$

From this we conclude for the probability that there exists a pair $\{v, w\}$ of vertices from different classes which is contained in more than $2 \cdot \mathbb{E}[R_{v,w}]$ random r-element sets, the upper bound

$$\binom{r}{2} \cdot N^2 \cdot e^{-(c/3)N^{k-2}} < \frac{1}{3} \tag{16}$$

for n large enough.

By (14)–(16) we infer that there exists a family $\mathcal{G}'$ of r-element subsets of size at least $c \cdot N^k/2$, where the number of distinct r-element sets in $\mathcal{G}'$, which have a k-element set in common, is at most $3 \cdot c^2 \cdot \binom{r}{k} \cdot N^k$ and where every pair $\{v, w\}$ of distinct vertices is contained in at most $2 \cdot c \cdot N^{k-2}$ r-element sets of $\mathcal{G}'$.

For $c < 1/(12 \cdot \binom{r}{k})$ we delete from each pair of distinct r-element sets, which have a k-element set in common, one of the r-element sets, and we obtain a subfamily $\mathcal{G} \subseteq \mathcal{G}'$ of r-element sets of size at least $c \cdot N^k/4$, which pairwise do not have a k-element set in common and where every pair $\{v, w\}$ of distinct vertices is contained in at most $2 \cdot c \cdot N^{k-2}$ r-element sets of $\mathcal{G}$. □

We know by (9) that, for $0 < \delta \leq (1/(5 \cdot \ell^2))^k$ and for n sufficiently large, every class V_i, $i \in [\ell]$, has size at least $4 \cdot n/(5 \cdot \ell)$, thus we may select from every class V_i, $i \neq j$, two disjoint subsets A_i, and B_i, where each set A_i has size m, and each set B_i has size $n/(3 \cdot \ell)$, and both A_i and B_i are disjoint from the set $\{v, w_1, \ldots, w_{k-3}\}$. Moreover, we select from the class V_j the sets $A_j = \{a_1, \ldots, a_m\}$ and $A_j^* = \{b_1, \ldots, b_m\}$ and a subset $B_j \subset V_j$ of size $n/(3 \cdot \ell)$, which is disjoint from $A_j \cup A_j^* \cup \{v, w_1, \ldots, w_{k-3}\}$.

For $r = \ell$, we apply Claim 11 to the complete, ℓ-partite, ℓ-uniform hypergraph with vertex classes $A_1, \ldots, A_\ell$, where $|A_i| = m := \beta \cdot n/(2 \cdot b_\ell)$, with $c = 1/(13 \cdot \binom{\ell}{k})$ and we obtain a family $\mathcal{G}$ on $A_1 \dot\cup \cdots \dot\cup A_\ell$ of crossing ℓ-element sets with

$$|\mathcal{G}| = \frac{1}{52 \cdot \binom{\ell}{k}} \cdot m^k,$$

where pairwise the hyperedges in $\mathcal{G}$ do not have any k-element subset in common, and where every pair v, w of distinct vertices in $V(\mathcal{G})$ is contained in at most $(2/(13 \cdot \binom{\ell}{k})) \cdot m^{k-2}$ many hyperedges from $\mathcal{G}$.

For each hyperedge g in $\mathcal{G}$, we construct copies of the hypergraph $F_{\ell+1}^k$ with core containing the set g. Enumerate the ℓ-element sets (hyperedges) in $\mathcal{G}$ by $g_1, \ldots, g_{(1/(52\binom{\ell}{k}))m^k}$. For each ℓ-element set g_i, $i \in [(1/(52 \cdot \binom{\ell}{k})) \cdot m^k]$ choose a k-element subset K_i with $|K_i \cap A_j| = 1$. This k-element set is the core-hyperedge of g_i.

Let $B = B_1 \dot\cup \cdots \dot\cup B_\ell$, i.e., $|B| = n/3$ since $|B_i| = n/(3 \cdot \ell)$, $i \in [\ell]$, and note that $B \cap (A_1 \cup \cdots \cup A_\ell) = \varnothing$. Partition the set B into $(\binom{\ell+1}{2} - 1)$ mutually disjoint subsets $B_{(x,y)}$, $1 \leq x < y \leq \ell + 1$ but $(x, y) \neq (j, j + 1)$, each of size

$$|B_{(x,y)}| \geq \frac{n}{3 \cdot \ell^2} \quad \text{such that} \quad |B_{(x,y)} \cap B_i| \geq \frac{n}{3 \cdot \ell^3} \text{ for all } i \in [\ell],$$

which leads to at least

$$\left(\frac{n}{3 \cdot \ell^3}\right)^{k-2} \gg \left(\frac{2}{13 \cdot \binom{\ell}{k}}\right) \cdot m^{k-2}$$

distinct crossing $(k-2)$-element sets.

For each pair (x, y), $1 \leq x < y \leq \ell+1$ but $(x, y) \neq (j, j+1)$, choose $(2/(13 \cdot \binom{\ell}{k})) \cdot m^{k-2}$ pairwise distinct $(k-2)$-element subsets from the set $B_{(x,y)}$, and enumerate these as $u_{(x,y)}(1), \ldots, u_{(x,y)}((2/(13 \cdot \binom{\ell}{k}))m^{k-2})$.

For each pair v, w of distinct vertices enumerate all ℓ-subsets in $\mathcal{G}$, which contain both vertices v and w by $L_{(v,w)}(1), \ldots, L_{(v,w)}(n(v, w))$, where $n(v, w) \leq 2/(13 \cdot \binom{\ell}{k}) \cdot m^{k-2}$.

With any hyperedge $g_p \in \mathcal{G}$ we associate an ℓ-tuple $\hat{g}_p$, such that for $\hat{g}_p = (v_1, \ldots, v_\ell)$, we have $g_p = \{v_1, \ldots, v_\ell\} \in \mathcal{G}$, where $v_i \in A_i$ for $i \in [\ell]$ (and therefore $v_j = a_s$ for some $s \in [m]$). We enlarge $\hat{g}_p$ by the vertex b_s to $\hat{g}_p^* = (v_1, \ldots, v_{j-1}, a_s, b_s, v_{j+1}, \ldots, v_\ell)$, where $\{a_s, b_s\}$ is an edge from the matching M. Let $(\hat{g}_p^*)_i$ denote the entry of $\hat{g}_p^*$ in coordinate i, $i \in [\ell + 1]$.

For each pair v, w from $\hat{g}_p^*$ of distinct vertices with $v \in A_x$ and $w \in A_y$, $x \neq y$ and $(x, y) \neq (j, j+1)$, hence $\{v, w\} \neq \{a_s, b_s\}$, we extend the 2-element set $\{v, w\}$ by the $(k-2)$-element set $u_{(x,y)}(i)$, if $g_p = L_{(v,w)}(i)$ for some i and not both vertices v and w are contained in the core hyperedge K_p of g_p. Moreover, we extend the 2-element set $\{a_s, b_s\}$ by the $(k-2)$-element set $\{v, w_1, \ldots, w_{k-3}\}$, which is a hyperedge in H.

These $(1 + \binom{\ell+1}{2} - \binom{k}{2})$ many $(k-2)$-element sets $u_{(x,y)}(i)$ and $\{v, w_1, \ldots, w_{k-3}\}$ are pairwise disjoint by construction, and we obtain a copy $F(\hat{g}_p)$ of $F^k_{\ell+1}$ with core K_p.

We construct such copies $F(\hat{g}_p)$ of $F^k_{\ell+1}$ for every $g_p \in \mathcal{G}$. For $s \in [m]$, define the families $\mathcal{F}_s := \{F(\hat{g}_p) : g_p \in \mathcal{G}, (\hat{g}_p)_j = a_s\}$.

By construction, distinct copies $F(\hat{g}_p)$ and $F(\hat{g}'_p)$ of $F^k_{\ell+1}$ from the same subfamily $\mathcal{F}_s$ intersect in the k-element set $\{a_s, b_s, v, w_1, \ldots, w_{k-3}\} \in E(H)$ only, while copies $F(\hat{g}_p)$ and $F(\hat{g}'_p)$ of $F^k_{\ell+1}$ from distinct subfamilies $\mathcal{F}_s$ and $\mathcal{F}_t$, $s \neq t$, respectively, do not have any k-element set in common.

Thus, we have found at least

$$|\mathcal{G}| \geq \frac{1}{52 \cdot \binom{\ell}{k}} \cdot m^k \geq \frac{1}{52 \cdot \binom{\ell}{k}} \cdot \left(\frac{\beta \cdot n}{2 \cdot b_\ell}\right)^k = \frac{\beta^k \cdot n^k}{52 \cdot \binom{\ell}{k} \cdot (2 \cdot b_\ell)^k} \tag{17}$$

copies of $F^k_{\ell+1}$ in the complete ℓ-partite k-uniform hypergraph $K[V_1, \ldots, V_\ell]$ (on the same vertex set as H). Not all of these copies of $F^k_{\ell+1}$ might be present in H as subhypergraphs, as by (8) at most $\delta \cdot n^k$ crossing hyperedges are missing in H. But as all common hyperedges $\{a_s, b_s, v, w_1, \ldots, w_{k-3}\}$, $s \in [m]$, are present in H, we obtain for $0 < \delta \ll \beta$ at least

$$\frac{1}{52 \cdot \binom{\ell}{k} \cdot (2 \cdot b_\ell)^k} \cdot \beta^k \cdot n^k - \delta \cdot n^k \geq \frac{1}{53 \cdot \binom{\ell}{k} \cdot (2 \cdot b_\ell)^k} \cdot \beta^k \cdot n^k$$

subhypergraphs $F^k_{\ell+1}$ in H with the desired properties, as claimed in Lemma 10. □

Now for $F = F^k_{\ell+1}$ or $F = H^k_{\ell+1}$ we show how the existence of the family $\mathcal{F}$, as guaranteed by Lemmas 8 and 10, implies that Case 1 never holds. Let $\mathcal{F}' \subseteq \mathcal{F}$ with $\mathcal{F}'_s \subseteq \mathcal{F}_s$, $s \in [m]$, be a subfamily of size the minimum guaranteed by Lemmas 8 and 10, i.e., for $0 < \beta \ll 1$

$$|\mathcal{F}'| \geq \frac{1}{53 \cdot \binom{\ell}{k} \cdot (2 \cdot b_\ell)^k} \cdot \beta^k \cdot n^k.$$

In what follows we estimate the number of F-free 3-colorings of the set of hyperedges of H. Once we fix the color of the common hyperedge in a subfamily $\mathcal{F}'_s$, $s \in [m]$, we may color the set of remaining hyperedges of any single subhypergraph F in $\mathcal{F}'_s$ in at most $(3^{e(F)-1} - 1)$ instead of at most $3^{e(F)-1}$ ways, as otherwise we obtain a monochromatic subhypergraph F. Applying the same considerations to all common hyperedges in every subfamily $\mathcal{F}'_s$, $s \in [m]$, and the corresponding subhypergraphs F, we obtain the following possibilities for coloring the set of hyperedges of H:

- every common hyperedge in a subfamily $\mathcal{F}'_s$, $s \in [m]$, may be colored in at most 3 ways, and
- by Lemma 10 there exist at least $\frac{1}{53 \cdot \binom{\ell}{k} \cdot (2 \cdot b_\ell)^k} \cdot \beta^k \cdot n^k$ pairwise distinct subhypergraphs F in H, and hence at least

  $$(e(F) - 1) \cdot \frac{1}{53 \cdot \binom{\ell}{k} \cdot (2 \cdot b_\ell)^k} \cdot \beta^k \cdot n^k$$

  pairwise distinct hyperedges in H distinct from the common hyperedges, such that any of the $(e(F) - 1)$ hyperedges of a single subhypergraph F may be colored in at most $(3^{e(F)-1} - 1)$ instead of at most $3^{e(F)-1}$ ways, and

- finally, the set of the remaining hyperedges may be colored arbitrarily by at most 3 colors.

Thus, for $0 < \delta \ll \beta$ and n sufficiently large, we bound from above the number of hyperedge 3-colorings of H by

$$3^{\mathrm{ex}(n,F)+\delta n^k-(e(F)-1)\cdot\frac{1}{53\cdot\binom{\ell}{k}\cdot(2\cdot b_\ell)^k}\cdot\beta^k\cdot n^k}\cdot(3^{e(F)-1}-1)^{\frac{1}{53\cdot\binom{\ell}{k}\cdot(2\cdot b_\ell)^k}\cdot\beta^k\cdot n^k}$$

$$=3^{\mathrm{ex}(n,F_{\ell+1}k)+\delta n^k}\cdot\left(\frac{3^{e(F)-1}-1}{3^{e(F)-1}}\right)^{\frac{1}{53\cdot\binom{\ell}{k}\cdot(2\cdot b_\ell)^k}\cdot\beta^k\cdot n^k}\ll 3^{\mathrm{ex}(n,F^k_{\ell+1})},$$

which contradicts the assumption $c_{3,F}(H)\geq 3^{\mathrm{ex}(n,F)}$. Therefore, for both cases $F=F^k_{\ell+1}$ or $F=H^k_{\ell+1}$, we have shown that Case 1 never holds.

