An Introduction to the Contraction Method

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Bucket Selection

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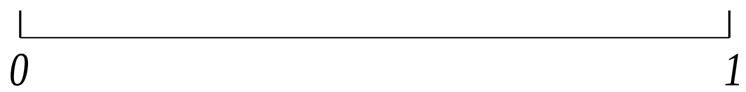

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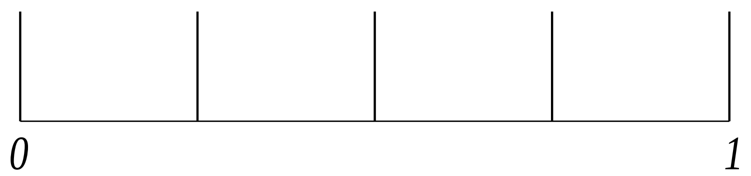

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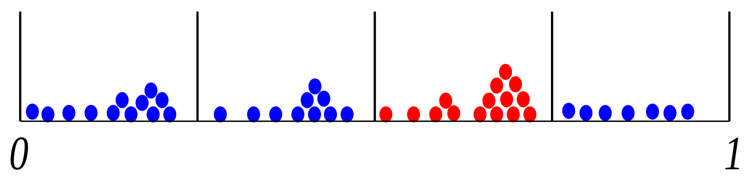

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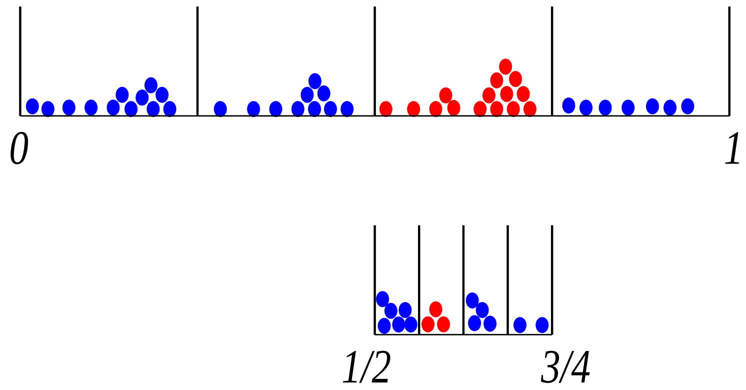

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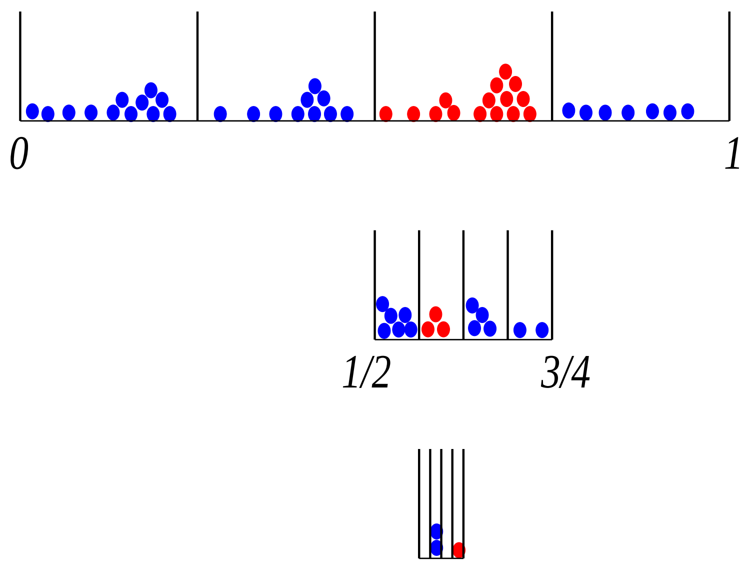

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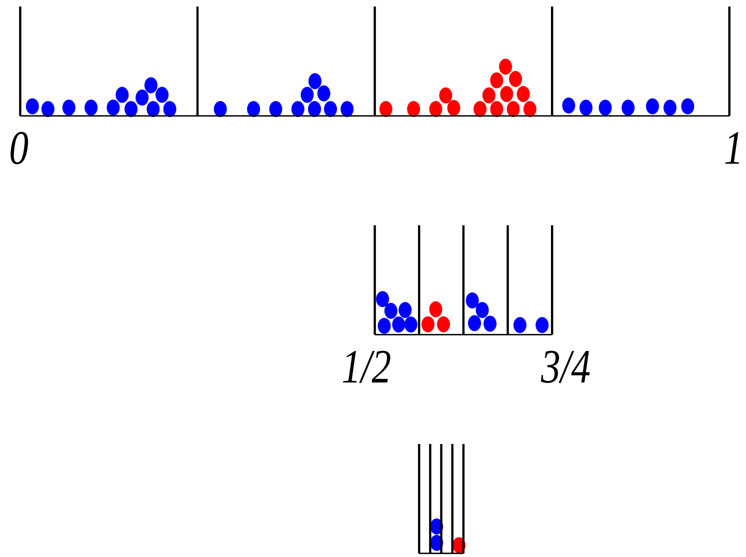

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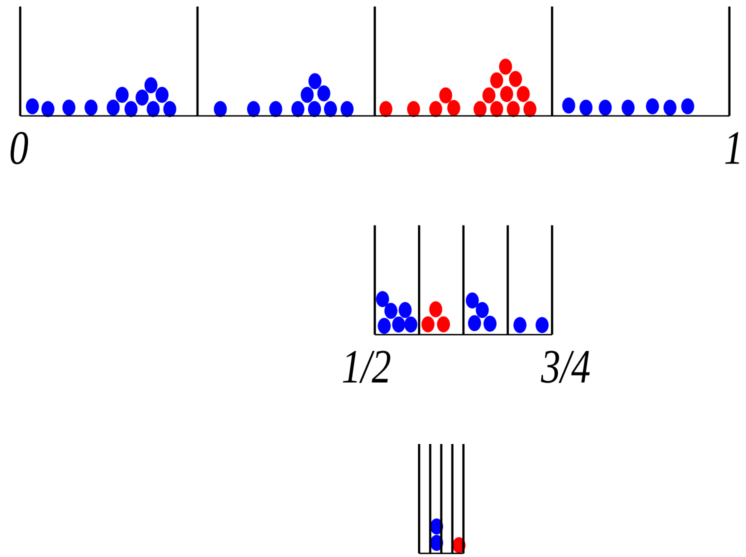

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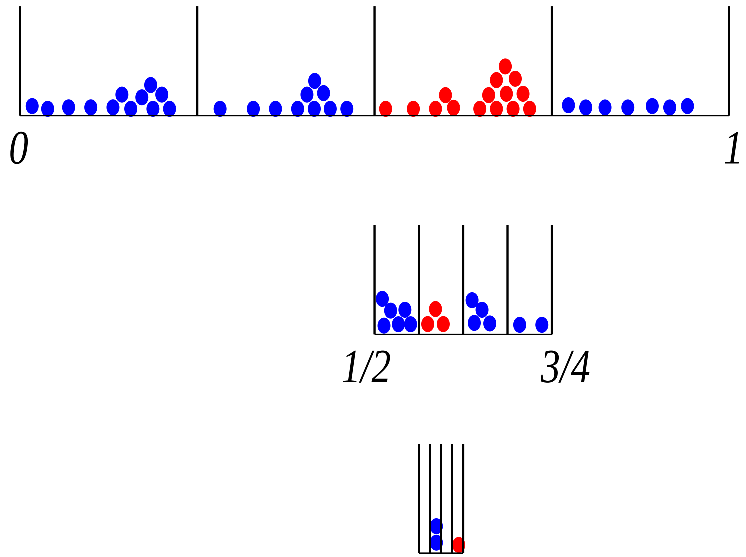

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$I_n = \#$ items in relevant bucket

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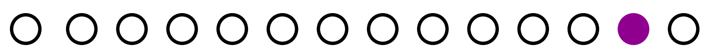

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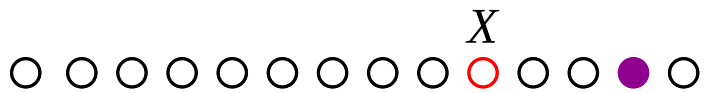

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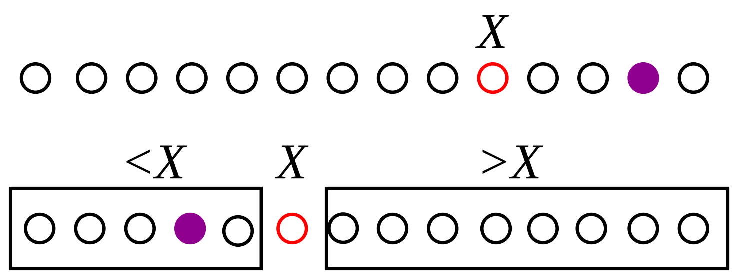