3.0.2. *Case 2: H satisfies $\exists i\in[\ell]$ and $\exists v\in V_i$: $|E_{\mathrm{defect}}(v)|\geq\beta\cdot n^{k-1}$.* In the following we denote $F_{\ell+1}=H^k_{\ell+1}$ or $F_{\ell+1}=F^k_{\ell+1}$.

We introduce some further notation. For a vertex $v\in V_i$, let $e\in E(H)$ (with $v\in e$) be a hyperedge of some type τ with respect to the partition $\mathcal{P}$. If e intersects the class V_i in a further vertex distinct from v, we say that the vertex v *covers* class V_i *via type* τ and hyperedge e, moreover, we say that vertex v covers class V_i via type τ *by* $|E^\tau(v)|$ hyperedges.

As we are not in Case 1, we know that for each $j\in[\ell]$ and for each vertex $v\in V_j$ we have $|E_{\mathrm{bad}}(v)|\leq\beta\cdot n^{k-1}$. Case 2 asserts the existence of a vertex $v\in V$ such that $|E_{\mathrm{defect}}(v)|\geq\beta\cdot n^{k-1}$ with respect to the maximal partition $\mathcal{P}$. Recall that in the hypergraph H there are at most $c_\ell=\binom{\ell-1}{k-1}$ types of crossing hyperedges and at most $d_\ell=\binom{\ell-1}{k-2}$ types of defective hyperedges incident to the vertex v.

Lemma 12. *Let $\ell\geq 2$ be a fixed integer. Let $\mathcal{P}$ be a partition with $V(H)=V_1\dot\cup\cdots\dot\cup V_\ell$, that maximizes the number of crossing hyperedges in the hypergraph H.*

If there exists a vertex v such that $|E_{\mathrm{defect}}(v)|\geq\beta\cdot n^{k-1}$ with respect to the partition $\mathcal{P}$, then v covers each class V_i, $i\in[\ell]$, via some crossing type by at least

$$\beta\cdot n^{k-1}/(d_\ell\cdot c_\ell) \tag{18}$$

hyperedges.

Proof. Since $|E_{\mathrm{defect}}(v)|\geq\beta\cdot n^{k-1}$, there exists a defective type τ such that $|E^\tau(v)|\geq\beta\cdot n^{k-1}/d_\ell$. Assume without loss of generality that $v\in V_1$ and all hyperedges of type τ incident to vertex v intersect class V_1 in one further vertex, and each class $V_2,\dots,V_{k-1}$ in exactly one vertex. Thus, the remaining classes $V_k,\dots,V_\ell$ are disjoint from hyperedges of type τ incident to vertex v, and v covers via type τ each class $V_1,\dots,V_{k-1}$ by $|E^\tau(v)|\geq\beta\cdot n^{k-1}/d_\ell$ hyperedges. By assumption, vertex v covers class V_1 by at least $\beta\cdot n^{k-1}$ defective hyperedges. If we would move vertex v to some class V_j with $j\geq k$, then, by the maximality of the partition $\mathcal{P}$, we would not increase the number of crossing hyperedges. Therefore, we conclude that $|E_{\mathrm{cross}}(v)|\geq|E^\tau(v)|$, hence

$$|E_{\mathrm{cross}}(v)|\geq\beta\cdot n^{k-1}/d_\ell. \tag{19}$$

Then, there must be a crossing type τ' of hyperedges incident to vertex v, and intersecting class V_j in one vertex, such that $|E^{\tau'}(v)|\geq\beta\cdot n^{k-1}/(c_\ell\cdot d_\ell)$. In fact, we even have $|E^{\tau'}(v)|\geq\beta\cdot n^{k-1}/(d_\ell\cdot\binom{\ell-2}{k-2})$, as moving vertex v to the class V_j

would destroy only those types of crossing hyperedges, which are incident to v and intersect class V_j in one vertex. Thus, we have shown that vertex v covers every class V_i, $i \in [\ell]$, by at least $\beta \cdot n^{k-1}/(d_\ell \cdot c_\ell)$ hyperedges, and (18) follows. □

Next we partition the set $\mathcal{C}$ of allowed 3-colorings of the set of hyperedges of H into two sets $\mathcal{C}_1$ and $\mathcal{C}_2$, i.e., $\mathcal{C} = \mathcal{C}_1 \dot\cup \mathcal{C}_2$. Having shown that the size of $\mathcal{C}_1$ is small, we then concentrate on $\mathcal{C}_2$ and perform the inductive step.

Let $\mathcal{C}_1$ consist of the set of all 3-colorings of the set of hyperedges of H such that vertex v covers every class $V_i, i \in [\ell]$, via some defective (for $i = 1$) or crossing (for $i \geq 2$) type by at least $\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)$ hyperedges, where all these hyperedges are colored the same, i.e., they are all either blue, green, or red.

Fix some coloring c from $\mathcal{C}_1$. Let $E_i(v)$ be the set of all hyperedges incident to vertex v of some type, defective or crossing, such that v covers class V_i by $E_i(v)$, $i \in [\ell]$, with

$$|E_i(v)| \geq \beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell),$$

and all hyperedges in $\cup_{i=1}^{\ell} E_i(v)$ are colored the same, say, in green. Let

$$H_v = (V, \cup_{i=1}^{\ell} E_i(v))$$

denote the subhypergraph of H, containing all these green hyperedges incident to vertex v.

In the following we show the upper bound $|\mathcal{C}_1| \leq 3^{\mathrm{ex}(n, F_{\ell+1})-1}$. The reason for this small size is, that many pairwise hyperedge-disjoint subhypergraphs F_ℓ (and not $F_{\ell+1}$) arise, and for any such subhypergraph F_ℓ its set of hyperedges cannot be colored completely in green, as otherwise we may build together with the green hyperedges from $\dot\cup_{i=1}^{\ell} E_i(v)$ monochromatic subhypergraphs $F_{\ell+1}$.

Let us make this precise. For every class V_i, $i \in [\ell]$, consecutively we fix exactly $(k-2)$ pairwise distinct vertices $w_1^i, \ldots, w_{k-2}^i$, all distinct from vertex v, such that $\cup_{g<i}\{w_1^g, \ldots, w_{k-2}^g\}$ is disjoint from $\{w_1^i, \ldots, w_{k-2}^i\}$, and which satisfy for n sufficiently large

$$|S_i| = |L_{H_v}(v, w_1^i, \ldots, w_{k-2}^i) \cap V_i| \geq \beta \cdot n/(200 \cdot d_\ell \cdot c_\ell),$$

where $S_i := L_{H_v}(v, w_1^i, \ldots, w_{k-2}^i) \cap V_i$; that is, we concentrate on "green neighbourhoods" of vertex v in class V_i, $i \in [\ell]$. Assume in the following for simplicity that $|S_1| = \cdots = |S_\ell| = s := \beta \cdot n/(200 \cdot d_\ell \cdot c_\ell)$ and that s is divisible by $(\ell - 1)$. Consider the on the vertex partition $S_1 \dot\cup \ldots \dot\cup S_\ell$ defined complete ℓ-partite k-uniform hypergraph G. We look for subhypergraphs F_ℓ which are contained in H. Each such subhypergraph F_ℓ, $F_\ell = H_\ell^k$ or $F_\ell = F_\ell^k$, cannot be completely green, as otherwise we obtain in H a green subhypergraph $F_{\ell+1}$ by using the green hyperedges $\{v, w_i, w_1^i, \ldots, w_{k-2}^i\}$, $i \in [\ell]$, where $w_i \in S_i$ is a vertex from the core of F_ℓ.

First we count the number of such hyperedge-disjoint copies of F_ℓ in G (these need not be subhypergraphs in H).

Lemma 13. *The induced ℓ-partite k-uniform hypergraph $H[S_1, \ldots, S_\ell]$ (and therefore the hypergraph H as well) contains at least*

$$c \cdot \beta^k \cdot n^k$$

pairwise hyperedge-disjoint subhypergraphs F_ℓ, where $c > 0$ is a constant depending only on k and ℓ, i.e.,

$$c = \frac{\binom{\ell-1}{k-1}}{2 \cdot k \cdot \ell \cdot e(F_\ell) \cdot (200 \cdot d_\ell \cdot c_\ell)^k}.$$

Proof. For finding these subhypergraphs F_ℓ, we use the bounds on the number of hyperedges in different Turán hypergraphs, see (2). In particular, $\mathcal{T}_\ell^{(k)}(n)$ is extremal for $F_{\ell+1}$ while $\mathcal{T}_{\ell-1}^{(k)}(n)$ is extremal for F_ℓ when $\ell \geq k+1$. Moreover, for $\ell = k$ it follows by a result of Erdős [7], that $\mathrm{ex}(n, H_k^k) = o(n^k)$, while $\mathrm{ex}(n, F_k^k) = 1$ is trivial. Thus we have $\mathrm{ex}(\ell \cdot s, F_{\ell+1}) - \mathrm{ex}(\ell \cdot s, F_\ell) = \Theta(s^k)$ for $\ell \geq k$. Namely, with $\mathrm{ex}(\ell \cdot s, F_{\ell+1}) - \mathrm{ex}(\ell \cdot s, F_\ell) > 0$ we know that the complete ℓ-partite k-uniform hypergraph $G = K[S_1, \ldots, S_\ell]$ contains at least one copy of F_ℓ, and we remove from G all its $e(F_\ell) = |E(F_\ell)|$ many hyperedges. We may repeat this procedure at least $\xi \cdot s^k$ times, where $\xi = \xi(k, \ell) > 0$ is a constant depending only on k and ℓ for $\ell \geq k$, and this ξ can be computed for $\ell > k$ by the bounds in (2), namely

$$\begin{aligned} \mathrm{ex}(\ell \cdot s, F_{\ell+1}) - \mathrm{ex}(\ell \cdot s, F_\ell) &= \binom{\ell}{k} \cdot s^k - \binom{\ell-1}{k} \cdot \left(\frac{\ell \cdot s}{\ell-1}\right)^k \\ &= \frac{\ell}{k} \cdot \binom{\ell-1}{k-1} \cdot s^k - \frac{\ell-k}{k} \cdot \binom{\ell-1}{k-1} \cdot \left(\frac{\ell \cdot s}{\ell-1}\right)^k \\ &= s^k \cdot \frac{\binom{\ell-1}{k-1}}{k} \cdot \left(\ell - (\ell-k) \cdot \left(\frac{\ell}{\ell-1}\right)^k\right) > s^k \cdot \frac{\binom{\ell-1}{k-1}}{k \cdot \ell}, \end{aligned} \quad (20)$$

where the last inequality can be seen as follows: the function $f(k) := (\ell-k) \cdot (\ell/(\ell-1))^k$ is strictly decreasing for $2 \leq k \leq \ell$, which follows from $f(k+1)/f(k) < 1$, hence $\ell - f(k) \geq \ell - f(2) = \ell/(\ell-1)^2 > 1/\ell$.

Having done so, we find in the hypergraph G, assuming that $|S_i| = s$ for every $i \in [\ell]$, at least $\xi \cdot s^k$ hyperedge-disjoint copies of F_ℓ, where by (20) it is

$$\xi > \frac{\binom{\ell-1}{k-1}}{k \cdot \ell \cdot e(F_\ell)}. \quad (21)$$

These $\xi \cdot s^k$ hyperedge-disjoint copies of F_ℓ might not be subhypergraphs F_ℓ in H, as some hyperedges are missing. However, in the hypergraph H less than $\delta \cdot n^k$ crossing hyperedges are missing, hence also in the induced subhypergraph $H[S_1, \ldots, S_\ell]$ of H less than $\delta \cdot n^k$ crossing hyperedges are missing, cf. (8). But with $\delta \ll \beta$, see also (11), we loose only at most half of the copies F_ℓ already found. Thus, we always find at least

$$\xi \cdot s^k - \delta \cdot n^k \geq (\xi/2) \cdot (\beta \cdot n/(200 \cdot d_\ell \cdot c_\ell))^k \geq c \cdot \beta^k \cdot n^k \quad (22)$$

subhypergraphs F_ℓ in H, where $c = (\xi/2) \cdot (1/(200 \cdot d_\ell \cdot c_\ell))^k$. □

Let the pairwise hyperedge-disjoint subhypergraphs of $F_{\ell+1}$ in H, as guaranteed by Lemma 13, be enumerated by $H_1, \ldots, H_{c\beta^k n^k}$. Recall that every subhypergraph H_j, $j \in [c \cdot \beta^k \cdot n^k]$, with core $w_1, \ldots, w_\ell$, $w_i \in S_i$ for $i \in [\ell]$, together with the hyperedges $\{w_i, v, w_1^i, \ldots, w_{k-2}^i\}$, $i \in [\ell]$, builds a subhypergraph $F_{\ell+1}$. Moreover, the latter ℓ hyperedges are all colored the same for *every* coloring $c \in \mathcal{C}_1$. Now, we estimate the number $|\mathcal{C}_1|$ of colorings as follows:

- there are 3 ways to choose the color in which the hyperedges in $\cup_{i=1}^{\ell} E_i(v)$ should be monochromatic, say, in green, and for each class V_i, $i \in [\ell]$, there are at least $\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)$ defective or crossing green hyperedges incident to vertex v that cover class V_i, which yields at most

$$\left(\sum_{i=\beta n^{k-1}/(100 d_\ell c_\ell)}^{\binom{n}{k-1}} \binom{\binom{n}{k-1}}{i} \right)^{\ell} \leq 2^{\ell n^{k-1}}$$

  choices for the green hyperedges in $\cup_{i=1}^{\ell} E_i(v)$, and
- at least $c \cdot \beta^k \cdot n^k$ subhypergraphs F_ℓ together with these green hyperedges yield at least $c \cdot \beta^k \cdot n^k$ copies of $F_{\ell+1}$, and we may therefore color the set of hyperedges in every copy of $F_{\ell+1}$ in at most $(3^{e(F_\ell)} - 1)$ instead of $3^{e(F_\ell)}$ ways, as a subhypergraph F_ℓ cannot be monochromatic in green, hence, we consider $e(F_\ell) \cdot c \cdot \beta^k \cdot n^k$ hyperedges, which may be colored in at most

$$(3^{e(F_\ell)} - 1)^{c\beta^k n^k}$$

  ways, and
- the set of remaining hyperedges may be colored arbitrarily by 3 colors.