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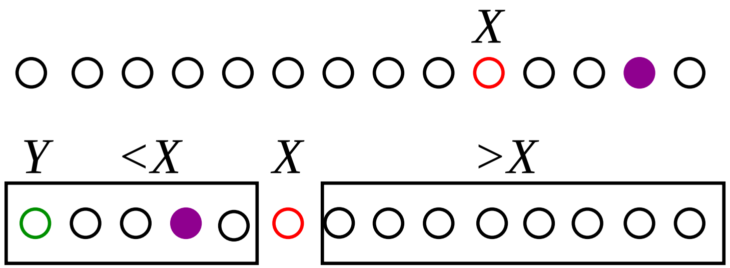

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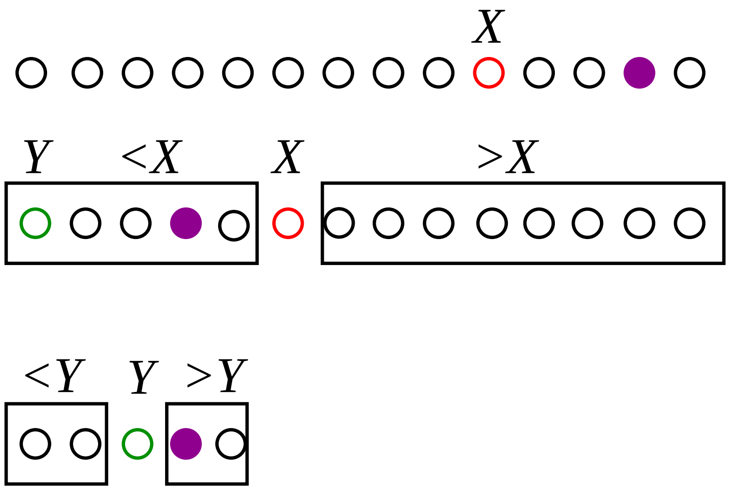

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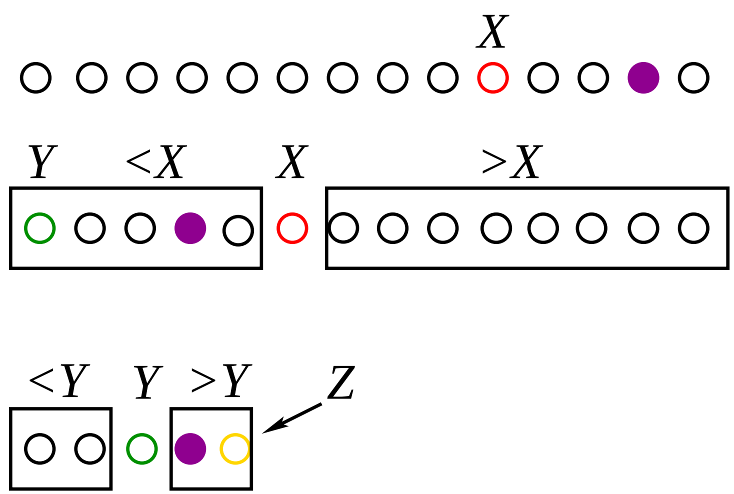

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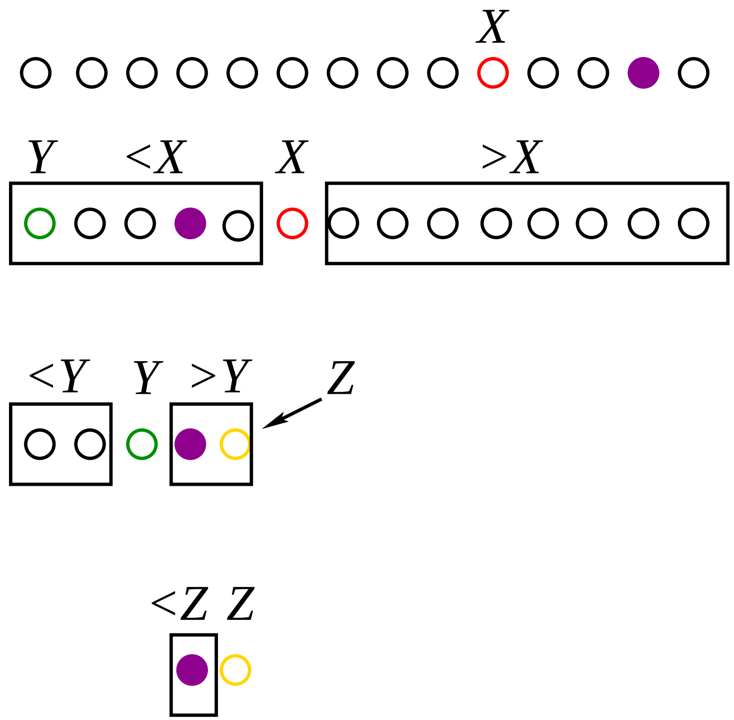