Let $\lambda > 0$, which depends on ℓ only, such that

$$3^{e(F_\ell)-\lambda} = 3^{e(F_\ell)} - 1.$$

With $0 < \delta \leq \lambda \cdot c \cdot \beta^k/2$ and n sufficiently large, we obtain the following upper bound

$$\begin{aligned} |\mathcal{C}_1| &\leq 3 \cdot 2^{\ell n^{k-1}} \cdot 3^{(e(F_\ell)-\lambda)c\beta^k n^k} \cdot 3^{\mathrm{ex}(n,F_{\ell+1})+\delta n^k - e(F_\ell)c\beta^k n^k} \\ &\leq 3 \cdot 2^{\ell n^{k-1}} \cdot 3^{\mathrm{ex}(n,F_{\ell+1})+\delta n^k - c\lambda\beta^k n^k} \leq 3^{\mathrm{ex}(n,F_{\ell+1})-1}. \end{aligned} \tag{23}$$

Now we turn to the colorings in $\mathcal{C}_2$ and show that by removing vertex v we obtain for the subhypergraph $H' := H - \{v\}$ on $(n-1)$ vertices the lower bound

$$c_{3,F_{\ell+1}}(H') \geq 3^{\mathrm{ex}(n-1,F_{\ell+1})+m+1}, \tag{24}$$

thus, showing the induction hypothesis (5).

By (23) we already know that

$$|\mathcal{C}_2| = |\mathcal{C}| - |\mathcal{C}_1| \geq 3^{\mathrm{ex}(n,F_{\ell+1})+m} - 3^{\mathrm{ex}(n,F_{\ell+1})-1} \geq 3^{\mathrm{ex}(n,F_{\ell+1})+m-1}. \tag{25}$$

Next we estimate the number of 3-colorings in $\mathcal{C}_2$ restricted to the set of all hyperedges incident to vertex v. By (9) we know that $|V_i| \leq n/\ell + \ell^2 \cdot \delta^{1/k} \cdot n$, $i \in [\ell]$.

Observe, that in the Turán hypergraph $\mathcal{T}_\ell^{(k)}(n)$ there are only crossing hyperedges, hence in $\mathcal{T}_\ell^{(k)}(n)$ there are at least $\binom{\ell-1}{k-1} \cdot \lfloor n/\ell \rfloor^{k-1}$ hyperedges incident to vertex v. However, in the hypergraph H possibly we have not only crossing types of hyperedges incident to vertex v, i.e., all types may be present. But, as shown in Case 1, the number of bad hyperedges incident to vertex v in H is at most $\beta \cdot n^{k-1}$. In addition to the crossing hyperedges incident to vertex v we possibly have defective hyperedges incident to v, thus, in the worst-case we have to treat the situation that vertex v may be incident to at most

$$\binom{\ell}{k-1} \cdot (n/\ell + \ell^2 \cdot \delta^{1/k} \cdot n)^{k-1} + \beta \cdot n^{k-1}$$

hyperedges. However, our advantage is that we consider colorings from $\mathcal{C}_2$, which imply certain restrictions on the colorings of the hyperedges of these types, and only a certain amount of these hyperedges may be colored arbitrarily by 3 colors.

Let $c \in \mathcal{C}_2$ be a fixed coloring. We know that vertex v covers every class V_i, $i \in [\ell]$ (via some type). On the other hand, for every color $\text{col} \in \{\text{green}, \text{blue}, \text{red}\}$, there *must* exist one class $V_{i_{\text{col}}}$, such that whenever v covers V_{col} via some type τ, the number of hyperedges from $E^{\tau}(v)$ colored in col is at most $\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)$. We say such type τ misses color col. Moreover, we say in the case as above that class V_{col} is *missed* by the color col (or the color col *misses* the class V_{col}), i.e., whenever v covers V_{col} by some type τ, τ misses the class V_{col}. Note also, that there are exactly $\binom{\ell-1}{k-2} = \binom{\ell-2}{k-2} + \binom{\ell-2}{k-3}$ defective or crossing types possible incident to vertex v and missing the class V_{col} and the color col. Furthermore notice that if a defective or crossing type (for v) misses some color, then it misses $k-1$ classes, i.e., v covers exactly $k-1$ classes.

We are aiming to show, that the number of colorings in $\mathcal{C}_2$ of the set of hyperedges incident to vertex v is bounded from above by

$$\ell^3 \cdot \binom{\binom{n}{k-1}}{\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)}^{3(c_\ell+d_\ell)} \cdot 2^{A(n/\ell+\ell^2\delta^{1/k}n)^{k-1}} \cdot 3^{\beta n^{k-1}+B(n/\ell+\ell^2\delta^{1/k}n)^{k-1}}, \tag{26}$$

where A is the number of defective or crossing types that miss exactly one color. Analogously, B is the number of defective or crossing types of hyperedges (incident to vertex v,) that do not miss any color. Notice that we are only interested in those hyperedges and therefore their types that contain vertex v.

Note that our benchmark is the Turán hypergraph $\mathcal{T}_\ell^{(k)}(n)$, where the number of 3-colorings of the set of hyperedges incident to vertex v is precisely

$$3^{\delta(\mathcal{T}_\ell^{(k)}(n))}. \tag{27}$$

Our goal is to show that the upper bound (26) is much less than (27). Thus, noting that

$$\text{ex}(n-1, F_{\ell+1}) = \text{ex}(n, F_{\ell+1}) - \delta(\mathcal{T}_\ell^{(k)}(n)) = e(\mathcal{T}_\ell^{(k)}(n)) - \delta(\mathcal{T}_\ell^{(k)}(n)), \tag{28}$$

concludes the inductive step, i.e., with (25) this shows (24).

First of all, for a coloring $c \in \mathcal{C}_2$ we note that it is impossible that a class V_i is missed by all colors, as otherwise, vertex v covers class V_i by at most $3 \cdot \beta \cdot n^{k-1}/(100 \cdot c_\ell \cdot d_\ell)$ hyperedges, which contradicts our assumption $|E_{\text{defect}}(v)| \geq \beta \cdot n^{k-1}$ for $i = 1$, or (18) for $i \geq 2$. However, the same class may be missed by two of the 3 colors. This case will be considered after the more general one, in which we assume that color $\text{col} \in \{\text{green}, \text{red}, \text{blue}\}$ misses class $V_{i_{\text{col}}}$ with pairwise distinct indices i_{col}.

3.0.3. *Distinct Colors Miss Distinct Classes.* For convenience, let green miss $V_{i_{\text{green}}}$, red miss $V_{i_{\text{red}}}$ and blue miss $V_{i_{\text{blue}}}$, where i_{green}, i_{red}, i_{blue} are pairwise distinct. The number of choices for these classes is $(\ell)_3$. Next we calculate, which types of hyperedges can be colored with how many colors.

The number of defective or crossing types of hyperedges, which are incident to vertex v and intersect all classes $V_{i_{\text{green}}}, V_{i_{\text{red}}}$, and $V_{i_{\text{blue}}}$ is exactly $\binom{\ell-3}{k-4}$. Note that $\binom{\ell-3}{i} = 0$ for $i < 0$. Every such type misses *every* class $V_{i_{\text{col}}}$ which it intersects, thus for each color and each such type the number of these hyperedges is less than

$\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)$, hence, the number of possible 3-colorings of this set of hyperedges is less than

$$\left(\sum_{i<\beta n^{k-1}/(100 d_\ell c_\ell)} \binom{\binom{n}{k-1}}{i} \right)^{3\binom{\ell-3}{k-4}} \leq \binom{\binom{n}{k-1}}{\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)}^{3\binom{\ell-3}{k-4}}, \tag{29}$$

where we used $\sum_{i=0}^{p} \binom{n}{i} \leq \binom{n}{p+1}$ for $p \leq n/4$ and $n \geq 8$.

The number of defective or crossing types of hyperedges, which are incident to vertex v and intersect exactly two of the classes $V_{i_{\text{green}}}, V_{i_{\text{red}}}$, and $V_{i_{\text{blue}}}$, is $3 \cdot \binom{\ell-3}{k-3}$. In this case, all but less than $2 \cdot \beta \cdot n^k/(100 \cdot d_\ell \cdot c_\ell)$ hyperedges of each of these types can be colored with only one color. Therefore, the number of 3-colorings of this set of hyperedges is at most

$$\left(\sum_{i<\beta n^{k-1}/(100 d_\ell c_\ell)} \binom{\binom{n}{k-1}}{i} \right)^{6\binom{\ell-3}{k-3}} \leq \binom{\binom{n}{k-1}}{\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)}^{6\binom{\ell-3}{k-3}}. \tag{30}$$

The number of defective or crossing types of hyperedges incident to vertex v that intersect exactly one of the classes $V_{i_{\text{green}}}, V_{i_{\text{red}}}$, and $V_{i_{\text{blue}}}$, is $A = 3 \cdot \binom{\ell-3}{k-2}$, where A is the number used in (26). Here, for each type, for every involved class $V_{i_{\text{col}}}$, $\text{col} \in \{\text{green}, \text{red}, \text{blue}\}$, for all but less than $\beta \cdot n^k/(100 \cdot d_\ell \cdot c_\ell)$ hyperedges we can use only 2 colors. This gives at most

$$\left(\sum_{i<\beta n^{k-1}/(100 d_\ell c_\ell)} \binom{\binom{n}{k-1}}{i} \right)^{A} \cdot 2^{A(n/\ell+\ell^2\delta^{1/k} n)^{k-1}}$$
$$\leq \binom{\binom{n}{k-1}}{\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)}^{3\binom{\ell-3}{k-2}} \cdot 2^{3\cdot\binom{\ell-3}{k-2}(n/\ell+\ell^2\delta^{1/k} n)^{k-1}} \tag{31}$$

3-colorings of this set of hyperedges.

The number of the remaining defective or crossing types of hyperedges incident to vertex v is exactly $B = \binom{\ell-3}{k-1}$, which is our constant B in (26). Here we may use all three colors, and combined with the at most $\beta \cdot n^{k-1}$ bad hyperedges incident to vertex v, this yields at most

$$3^{\beta n^{k-1} + B(n/\ell+\ell^2\delta^{1/k} n)^{k-1}} \tag{32}$$

colorings.

By Pascal's identity we have

$$3 \cdot \binom{\ell-3}{k-4} + 6 \cdot \binom{\ell-3}{k-3} + 3 \cdot \binom{\ell-3}{k-2} = 3 \cdot \binom{\ell-1}{k-2},$$

and we obtain by (29)–(32) at most

$$(\ell)_3 \cdot \binom{\binom{n}{k-1}}{\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)}^{3\binom{\ell-1}{k-2}} \cdot 2^{A(n/\ell+\ell^2\delta^{1/k} n)^{k-1}} \cdot 3^{\beta n^{k-1} + B(n/\ell+\ell^2\delta^{1/k} n)^{k-1}}$$
$$\leq (\ell)_3 \cdot \binom{\binom{n}{k-1}}{\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)}^{3\binom{\ell-1}{k-2}} \cdot \left(2^A \cdot 3^B\right)^{(n/\ell+\ell^2\delta^{1/k} n)^{k-1}} \cdot 3^{\beta n^{k-1}} \tag{33}$$

3-colorings of the set of all hyperedges incident to vertex v.

To see that (33) is strictly less than (27), we compare two quantities:

$$3^{\binom{\ell-1}{k-1}} \qquad \text{and} \qquad 2^A \cdot 3^B = 2^{3\binom{\ell-3}{k-2}} \cdot 3^{\binom{\ell-3}{k-1}}.$$

Let $\zeta > 0$ be a constant with $3^{2-\zeta} = 2^3$. By Pascal's identity we have

$$\begin{aligned} 2^A \cdot 3^B &= 2^{3\binom{\ell-3}{k-2}} \cdot 3^{\binom{\ell-3}{k-1}} = 3^{2\binom{\ell-3}{k-2}+\binom{\ell-3}{k-1}-\zeta\binom{\ell-3}{k-2}} \\ &= 3^{\binom{\ell-2}{k-1}+\binom{\ell-3}{k-2}-\zeta\binom{\ell-3}{k-2}} = 3^{\binom{\ell-1}{k-1}-\binom{\ell-2}{k-2}+\binom{\ell-3}{k-2}-\zeta\binom{\ell-3}{k-2}} \\ &= 3^{\binom{\ell-1}{k-1}-\binom{\ell-3}{k-3}-\zeta\binom{\ell-3}{k-2}} \end{aligned} \tag{34}$$

Thus, with (34) and using the entropy function $h(x)$, for $0 < \beta, \delta \ll 1$ expression (33) can be bounded from above by

$$\begin{aligned} &(\ell)_3 \cdot \binom{\binom{n}{k-1}}{\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)}^{3\binom{\ell-1}{k-2}} \cdot \left(2^A \cdot 3^B\right)^{(n/\ell+\ell^2\delta^{1/k}n)^{k-1}} \cdot 3^{\beta n^{k-1}} \\ &\le (\ell)_3 \cdot 2^{3\binom{\ell-1}{k-2}h(\beta/(100 d_\ell c_\ell))n^{k-1}} \cdot 3^{\beta n^{k-1}+(\binom{\ell-1}{k-1}-\binom{\ell-3}{k-3}-\zeta\binom{\ell-3}{k-2})(1/\ell+\delta^{1/k})^{k-1}n^{k-1}} \\ &\overset{(3)}{\le} 3^{\delta(T_\ell^{(k)}(n))-3}, \end{aligned} \tag{35}$$

as $h(x) \longrightarrow 0$ with $x \longrightarrow 0$.

3.0.4. *Two Colors Miss One Class.* Next we consider the case, when two colors miss some class V_i, and, moreover, there exists another class $V_j, j \neq i$, not covered by the third color, i.e., assume that $V_{i_{\text{green}}} = V_{i_{\text{red}}} \neq V_{i_{\text{blue}}}$, for which there are $3 \cdot (\ell)_2$ choices. We estimate the number of defective or crossing types similarly as above.

The number of defective or crossing types of hyperedges incident to vertex v and intersecting both classes $V_{i_{\text{green}}} = V_{i_{\text{red}}}$ and $V_{i_{\text{blue}}}$ is $\binom{\ell-2}{k-3}$. For each of these types we have for each color less than $\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)$ hyperedges incident to vertex v, thus, the number of 3-colorings of this set of hyperedges is at most

$$\left(\sum_{i<\beta n^{k-1}/(100 d_\ell c_\ell)} \binom{\binom{n}{k-1}}{i}\right)^{3\binom{\ell-2}{k-3}} \le \binom{\binom{n}{k-1}}{\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)}^{3\binom{\ell-2}{k-3}}. \tag{36}$$

The number of defective or crossing types of hyperedges incident to vertex v and intersecting the class $V_{i_{\text{green}}} = V_{i_{\text{red}}}$ and not intersecting the class $V_{i_{\text{blue}}}$ is $\binom{\ell-2}{k-2}$. For each of these types we have for each of the colors green and red less than $\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)$ hyperedges incident to vertex v and the remaining hyperedges are colored by blue. Thus, the number of 3-colorings of this set of hyperedges is at most

$$\left(\sum_{i<\beta n^{k-1}/(100 d_\ell c_\ell)} \binom{\binom{n}{k-1}}{i}\right)^{2\binom{\ell-2}{k-2}} \le \binom{\binom{n}{k-1}}{\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)}^{2\binom{\ell-2}{k-2}}. \tag{37}$$

The number of defective or crossing types of hyperedges incident to vertex v, intersecting class $V_{i_{\text{blue}}}$ and disjoint from class $V_{i_{\text{green}}}$ is $A = \binom{\ell-2}{k-2}$, and we can use two colors for coloring all but less than $\beta \cdot n^k/(100 \cdot d_\ell \cdot c_\ell)$ hyperedges incident to vertex v. This gives at most

$$\left(\sum_{i<\beta n^{k-1}/(100 d_\ell c_\ell)} \binom{\binom{n}{k-1}}{i}\right)^A \cdot 2^{A(n/\ell+\ell^2\delta^{1/k}n)^{k-1}}$$
$$\leq \binom{\binom{n}{k-1}}{\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)}^{\binom{\ell-2}{k-2}} \cdot 2^{A(n/\ell+\ell^2\delta^{1/k}n)^{k-1}} \tag{38}$$

3-colorings of this set of hyperedges.

The number of defective or crossing types of hyperedges incident to vertex v and disjoint from both classes $V_{i_{\text{green}}}$ and $V_{i_{\text{blue}}}$ is $B = \binom{\ell-2}{k-1}$, and we may use all 3 colors for the corresponding hyperedges, i.e., we obtain at most

$$3^{B(n/\ell+\ell^2\delta^{1/k}n)^{k-1}} \tag{39}$$

3-colorings.

Thus, by Pascal's identity we obtain by (36)–(39), using that there are at most $\beta \cdot n^{k-1}$ bad hyperedges incident to vertex v, at most

$$3\cdot(\ell)_2\cdot\binom{\binom{n}{k-1}}{\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)}^{3\binom{\ell-1}{k-2}} \cdot 2^{A(n/\ell+\ell^2\delta^{1/k}n)^{k-1}} \cdot 3^{\beta n^{k-1}+B(n/\ell+\ell^2\delta^{1/k}n)^{k-1}}$$
$$\leq 3 \cdot (\ell)_2 \cdot \binom{\binom{n}{k-1}}{\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)}^{3\binom{\ell-1}{k-2}} \cdot \left(2^A \cdot 3^B\right)^{(n/\ell+\ell^2\delta^{1/k}n)^{k-1}} \cdot 3^{\beta n^{k-1}} \tag{40}$$

3-colorings of the set of all hyperedges incident to vertex v. To see that (40) is less than (35), we compare the quantities

$$3^{\binom{\ell-1}{k-1}} \qquad \text{and} \qquad 2^A \cdot 3^B = 2^{\binom{\ell-2}{k-2}} \cdot 3^{\binom{\ell-2}{k-1}}.$$

Let $3^{1-\alpha} = 2$, i.e., $\alpha > 0$ is constant. We infer

$$2^{\binom{\ell-2}{k-2}} \cdot 3^{\binom{\ell-2}{k-1}} = 3^{\binom{\ell-2}{k-2}+\binom{\ell-2}{k-1}-\alpha\binom{\ell-2}{k-2}} = 3^{\binom{\ell-1}{k-1}-\alpha\binom{\ell-2}{k-2}}, \tag{41}$$

hence (40) becomes with (41) for $0 < \beta, \delta \ll 1$:

$$3 \cdot (\ell)_2 \cdot \binom{\binom{n}{k-1}}{\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)}^{3\binom{\ell-1}{k-2}} \cdot \left(2^A \cdot 3^B\right)^{(n/\ell+\delta^{1/k}n)^{k-1}} \cdot 3^{\beta n^{k-1}}$$
$$\leq 3 \cdot (\ell)_2 \cdot 2^{3\binom{\ell-1}{k-2}h(\beta/(100 d_\ell c_\ell))n^{k-1}} \cdot 3^{\beta n^{k-1}+(\binom{\ell-1}{k-1}-\alpha\binom{\ell-2}{k-2})(1/\ell+\delta^{1/k})^{k-1}n^{k-1}} \tag{42}$$
$$\overset{(11)}{\leq} 3^{\delta(\mathcal{T}_\ell^{(k)}(n))-3},$$

hence with (35) in both situations the number of 3-colorings of the set of hyperedges incident to vertex v is at most

$$3^{\delta(\mathcal{T}_\ell^{(k)}(n))-2},$$

which concludes the inductive hypothesis (24) and finishes Case 2.

3.0.5. *Case 3: H satisfies $\forall i \in [\ell]$ and $\forall v \in V_i$:* $|E_{bad}(v)\dot\cup E_{defect}(v)| \leq 2\cdot\beta\cdot n^{k-1}$. Here let $F_{\ell+1}$ be $H^k_{\ell+1}$ or $F^k_{\ell+1}$ for $\ell > k \geq 2$. The case $F_{k+1} = H^k_{k+1}$ is very similar to the general one (thus we only sketch it), while for $F_{k+1} = F^k_{k+1}$ a shortcut is necessary, that will be treated at the very end.

Here we are left with the last case, when most of the hyperedges, i.e., at least $\binom{\ell-1}{k-1}\cdot\lfloor n/\ell\rfloor^{k-1} - 2\cdot\beta\cdot n^{k-1}$ incident to *any* vertex v are crossing.

For a vertex v set $L_{\text{cross}}(v) := \{e\setminus\{v\}|\ v\in e,\ e\in E(H),\ \text{and } e \text{ is crossing}\}$, and for a type τ let $L^\tau(v) := \{e\setminus\{v\}:\ e\in E(H), e\,\text{has type}\,\tau\ , v\in e\}$. By assumption $H \neq \mathcal{T}^{(k)}_\ell(n)$, hence there exists a non-crossing hyperedge $e\in E(H)$ with respect to the maximal partition $\mathcal{P}$. Let v_1, v_2 be distinct vertices that belong to this hyperedge e and are contained in the same class, say V_1. As there are at most $2\cdot\beta\cdot n^{k-1}$ bad or defective hyperedges incident to any vertex v, with (9) we know for $0<\delta\ll\beta$ that

$$\begin{aligned}
&|L_{\text{cross}}(v_1)\cap L_{\text{cross}}(v_2)|\\
&\geq 2\cdot\binom{\ell-1}{k-1}\cdot\left\lfloor\frac{n}{\ell}\right\rfloor^{k-1} - 4\cdot\beta\cdot n^{k-1} - \binom{\ell-1}{k-1}\cdot\left(\frac{n}{\ell}+\ell^2\cdot\delta^{1/k}\cdot n\right)^{k-1}\\
&\geq \binom{\ell-1}{k-1}\cdot\left(\frac{n}{\ell}\right)^{k-1} - 5\cdot\beta\cdot n^{k-1},
\end{aligned}\tag{43}$$

recalling the minimum degree condition (6) for H. Again by (9), we infer for $0<\delta\ll\beta$, that for *any* crossing type τ in fact, v_1 and v_2 have large common neighbourhoods, i.e.,

$$|L^\tau(v_1)\cap L^\tau(v_2)| \geq \left(\frac{n}{\ell}\right)^{k-1} - 6\cdot\beta\cdot n^{k-1},\tag{44}$$

as otherwise (43) is violated.

For crossing types τ let $L^\tau(v_1,v_2) := L^\tau(v_1)\cap L^\tau(v_2)$. Moreover, for a vertex v and some set L of $(k-1)$-element sets let $E(v,L) := \{f\cup\{v\}\,|\,f\in L\}$.

Again, our argument splits into two cases. We distinguish between two subsets $\mathcal{C}_1$ and $\mathcal{C}_2$ of the set $\mathcal{C}$ of (in Case 3) allowed hyperedge-colorings of H, i.e., $\mathcal{C} = \mathcal{C}_1\dot\cup\mathcal{C}_2$. Let $\mathcal{C}_1$ be the set of all $F_{\ell+1}$-free 3-colorings of the set of hyperedges of H such that for every crossing type τ that intersects V_1, there exists a subset $L_\tau\subseteq L^\tau(v_1,v_2)$ with $|L_\tau|\geq\varepsilon\cdot(n/\ell)^{k-1}$, for fixed $\varepsilon>0$, and all hyperedges in $\cup_{\tau\ \text{crossing}}(E(v_1,L_\tau)\cup E(v_2,L_\tau))$ and the hyperedge e are colored the same, say in green.

We show first that the size of $\mathcal{C}_1$ is small, i.e., $|\mathcal{C}_1|\leq 3^{\text{ex}(n,F_{\ell+1})-1}$, to finally concentrate on $\mathcal{C}_2$. The reason for the small size of $\mathcal{C}_1$ is, that many pairwise hyperedge-disjoint subhypergraphs $F_{\ell-1}$ (and not F_ℓ) arise, and for each its set of hyperedges cannot be colored completely in green, as otherwise, having such a subhypergraph $F_{\ell-1}$, we may build together with the green hyperedges in $\{e\}\cup\cup_{\tau\ \text{cross}}(E(v_1,L_\tau)\cup E(v_2,L_\tau))$ a green subhypergraph $F_{\ell+1}$.

Consider a coloring from $\mathcal{C}_1$. Then, for every crossing type τ of hyperedges containing vertex v_1, there exist subsets $L_\tau\subseteq L^\tau(v_1,v_2)$ with $|L_\tau| = \varepsilon\cdot(n/\ell)^{k-1}$, where

$$0\ll\delta\ll\beta\ll\varepsilon\ll 1,\tag{45}$$

and all k-element sets in $E(v_1,L_\tau)\cup E(v_2,L_\tau)$ are colored green.

For every $j\in\{2,\ldots,\ell\}$, fix a crossing type τ_j intersecting V_1 and V_j.

Fix $j \in [\ell]$. Assume without loss of generality that $j \le k$ and that the hyperedges of type $\tau := \tau_j$ incident to vertex v_1 intersect each class V_i, $i \in [k]$. Let W_1 and W_2 be the set of all sets $w^1 = \{v_1, w_1^1, \ldots, w_{k-2}^1\}$ and $w^2 = \{v_2, w_1^2, \ldots, w_{k-2}^2\}$, respectively, with $|w^1 \cap V_i| = |w^2 \cap V_i| = 1$, $i \in \{2, \ldots, j-1, j+1, \ldots, k\}$. Let $L_\tau(w^i)$, $i \in [2]$, be the corresponding link-sets (with hyperedges from L_τ only) in class V_j, i.e.,

$$L_\tau(w^i) = \{v \in V_j : (w^i \cup \{v\} \setminus \{v_i\}) \in L_\tau\}.$$

Let $d_\tau(v)$ for $v \in V_j$ be the number of hyperedges in $\{\{v_1\} \cup f \colon f \in L_\tau\}$ incident to vertex v.

Then, for $0 < \delta < (1/\ell)^{3k}$ we infer by using the Cauchy-Schwarz inequality

$$\begin{aligned} \sum_{(w^1,w^2)\in W_1\times W_2} |L_\tau(w^1) \cap L_\tau(w^2)| &= \sum_{v\in V_j} (d_\tau(v))^2 \\ \ge \frac{\left(\sum_{v\in V_j} d_\tau(v)\right)^2}{|V_j|} &= |L_\tau|^2/|V_j| \ge \frac{\left(\varepsilon\cdot\left(\frac{n}{\ell}\right)^{k-1}\right)^2}{\frac{n}{\ell} + \ell^2\cdot\delta^{1/k}\cdot n} \ge \frac{\varepsilon^2}{2}\cdot\left(\frac{n}{\ell}\right)^{2k-3}. \end{aligned} \tag{46}$$

The number of pairs $(w^1, w^2) \in W_1 \times W_2$, which have at least one common entry, is less than

$$k \cdot \left(\frac{n}{\ell} + \ell^2 \cdot \delta^{1/k} \cdot n\right)^{2k-5},$$

and their contribution to the first sum in (46) is at most

$$k \cdot \left(\frac{n}{\ell} + \ell^2 \cdot \delta^{1/k} \cdot n\right)^{2k-4} = o(n^{2k-3}),$$

hence we know by (46) that for n sufficiently large

$$\sum_{(w^1,w^2)\in W_1\times W_2;\, w^1\cap w^2=\varnothing} |L_\tau(w^1) \cap L_\tau(w^2)| \ge \frac{\varepsilon^2}{4}\cdot\left(\frac{n}{\ell}\right)^{2k-3}.$$

This implies that for $0 < \delta < ((2^{1/(2k-4)} - 1)/\ell^3)^k$ there exists a pair $(w^1, w^2) \in W_1 \times W_2$ with $w^1 \cap w^2 = \varnothing$ and we have

$$|L_\tau(w^1) \cap L_\tau(w^2)| \ge \frac{\frac{\varepsilon^2}{4}\cdot\left(\frac{n}{\ell}\right)^{2k-3}}{\left(\frac{n}{\ell} + \ell^2\cdot\delta^{1/k}\right)^{2k-4}} \ge \frac{\varepsilon^2}{8}\cdot\frac{n}{\ell}.$$

Doing this for every $j \in \{2, \ldots, \ell\}$, by the above averaging argument and (44), using our assumption $|L_\tau| = \varepsilon \cdot (n/\ell)^{k-1}$ for every crossing type τ, for every class V_j, $j \in \{2, \ldots \ell\}$, there is a crossing type τ_j of hyperedges containing vertex v_1 (and similarly containing vertex v_2), such that for $i \in [2]$ consecutively we may find $(k-2)$ distinct vertices $w_1^{(i,j)}, \ldots, w_{k-2}^{(i,j)}$ with $\{w_1^{(i,j)}, \ldots, w_{k-2}^{(i,j)}\}$ disjoint from $\{v_1, v_2\}$ and the hyperedge e, and also

$$\{w_1^{(i,j)}, \ldots, w_{k-2}^{(i,j)}\} \cap \{w_1^{(i',j')}, \ldots, w_{k-2}^{(i',j')}\} = \varnothing,$$

whenever $(i, j) \ne (i', j')$, and there exist subsets $S_j \subset V_j$ with

$$|S_j| \ge \varepsilon^2 \cdot n/(10 \cdot \ell),$$

such that for all $i \in [2]$, and $j \in \{2, \ldots, \ell\}$, and $s_j \in S_j$ we have

$$\{v_i, w_1^{(i,j)}, \ldots, w_{k-2}^{(i,j)}, s_j\} \in E(v_i, L_{\tau_j}). \tag{47}$$

Moreover, all sets S_j, $j \in [\ell]$, are disjoint from any set $\{w_1^{(i',j')}, \ldots, w_{k-2}^{(i',j')}\}$ and from the hyperedge e.

By possibly omitting some vertices, we assume that $|S_2| = \cdots = |S_\ell| = s := \varepsilon^2 \cdot n/(10 \cdot \ell)$.

For $\ell > k$, we consider the complete $(\ell-1)$-partite k-uniform hypergraph G with vertex partition $S_2 \dot\cup \ldots \dot\cup S_\ell$. No subhypergraph $F_{\ell-1}$ in G with core $s_2, \ldots, s_\ell$, $s_i \in S_i$, can be colored completely in green, as otherwise we obtain a green subhypergraph $F_{\ell+1}$ in H by using the hyperedge e and all hyperedges $\{v_i, w_1^{(i,j)}, \ldots, w_{k-2}^{(i,j)}, s_j\}$, $s_j \in S_j$, $i \in [2]$ and $j \in \{2, \ldots, \ell\}$.

By Lemma 13, cf. (20)-(22), for $\ell > k$ there exist $\xi \cdot s^k$ with

$$\xi > \frac{\binom{\ell-2}{k-1}}{2 \cdot k \cdot (\ell-1) \cdot e(F_{\ell-1})}$$

hyperedge-disjoint copies of $F_{\ell-1}$ in G.

These copies of $F_{\ell-1}$ might not be subhypergraphs $F_{\ell-1}$ in $H[S_2, \ldots, S_\ell]$. However, in the hypergraph H hence also in the subhypergraph $H[S_2, \ldots, S_\ell]$ less than $\delta \cdot n^k$ crossing hyperedges are missing, cf. (8). With $\delta \le (\xi/2) \cdot (\varepsilon^2/(10 \cdot \ell))^k$, we always find at least

$$\xi \cdot s^k - \delta \cdot n^k = \xi \cdot (\varepsilon^2 \cdot n/(10 \cdot \ell))^k - \delta \cdot n^k \overset{(45)}{\ge} (\xi/2) \cdot (\varepsilon^2 \cdot n/(10 \cdot \ell))^k \ge c \cdot \varepsilon^{2k} \cdot n^k \quad (48)$$

subhypergraphs $F_{\ell-1}$ in H, where $c = \xi/(2 \cdot 10^k \cdot \ell^k)$. Let the hyperedge-disjoint subhypergraphs of $F_{\ell-1}$ in G be enumerated by $H_1, \ldots, H_{c\varepsilon^{2k} n^k}$.

Every subhypergraph H_j, $j \in [c \cdot \varepsilon^{2k} \cdot n^k]$, together with the hyperedge e and the hyperedges $\{s_j, v_i, w_1^{(i,j)}, \ldots, w_{k-2}^{(i,j)}\}$, $i \in [2]$ and $j \in \{2, \ldots, \ell\}$, and vertices $s_j \in S_j$ from the core of $F_{\ell-1}$ yields a subhypergraph $F_{\ell+1}$. Moreover, the latter hyperedges are all colored the same for *every* coloring $c \in \mathcal{C}_1$.

Now, we estimate the cardinality of $\mathcal{C}_1$ as follows:

- there are 3 choices for the color in which the $(2 \cdot \ell - 1)$ hyperedges should be colored, say, in green, and for every class V_j, $j \in \{2, \ldots, \ell\}$, there are at least $\varepsilon \cdot (n/\ell)^{k-1}$ green hyperedges that cover class V_j, which yields at most

$$\left(\sum_{i=\varepsilon(n/\ell)^{k-1}}^{\binom{n}{k-1}} \binom{\binom{n}{k-1}}{i} \right)^{\ell-1} \le 2^{\binom{n}{k-1}(\ell-1)} \le 2^{\ell n^{k-1}}$$

  choices for these green hyperedges, and
- at least $c \cdot \varepsilon^k \cdot n^k$ subhypergraphs $F_{\ell-1}$ together with at most $(2 \cdot \ell - 1)$ green hyperedges yield at least $c \cdot \varepsilon^{2k} \cdot n^k$ subhypergraphs $F_{\ell+1}$, and we may color the set of hyperedges in every subhypergraph $F_{\ell-1}$ in at most $(3^{e(F_{\ell-1})} - 1)$ instead of $3^{e(F_{\ell-1})}$ ways, as a subhypergraph $F_{\ell-1}$ cannot be monochromatic in green, hence in total, we consider $e(F_{\ell-1}) \cdot c \cdot \varepsilon^{2k} \cdot n^k$ many hyperedges, which may be colored in at most

$$(3^{e(F_{\ell-1})} - 1)^{c\varepsilon^{2k} n^k}$$

  ways, and
- the remaining hyperedges may be colored arbitrarily by 3 colors.

Let $\lambda > 0$ be such that

$$3^{e(F_{\ell-1})-\lambda} = 3^{e(F_{\ell-1})} - 1.$$

With $0 < \delta < \lambda \cdot c \cdot \varepsilon^{2k}/2$, for n sufficiently large, we obtain the following upper bound

$$
\begin{aligned}
|\mathcal{C}_1| &\le 3 \cdot 2^{\ell n^{k-1}} \cdot 3^{e(F_{\ell-1})-\lambda)c\varepsilon^{2k}n^k} \cdot 3^{\mathrm{ex}(n,F_{\ell+1})-e(F_{\ell-1})c\varepsilon^{2k}n^k+\delta n^k} \\
&\le 3 \cdot 2^{\ell n^{k-1}} \cdot 3^{\mathrm{ex}(n,F_{\ell+1})+\delta n^k - c\lambda\varepsilon^{2k}n^k} \\
&\overset{(45)}{\le} 3^{\mathrm{ex}(n,F_{\ell+1})-1}.
\end{aligned} \tag{49}
$$

Next, we explain how to adjust the arguments to the case when $F_{k+1} = H_{k+1}^k$. There we need an additional vertex set $S_1 \subset V_1$, which does not contain any vertices $(v_i, w_t^{(i,j)})$ previously chosen. Now we clearly find $\Theta(n^k)$, see [7], many hyperedge disjoint copies of H_k^k (and therefore of H_{k-1}^k) in $H[S_1, \dots, S_\ell]$. The remaining argument goes analogously.

Next we turn to the colorings in $\mathcal{C}_2 = \mathcal{C} \setminus \mathcal{C}_1$. By (49) we know for $\ell \ge k$ that

$$
|\mathcal{C}_2| = |\mathcal{C}| - |\mathcal{C}_1| \ge 3^{\mathrm{ex}(n,F_{\ell+1})+m} - 3^{\mathrm{ex}(n,F_{\ell+1})-1} \ge 3^{\mathrm{ex}(n,F_{\ell+1})+m-1}. \tag{50}
$$

Fix a coloring from $\mathcal{C}_2$. By (9), for each crossing type τ there are at most $(n/\ell + \ell^2 \cdot \delta^{1/k} \cdot n)^{k-1}$ hyperedges incident to any fixed vertex v. As we consider colorings from $\mathcal{C}_2$, there must exist a crossing type τ such that we have less than $\varepsilon \cdot (n/\ell)^{k-1}$ hyperedges $f \in E(H)$ incident to vertex v_1, where $f \setminus \{v_1\}$ is contained in $L^\tau(v_1, v_2)$, which have the same color, say green, as the hyperedge e. Let L be this set of $(k-1)$-element subsets $f \setminus \{v_1\}$, $f \in E(H)$. These hyperedges can be chosen in at most

$$
\sum_{i < \varepsilon \cdot (n/\ell)^{k-1}} \binom{\binom{n}{k-1}}{i} \le \binom{\binom{n}{k-1}}{\varepsilon \cdot (n/\ell)^{k-1}} \le 2^{h(\varepsilon)n^{k-1}}
$$

ways.

Thus, with (44) for this type τ we can color all but at most $2 \cdot \varepsilon \cdot (n/\ell)^{k-1} + 6 \cdot \beta \cdot n^{k-1}$ hyperedges from the set $E^\tau(v_1) \cup E^\tau(v_2)$ in at most 8 instead of 9 ways, as for every $(k-1)$-element set f from $L^\tau(v_1, v_2) \setminus L$ we cannot color both hyperedges $\{v_1\} \cup f$ and $\{v_2\} \cup f$ in green.

There are at most

$$
\left(\binom{\ell - 1}{k - 1} - 1 \right) \cdot \left(\frac{n}{\ell} + \ell^2 \cdot \delta^{1/k} \cdot n \right)^{k-1}
$$

crossing hyperedges incident to vertex v_1 of a type distinct from τ, which may be colored by at most 3 colors, hence for this set of hyperedges we obtain at most

$$
3^{(\binom{\ell-1}{k-1}-1)(n/\ell+\ell^2\delta^{1/k}n)^{k-1}}
$$

3-colorings, and similarly for vertex v_2.

There are 3 choices for the color of the hyperedge e, hence altogether, as there are at most $4 \cdot \beta \cdot n^{k-1}$ bad or defective hyperedges incident to vertex v_1 or v_2, we have at most

$$
3 \cdot \binom{\ell - 1}{k - 1} \cdot 2^{h(\varepsilon)n^{k-1}} 8^{(n/\ell+\ell^2\delta^{1/k}n)^{k-1}} \cdot 3^{2[2\beta n^{k-1} + (\binom{\ell-1}{k-1}-1)(n/\ell+\ell^2\delta^{1/k}n)^{k-1}]} \tag{51}
$$

different 3-colorings of hyperedges that contain either v_1 or v_2 (or both). For $0 < \delta \ll \beta \ll \varepsilon \ll 1$ the upper bound (51) is less than

$$
3^{\delta(T_\ell^{(k)}(n)) + \delta(T_\ell^{(k)}(n-1)) - 2}. \tag{52}
$$

Hence we can delete both vertices v_1 and v_2 and consider the subhypergraph $H' := H - \{v_1, v_2\}$. For $\ell > k$, with (28) and (50) the induction step is finished and yields

$$c_{3,F_{\ell+1}}(H') \geq \frac{|\mathcal{C}_2|}{3^{\delta(\mathcal{T}_\ell^{(k)}(n))+\delta(\mathcal{T}_\ell^{(k)}(n-1))-2}} \geq 3^{\mathrm{ex}(n-2,F_{\ell+1})+m+1},$$

which proves (5).

Finally, we turn to the case when $F_{k+1} = F^k_{k+1}$. We again assume that a non-crossing hyperedge e intersects V_1 in some vertices v_1, v_2. We consider then the $(k-1)$-uniform hypergraph $G := L_{\mathrm{cross}}(v_1) \cap L_{\mathrm{cross}}(v_2) \cap [V(H) \setminus (e \cup V_1)]^{k-1}$ (notice that crossing hyperedges are with respect to the vertex partition $\mathcal{P}$ and here we identify the hypergraph G with the set of its hyperedges). Using (44) we know

$$e(H') \geq \left(\frac{n}{k}\right)^{k-1} - 6 \cdot \beta \cdot n^{k-1}$$

and therefore G contains at least

$$\frac{1}{k} \cdot \left(\frac{n}{k}\right)^{k-1} - 7 \cdot \beta \cdot n^{k-1}$$

hyperedge-disjoint copies of the following $(k-1)$-partite $(k-1)$-uniform hypergraph F' with vertex set

$$V(F') = \{1, \ldots, (k-1)^2\}$$

and the hyperedge set

$$E(F') = \{\{1, \ldots, k-1\}, \ldots, \{(k-2)\cdot(k-1)+1, \ldots, (k-1)^2\}, \{1, k, \ldots, (k-2)\cdot(k-1)+1\}\},$$

as $\mathrm{ex}(n, F') = o(n^{k-1})$, cf. [7]. Clearly, a copy of F' in G will form together with the hyperedge e two copies of F^k_{k+1} that share only the hyperedge e. Thus, we find in this way at least

$$\frac{2}{k} \cdot \left(\frac{n}{k}\right)^{k-1} - 14 \cdot \beta \cdot n^{k-1}$$

copies of F^k_{k+1} in H that all have only one hyperedge e in common. This means that having fixed the color of e (3 possibilities), we can color any of the above copies of F^k_{k+1} in at most $3^k - 1$ instead of 3^k many ways. Therefore, we upper bound the number of colorings of the hyperedges that are incident to either v_1 or v_2 by

$$3 \cdot (3^k - 1)^{(2/k)\cdot(n/k)^{k-1} - 14\beta n^{k-1}} \cdot 3^{2(n/k + k^2\delta^{1/k} n)^{k-1} + 4\beta n^{k-1} + n^{k-2} - (2/k)\cdot(n/k)^{k-1} + 14\beta n^{k-1}}$$
$$\leq 3^{\delta(\mathcal{T}_\ell^{(k)}(n))+\delta(\mathcal{T}_\ell^{(k)}(n-1))-2}. \tag{53}$$

Thus, deleting v_1 and v_2 from H implies for $H' := H \setminus \{v_1, v_2\}$ with (53) that

$$c_{3,F^k_{k+1}}(H') \geq 3^{\mathrm{ex}(n-2,F^k_{k+1})+m+1},$$

and this finishes the Claim 3 and proves Lemma 6.

□

4. The Case of $r \geq 4$ Colors

First we prove for fixed $r \geq 4$ for the expanded, complete 2-graph $H^k_{\ell+1}$ the lower bounds $c_{r,H^k_{\ell+1}}(n) \gg r^{\mathrm{ex}(n,H^k_{\ell+1})}$ for n sufficiently large. Namely, we show the following

Lemma 14. *Let $r \geq 4$ be a fixed integer. Then, for the expanded, complete 2-graph $H^k_{\ell+1}$, $\ell \geq k \geq 2$, for n sufficiently large it is*

$$c_{r,H^k_{\ell+1}}(n) \gg r^{\mathrm{ex}(n,H^k_{\ell+1})}. \tag{54}$$

Proof. Let V be an n-element vertex set and assume for simplicity that 2ℓ divides n. Consider a partition $\mathcal{P}$ of the vertex set V into $(\ell+2)$ pairwise disjoint vertex sets $V_1, \ldots, V_{\ell-2}, W_1, \ldots, W_4$, where each class V_i, $i \in [\ell-2]$, has cardinality $|V_i| = n/\ell$, and every other class W_i, $i \in [4]$, satisfies $|W_i| = n/(2 \cdot \ell)$. Let H be the k-uniform $(\ell+2)$-partite hypergraph with respect to the partition $\mathcal{P}$, where all crossing hyperedges are present except for those that intersect more than two classes $W_i, W_j, i \neq j$. Let $\{1, \ldots, r\}$ be the set of colors.

All hyperedges in $E(H)$ which contain at most one vertex from $W_1 \cup \cdots \cup W_4$ can be colored with all r colors. All hyperedges in $E(H)$ which contain one vertex from each class W_1 and W_2 or from each class W_3 and W_4 are colored with $1, \ldots, r-1$. All hyperedges in $E(H)$ which contain one vertex from each class W_1 and W_3 or from each class W_2 and W_4 get colors $1, \ldots, r-2, r$. All hyperedges in $E(H)$ which contain one vertex from each class W_1 and W_4 or from each class W_2 and W_3 are colored with $r-1, r$. Note that the projection (link) of any three hyperedges on $W_1 \cup \cdots \cup W_4$ does not give a monochromatic graph triangle.

Then, by using Pascal's identity, the number of colorings of the set $E(H)$ of hyperedges of H is precisely

$$\begin{aligned} & r^{\binom{\ell-2}{k}(n/\ell)^k + 2\binom{\ell-2}{k-1}(n/\ell)^k} \cdot (r-1)^{\binom{\ell-2}{k-2}(n/\ell)^k} \cdot 2^{(1/2)\binom{\ell-2}{k-2}(n/\ell)^k} \\ = \; & \left(r \cdot \left(\frac{(r-1) \cdot \sqrt{2}}{r} \right)^{\frac{k(k-1)}{\ell(\ell-1)}} \right)^{\binom{\ell}{k}(n/\ell)^k} \\ \gg \; & r^{\binom{\ell}{k}(n/\ell)^k} \geq r^{\mathrm{ex}(n,H^k_{\ell+1})} \end{aligned}$$

for n sufficiently large.

Suppose for contradiction that for one of these colorings the hypergraph H contains a monochromatic $H^k_{\ell+1}$ with core $v_1, \ldots, v_{\ell+1}$. As all hyperedges in H are crossing, at least three vertices of the core of $H^k_{\ell+1}$ must be contained in the set $W_1 \cup \cdots \cup W_4$. Without loss of generality let v_1, v_2, v_3 be such vertices. By construction of H no two of these can be contained in the same vertex set W_i. For each pair $\{v_i, v_j\}$, $1 \leq i < j \leq 3$, there is a $(k-2)$-element set $S_{i,j}$ such that $\{v_i, v_j\} \cup S_{i,j}$ is a hyperedge in $H^k_{\ell+1}$, and again by construction $S_{i,j} \subseteq V_1 \cup \cdots \cup V_{\ell-2}$, but then the links $L_H(S_{i,j})$, $1 \leq i < j \leq 3$, yield a graph triangle in $W_1 \cup \cdots \cup W_4$, hence the hyperedges $\{v_i, v_j\} \cup S_{i,j}$, $1 \leq i < j \leq 3$, do not have all the same color. □

The lower bound in (54) can be improved for larger values of r. Say, for r divisible by 3 we take the hypergraph H as in the proof of Lemma 14 by using a better distribution of the colors as follows.

All hyperedges in $E(H)$ which contain at most one vertex from $W_1 \cup \cdots \cup W_4$ can be colored with all r colors. All hyperedges in $E(H)$ which contain one vertex from each class W_1 and W_2 or from each class W_3 and W_4 are colored with $1, \ldots, 2 \cdot r/3$. All hyperedges in $E(H)$ which contain one vertex from each class W_1 and W_3 or from each class W_2 and W_4 get colors $1, \ldots, r/3, 2 \cdot r/3 + 1, \ldots, r$. All hyperedges in $E(H)$ which contain one vertex from each class W_1 and W_4 or from each class W_2 and W_3 are colored with $r/3+1,\ldots,r$.

This gives (for r divisible by 3) the lower bound

$$c_{r,H^k_{\ell+1}}(n) \quad \geq \quad c_{r,H^k_{\ell+1}}(H) \geq \left(r^{\binom{\ell}{k}} \cdot \left(\frac{2 \cdot \sqrt{2r}}{3 \cdot \sqrt{3}} \right)^{\binom{\ell-2}{k-2}} \right)^{(n/\ell)^k} . \tag{55}$$

Next we prove the same lower bound for fixed $r \geq 4$ for the Fan(k)-hypergraph $F^k_{\ell+1}$.

Lemma 15. *Let $r \geq 4$ be a fixed integer. Then, for the Fan(k)-hypergraph $F^k_{\ell+1}$, $\ell \geq k \geq 2$, for positive integers n sufficiently large it is*

$$c_{r,F^k_{\ell+1}}(n) \quad \gg \quad r^{\mathrm{ex}(n,F^k_{\ell+1})}. \tag{56}$$

Proof. We consider the hypergraph H and all colorings from the proof of Lemma 14. Assume that we obtain a monochromatic subhypergraph $F^k_{\ell+1}$ with core-vertices $v_1,\ldots,v_{\ell+1}$ where $v_1,\ldots,v_k$ form the core-hyperedge in $F^k_{\ell+1}$. Then at least three of the core-vertices must be contained in the set $W_1 \cup \cdots \cup W_4$, say these are the vertices v_g, v_h, v_i, where $1 \leq g < h < i$. By construction of H we cannot have $i \leq k$, as $v_1,\ldots,v_k$ form a hyperedge of $F^k_{\ell+1}$. Note that every pair of distinct vertices from $\{v_g, v_h, v_i\}$ is contained in a hyperedge of $F^k_{\ell+1}$. Therefore, as in the proof of Lemma 14 we obtain a monochromatic triangle in $W_1 \cup \cdots \cup W_4$, which is not possible. □

Similarly, one can obtain the same lower bound for $c_{r,F^k_{\ell+1}}(n)$ as in (55).

5. Upper Bounds on $c_{r,F}(n)$ for $r \geq 4$

The next result gives an upper bound on $c_{r,F}(n)$ for any fixed, uniform, linear hypergraph F with positive Turán density and for any fixed integer $r \geq 4$.

Theorem 16. *Let F be any linear, k-uniform hypergraph F with $\mathrm{ex}(t,F) = (\pi_F + o(1)) \cdot \binom{t}{k}$ for a constant $\pi_F > 0$. For fixed integers $r \geq 4$ it is*

$$c_{r,F}(n) \leq (\pi_F \cdot r)^{\binom{n}{k}+o(n^k)} \qquad \text{if } \pi_F \cdot r \geq e,$$

and

$$c_{r,F}(n) \leq e^{(r/e)(\pi_F+o(1))\binom{n}{k}} \qquad \text{if } \pi_F \cdot r < e.$$

For F being the k-uniform expanded, complete 2-graph $H^k_{\ell+1}$ or the Fan(k)-hypergraph $F^k_{\ell+1}$, $\ell \geq k \geq 2$, Theorem 16 gives

$$c_{r,F}(n) \leq \left(\frac{(\ell)_k}{\ell^k} \cdot r \right)^{\binom{n}{k}+o(n^k)},$$

where $\pi_F = (\ell)_k/\ell^k$, compare with the lower bound (55), which for example for $\ell = k$ and r divisible by 3 large enough, i.e., $r \geq e \cdot k^k/k!$, yields

$$\left(\frac{2 \cdot \sqrt{2} \cdot r^{3/2}}{3 \cdot \sqrt{3}} \right)^{(n/k)^k} \leq c_{r,F}(n) \leq \left(\frac{k!}{k^k} \cdot r \right)^{\binom{n}{k}+o(n^k)}.$$

Proof. The arguments are similar to those in the proof of Theorem 2. Let $\gamma > 0$ be arbitrary and set $\varepsilon = \varepsilon(\gamma) > 0$ with $\varepsilon < \gamma/4$ such that Lemma 5 is satisfied. Moreover, let $t_0 = \max\{1/\varepsilon, t'\}$, where t' is sufficiently large, so that $\mathrm{ex}(t, F) = (\pi_F + o(1)) \cdot \binom{t}{k}$ for $t \geq t'$. Let $T_0 = T_0(r, t_0, \varepsilon)$ and $N_0 = N_0(r, t_0, \varepsilon)$ be given by Theorem 3 and let $m_0 = m_0(\gamma)$ be given by Lemma 5. Set $n_0 := \max\{N_0, T_0 \cdot m_0\}$ and let H be a hypergraph on $n \geq n_0$ vertices.

Consider any fixed r-coloring of the set of hyperedges of H without a monochromatic copy of the hypergraph F. By Theorem 3 there exists a partition $V(H) = V_1 \dot\cup \cdots \dot\cup V_t$ of the vertex set $V(H)$, $t_0 \leq t \leq T_0$, which is ε-regular with respect to each color, where without loss of generality $|V_i| = n/t$, $i \in [t]$.

For $\gamma > 0$ and $\mathrm{col} \in [r]$ let $H_{\mathrm{col}}(\gamma)$ be the corresponding cluster-hypergraphs on the vertex set $[t]$, i.e., $H_{\mathrm{col}}(\gamma)$ corresponds to all hyperedges of color $\mathrm{col} \in [r]$, which are contained in ε-regular k-tuples of density at least γ. Furthermore, for $s \in [r]$ let e_s be the number of k-tuples $(i_1, \ldots, i_k)$, $1 \leq i_1 < \cdots < i_k \leq t$, which are hyperedges in exactly s of the cluster-hypergraphs $H_{\mathrm{col}}(\gamma)$ with $\mathrm{col} \in [r]$. By our assumption and by Lemma 5 each hypergraph $H_{\mathrm{col}}(\gamma)$ is F-free, hence contains at most $\mathrm{ex}(t, F) = (\pi_F + o(1)) \cdot \binom{t}{k}$ hyperedges:

$$\sum_{s=1}^{r} s \cdot e_s \leq r \cdot \mathrm{ex}(t, F) \leq r \cdot (\pi_F + o(1)) \cdot \binom{t}{k}. \tag{57}$$

Next we count the number of r-colorings of the set of hyperedges of H, which yield the partition $\mathcal{P}$ of the vertex set with $V(H) = V_1 \dot\cup \cdots \dot\cup V_t$ and the cluster-hypergraphs $H_{\mathrm{col}}(\gamma)$, $\mathrm{col} \in [r]$. To do so, first we bound from above the number of hyperedges $e \in E(H)$, which are non-crossing with respect to the partition $\mathcal{P}$, or are contained in an ε-irregular k-tuple $(V_{i_1}, \ldots, V_{i_k})$, or, for one color class, are contained in a k-tuple $(V_{i_1}, \ldots, V_{i_k})$ of density less than γ, $1 \leq i_1 < \cdots < i_k \leq t$.

The number of non-crossing hyperedges $e \in E(H)$ is with $t \geq t_0 \geq 1/\varepsilon$ and $\varepsilon < \gamma/4$ at most

$$t \cdot \binom{n/t}{2} \cdot \binom{n}{k-2} < \frac{1}{2 \cdot t} \cdot n^k < \frac{\gamma}{8} \cdot n^k. \tag{58}$$

The number of crossing hyperedges $e \in E(H)$, which are contained in one of the at most $r \cdot \varepsilon \cdot \binom{t}{k}$ ε-irregular k-tuples $(V_{i_1}, \ldots, V_{i_k})$, $1 \leq i_1 < \cdots < i_k \leq t$, is for $k \geq 2$ and $\varepsilon < \gamma/4$ at most

$$r \cdot \varepsilon \cdot \binom{t}{k} \cdot \left(\frac{n}{t}\right)^k < \frac{r \cdot \varepsilon}{2} \cdot n^k < \frac{r \cdot \gamma}{8} \cdot n^k. \tag{59}$$

The number of crossing hyperedges $e \in E(H)$, which for one of the r color classes are contained in k-tuples $(V_{i_1}, \ldots, V_{i_k})$ of density less than γ, $1 \leq i_1 < \cdots < i_k \leq t$, is for $k \geq 2$ at most

$$r \cdot \binom{t}{k} \cdot \gamma \cdot \left(\frac{n}{t}\right)^k < \frac{r \cdot \gamma}{2} \cdot n^k. \tag{60}$$

Hence, the total number of all these hyperedges is by (58)–(60) less than

$$r \cdot \gamma \cdot n^k, \tag{61}$$

and this set of hyperedges can be chosen in at most

$$\binom{\binom{n}{k}}{r \cdot \gamma \cdot n^k} < \binom{n^k/k!}{r \cdot \gamma \cdot n^k} \leq 2^{h(k! r \gamma) n^k / k!} \tag{62}$$

ways and can be colored by r colors, which gives the following upper bound on the number of colorings of this set of hyperedges

$$r^{2\gamma n^k}. \tag{63}$$

With (62) and (63), the number of r-colorings of the set of hyperedges of H, which yield the vertex partition $V(H) = V_1\dot{\cup}\cdots\dot{\cup}V_t$ and the cluster-hypergraphs $H_1(\gamma), \ldots, H_r(\gamma)$, can be bounded from above by

$$\binom{\binom{n}{k}}{r\cdot\gamma\cdot n^k}\cdot r^{r\gamma n^k}\cdot\left(\prod_{s=1}^{r} s^{e_s}\right)^{(n/t)^k} \quad\leq\quad 2^{h(k!r\gamma)n^k/k!}\cdot r^{r\gamma n^k}\cdot\left(\prod_{s=1}^{r} s^{e_s}\right)^{(n/t)^k} \tag{64}$$

Since

$$\sum_{s=1}^{r} e_s \leq \binom{t}{k}$$

we may view $\prod_{s=1}^{r} s^{e_s}$ as a product of at most $\binom{t}{k}$ factors. The sum of those factors equals $\sum_{s=1}^{r} s\cdot e_s$, which due to (57) is bounded from above by $r\cdot(\pi_F+o(1))\cdot\binom{t}{k}$. Since a product of positive reals with bounded sum of the factors is maximized, when all factors are equal, one can show that

$$\prod_{s=1}^{r} s^{e_s} \leq ((\pi_F+o(1))\cdot r)^{\binom{t}{k}} \quad \text{if } \pi_F\cdot r\geq e\,, \tag{65}$$

and

$$\prod_{s=1}^{r} s^{e_s} \leq e^{(r/e)(\pi_F+o(1))\binom{t}{k}} \quad \text{if } \pi_F\cdot r< e\,, \tag{66}$$

see, e.g., [1, Lemma 4.3].

The number t of partition classes is at most T_0, hence there are at most n^{T_0} partitions of the vertex set V into at most T_0 classes. Given such a partition, we have at most $2^{r\binom{T_0}{k}} < 2^{rT_0^k}$ choices for the cluster-hypergraphs $H_1(\gamma), \ldots, H_r(\gamma)$. With (64) and (65) we obtain for $\pi_F\cdot r > e$:

$$\begin{aligned} c_{r,F}(n) &\leq n^{T_0}\cdot 2^{rT_0^k}\cdot 2^{h(k!r\gamma)n^k/k!}\cdot r^{r\gamma n^k}\cdot((\pi_F+o(1))\cdot r)^{\binom{t}{k}(n/t)^k}\\ &\leq n^{T_0}\cdot 2^{rT_0^k}\cdot 2^{h(k!r\gamma)n^k/k!}\cdot r^{r\gamma n^k}\cdot((\pi_F+o(1))\cdot r)^{\binom{n}{k}}\\ &\leq (\pi_F\cdot r)^{\binom{n}{k}+o(n^k)}\,, \end{aligned} \tag{67}$$

as $\gamma > 0$ can be chosen to be arbitrary small and the entropy function satisfies $h(\gamma)\to 0$ as $\gamma\to 0$.

Similarly, with (64) and (65) we obtain for $\pi_F\cdot r < e$:

$$e^{(r/e)(\pi_F+o(1))\binom{n}{k}},$$

as claimed. □

Remark 17. The linearity assumption in Theorem 16 can be dropped if we use the strong hypergraph regularity lemma of Rödl and Schacht in conjunction with counting lemma [28, 27], similarly as was done in [19, Theorem 2]. The remaining adjustments are straightforward.

6. Concluding remarks

In this paper we determined exactly the function $c_{r,F}(n)$ for $r = 2, 3$ and n large for the expanded complete 2-graphs and for the so-called Fan(k)-hypergraphs. Equipped with Theorem 2 from [19], one should be able to exactly determine the function $c_{r,F}(n)$ for various other hypergraphs F, where it is known that they are stable and the extremal hypergraphs result is proven. Such natural candidates are the (1-stable) hypergraphs F from [10, 13]. There it is plausible that $c_{r,F}(n) = r^{\mathrm{ex}(n,F)}$ for $r = 2$ or $r = 3$ and n large.

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7. Appendix

We say that two k-uniform hypergraphs H_1 and H_2 on n vertices are *ε-close* if there exists a bijection $\varphi\colon V(H_1) \longrightarrow V(H_2)$ such that $|E(H_1)\Delta\varphi(E(H_2))| \le \varepsilon \cdot n^k$, where Δ denotes the symmetric difference. Next we state the notion of s-stability, which was used by Pikhurko [25].

Definition 18 (s-stability)**.** *Given a k-uniform hypergraph F. Call F s-stable, if for every $\varepsilon > 0$ there exists an $\omega = \omega(\varepsilon) > 0$ and an integer n_0 such that for arbitrary F-free k-uniform hypergraphs $H_1, \dots, H_{s+1}$ each of the same order $n \ge n_0$ and each having at least $\pi_F \cdot \binom{n}{k} - \omega \cdot n^k$ hyperedges, there are two which are ε-close.*

We, however, prove the structural theorem below only in the case when 1-stability is given.

Proof of Theorem 2. Let F be a fixed, linear hypergraph which is stable. We prove the Theorem only for $r = 3$, as the proof for $r = 2$ is very similar. Let $\delta > 0$ be given. Fix γ sufficiently small with $0 < \gamma < 1$ such that

$$194 \cdot \gamma + 66 \cdot h(k! \cdot 2 \cdot \gamma)/k! < \frac{\delta}{3} \quad \text{and} \quad 66 \cdot \gamma + 22 \cdot h(k! \cdot 2 \cdot \gamma)/k! < \omega(\delta/3)\,, \tag{68}$$

where $\omega(\delta/3)$ comes from the fact that F is stable, and $h(x)$ is the entropy function. Note that such a $\gamma > 0$ exists, since $h(x) \to 0$ as $x \to 0$. Let $\varepsilon = \varepsilon(\gamma) > 0$ with $\varepsilon < \gamma/4$ be such, that Lemma 5 is satisfied. Moreover, let $t_0 = \max\{1/\varepsilon, t'\}$, where t' is sufficiently large, $\alpha > 0$ sufficiently small, so that

$$\mathrm{ex}(t, F) \le (\pi_F + \alpha) \cdot \binom{t}{k}, \quad (\pi_F + \alpha) \cdot \binom{t}{k}/t^k > \pi_F/k!, \quad \alpha < k! \cdot \gamma \tag{69}$$

for every $t \geq t'$, and so that $t' \geq n_0$, where n_0 is as asserted by Definition 18 for $\varepsilon = \delta/3$. Let $T_0 = T_0(3, t_0, \varepsilon)$ and $N_0 = N_0(3, t_0, \varepsilon)$ be according to Theorem 3 and let $m_0 = m_0(\gamma)$ be according to Lemma 5. Finally, set $n_0 := \max\{N_0, T_0 \cdot m_0\}$.

Let H be a hypergraph on $n \geq n_0$ vertices, which admits at least $3^{\mathrm{ex}(n,F)}$ many F-free 3-colorings of its set of hyperedges. Let us denote the colors by red, blue, and green.

Consider any fixed F-free 3-coloring of the set of hyperedges of H. By Theorem 3 for $r = 3$ there exists a positive integer $T_0 = T_0(3, t_0, \varepsilon)$ and there exists a partition $\mathcal{P}$ with $V(H) = V_1 \dot\cup \cdots \dot\cup V_t$ of the vertex set $V(H)$, $t_0 \leq t \leq T_0$, which is ε-regular with respect to each color class, where $|V_i| \leq \lceil n/t \rceil$, $i \in [t]$. To simplify the calculations, we assume for convenience that $|V_i| = n/t \in \mathbb{N}$, $i \in [t]$. This assumption does not change our computations asymptotically.

Let $H_{\mathrm{red}}(\gamma), H_{\mathrm{blue}}(\gamma)$ and $H_{\mathrm{green}}(\gamma)$ be the corresponding cluster-hypergraphs on the vertex set $[t]$, i.e., $H_{\mathrm{col}}(\gamma)$ corresponds to the set of all those hyperedges with color col $\in \{\mathrm{red}, \mathrm{blue}, \mathrm{green}\}$, which are contained in ε-regular k-tuples of density at least γ. By our assumption and by Lemma 5 each hypergraph $H_{\mathrm{col}}(\gamma)$ is F-free, hence with (69) each contains at most $\mathrm{ex}(t, F) \leq (\pi_F + \alpha) \cdot \binom{t}{k}$ hyperedges.

Next we count the number of 3-colorings of the set of hyperedges of H, which yield the partition $\mathcal{P}$ of the vertex set with $V(H) = V_1 \dot\cup \cdots \dot\cup V_t$ and the cluster-hypergraphs $H_{\mathrm{red}}(\gamma)$, $H_{\mathrm{blue}}(\gamma)$, and $H_{\mathrm{green}}(\gamma)$. To do so, first we bound from above the number of hyperedges $e \in E(H)$, which are non-crossing with respect to the partition $\mathcal{P}$, or are contained in an ε-irregular k-tuple $(V_{i_1}, \ldots, V_{i_k})$, or, for one color class, are contained in a k-tuple $(V_{i_1}, \ldots, V_{i_k})$ of density less than γ, $1 \leq i_1 < \cdots < i_k \leq t$.

The number of non-crossing hyperedges $e \in E(H)$ is with $t \geq t_0 \geq 1/\varepsilon$ and $\varepsilon < \gamma/4$ for $k \geq 2$ at most

$$t \cdot \binom{n/t}{2} \cdot \binom{n}{k-2} < \frac{1}{2 \cdot t} \cdot n^k < \frac{\gamma}{8} \cdot n^k. \tag{70}$$

The number of crossing hyperedges $e \in E(H)$, which are contained in one of the at most $3 \cdot \varepsilon \cdot \binom{t}{k}$ ε-irregular k-tuples $(V_{i_1}, \ldots, V_{i_k})$, $1 \leq i_1 < \cdots < i_k \leq t$, is for $k \geq 2$ and $\varepsilon < \gamma/4$ at most

$$3 \cdot \varepsilon \cdot \binom{t}{k} \cdot \left(\frac{n}{t}\right)^k < \frac{3 \cdot \varepsilon}{2} \cdot n^k < \frac{3 \cdot \gamma}{8} \cdot n^k. \tag{71}$$

The number of crossing hyperedges $e \in E(H)$, which for one of the three color classes are contained in k-tuples $(V_{i_1}, \ldots, V_{i_k})$ of density less than γ, $1 \leq i_1 < \cdots < i_k \leq t$, is for $k \geq 2$ at most

$$3 \cdot \binom{t}{k} \cdot \gamma \cdot \left(\frac{n}{t}\right)^k < \frac{3 \cdot \gamma}{2} \cdot n^k. \tag{72}$$

Hence, the total number of all these hyperedges is by (70)–(72) less than

$$2 \cdot \gamma \cdot n^k, \tag{73}$$

and, using that $\binom{n}{\alpha n} \leq 2^{h(\alpha)n}$, for this set of hyperedges can be chosen in at most

$$\binom{\binom{n}{k}}{2 \cdot \gamma \cdot n^k} \quad < \quad \binom{n^k/k!}{2 \cdot \gamma \cdot n^k} \leq 2^{h(k! 2\gamma) n^k/k!} \tag{74}$$

ways and each can be colored by 3 colors, which gives the following upper bound on the number of colorings of this set of hyperedges

$$3^{2\gamma n^k}. \tag{75}$$

Next we consider the set of all remaining hyperedges in $E(H)$, i.e., those, which are contained in ε-regular k-tuples $(V_{i_1}, \dots, V_{i_k})$ of density at least γ for every color class, $1 \le i_1 < \dots < i_k \le t$.

If $\{i_1, \dots, i_k\}$ is a hyperedge in exactly s, $s \in [3]$, of the cluster-hypergraphs $H_{\mathrm{red}}(\gamma), H_{\mathrm{blue}}(\gamma), H_{\mathrm{green}}(\gamma)$, then in the hypergraph H every remaining hyperedge in the ε-regular k-tuple $(V_{i_1}, \dots, V_{i_k})$, is colored by one of s possible colors. As $e_H(V_{i_1}, \dots, V_{i_k}) \le (n/t)^k$, we can color these hyperedges in at most

$$s^{(n/t)^k} \tag{76}$$

ways. Let e_s be the number of k-tuples $\{i_1, \dots, i_k\}$, $1 \le i_1 < \dots < i_k \le t$, which are hyperedges in exactly s cluster-hypergraphs. Hence, the number of 3-colorings, given the partition $V(H) = V_1 \dot\cup \dots \dot\cup V_t$ and the cluster-hypergraphs $H_{\mathrm{red}}(\gamma), H_{\mathrm{blue}}(\gamma), H_{\mathrm{green}}(\gamma)$, is by (74)–(76) at most

$$2^{h(k!2\gamma)n^k/k!} \cdot 3^{2\gamma n^k} \cdot (1^{e_1} \cdot 2^{e_2} \cdot 3^{e_3})^{(n/t)^k} = 2^{h(k!2\gamma)n^k/k!} \cdot 3^{2\gamma n^k} \cdot (2^{e_2} \cdot 3^{e_3})^{(n/t)^k}. \tag{77}$$

By assumption and by Lemma 5, each cluster-hypergraph $H_{\mathrm{col}}(\gamma)$ is F-free, hence each contains at most $\mathrm{ex}(t, F)$ hyperedges, i.e., $e(H_{\mathrm{col}}(\gamma)) \le \mathrm{ex}(t, F)$ for every $\mathrm{col} \in \{\mathrm{red}, \mathrm{blue}, \mathrm{green}\}$. Observe that

$$\begin{aligned} 2 \cdot e_2 + 3 \cdot e_3 \le e_1 + 2 \cdot e_2 + 3 \cdot e_3 &= e(H_{\mathrm{red}}(\gamma)) + e(H_{\mathrm{blue}}(\gamma)) + e(H_{\mathrm{green}}(\gamma)) \\ &\le 3 \cdot \mathrm{ex}(t, F), \end{aligned} \tag{78}$$

thus with (69)

$$e_2 \le \frac{3 \cdot \left((\pi_F + \alpha) \cdot \binom{t}{k} - e_3\right)}{2}. \tag{79}$$

With $2 < 3^{7/11}$ we infer

$$2^{e_2} \cdot 3^{e_3} \overset{(79)}{\le} 3^{\frac{21}{22}(\pi_F + \alpha)\binom{t}{k} + e_3/22}. \tag{80}$$

Assume first that for every choice of an F-free 3-coloring of the set of hyperedges of H we have

$$e_3 < (\pi_F + \alpha) \cdot \binom{t}{k} - 66 \cdot \gamma \cdot t^k - 22 \cdot h(k! \cdot 2 \cdot \gamma) \cdot t^k/k!.$$

Then, we obtain

$$2^{e_2} \cdot 3^{e_3} \overset{(80)}{<} 3^{(\pi_F + \alpha)\binom{t}{k} - 3\gamma t^k - h(k!2\gamma)t^k/k!}. \tag{81}$$

Recalling that there are at most n^{T_0} partitions of the vertex set $V(H)$ into at most T_0 classes and that for $k \ge 2$ there are at most $2^{3\binom{T_0}{k}} < 2^{2T_0^k}$ choices for the cluster-hypergraphs $H_{\mathrm{red}}(\gamma), H_{\mathrm{blue}}(\gamma), H_{\mathrm{green}}(\gamma)$, we infer from (77) and (81) that the total number of such 3-colorings of H is at most

$$\begin{aligned} &n^{T_0} \cdot 2^{2T_0^k} \cdot 2^{h(k!2\gamma)n^k/k!} \cdot 3^{2\gamma n^k} \cdot \left(3^{(\pi_F + \alpha)\binom{t}{k} - 3\gamma t^k - h(k!2\gamma)t^k/k!}\right)^{(n/t)^k} \\ &\le n^{T_0} \cdot 2^{2T_0^k} \cdot 2^{h(k!2\gamma)n^k/k!} \cdot 3^{2\gamma n^k} \cdot 3^{(\pi_F + \alpha)\binom{n}{k} - 3\gamma n^k - h(k!2\gamma)n^k/k!} \\ &< n^{T_0} \cdot 2^{2T_0^k} \cdot 3^{(\pi_F + \alpha)\binom{n}{k} - \gamma n^k} < 3^{\mathrm{ex}(n, F)} \end{aligned}$$

for sufficiently large n, as $\gamma > 0$ is fixed, which contradicts our assumption. The first inequality holds, since $[n]_k/n^k$ is increasing for fixed k.

Hence, there exists an F-free 3-coloring of the set of hyperedges of H, and a partition $V(H) = V_1\dot{\cup}\cdots\dot{\cup}V_t$, $t \le T_0$, and cluster-hypergraphs $H_{\mathrm{red}}(\gamma)$, $H_{\mathrm{blue}}(\gamma)$, $H_{\mathrm{green}}(\gamma)$ such that

$$e_3 \ge (\pi_F + \alpha)\cdot\binom{t}{k} - 66\cdot\gamma t^k - 22\cdot h(k!\cdot 2\cdot\gamma)\cdot t^k/k! \tag{82}$$

holds. We infer

$$e_1 + e_2 \le e_1 + 2\cdot e_2 \overset{(78),(82)}{\le} 192\cdot\gamma\cdot t^k + 66\cdot h(k!\cdot 2\cdot\gamma)\cdot t^k/k!. \tag{83}$$

Let H_3 be the underlying subhypergraph of H which consists of all hyperedges which are contained in one of the ε-regular k-tuple of density at least γ in *every* color. Then H_3 must be an F-free hypergraph, as otherwise the corresponding k-tuples that contain the hyperedges of the copy of F form a copy of F in each of the cluster hypergraphs $H_{\mathrm{red}}(\gamma), H_{\mathrm{blue}}(\gamma), H_{\mathrm{green}}(\gamma)$, and hence there will be many monochromatic copies of F in H which is impossible.

Note that

$$|E(H)\setminus E(H_3)| \overset{(73),(83)}{\le} 2\cdot\gamma\cdot n^k + [192\cdot\gamma\cdot t^k + 66\cdot h(k!\cdot 2\cdot\gamma)\cdot t^k/k!](n/t)^k \overset{(68)}{<} \delta\cdot n^k/3,$$

and, by (82)

$$e(H) \ge e_3\cdot(n/t)^k \overset{(82)}{\ge} (\pi_F+\alpha)\cdot\binom{t}{k}\cdot(n/t)^k - \omega(\delta/3)\cdot n^k \overset{(69)}{\ge} \pi_F\cdot\binom{n}{k} - \omega(\delta/3)\cdot n^k.$$

Furthermore, the reasoning above applies for any $H \in \{H_1, H_2\}$. Thus, we obtain from the stability of F that

$$|E(H_1)\Delta E(H_2)| < \delta\cdot n^k.$$

□

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