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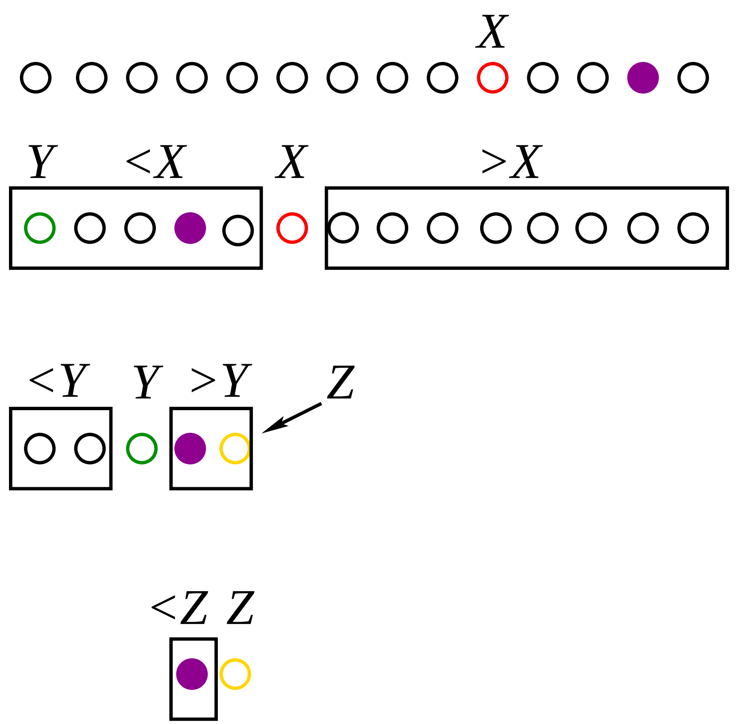

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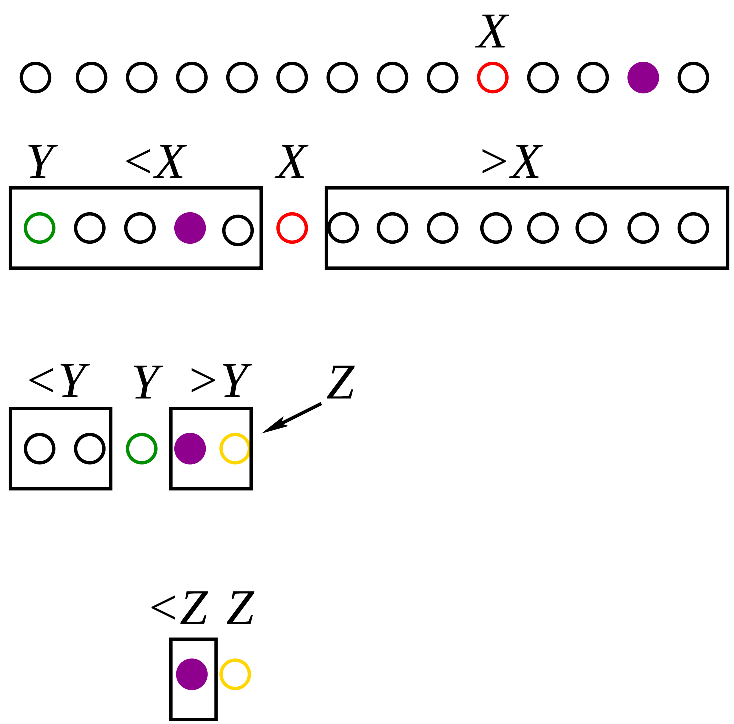

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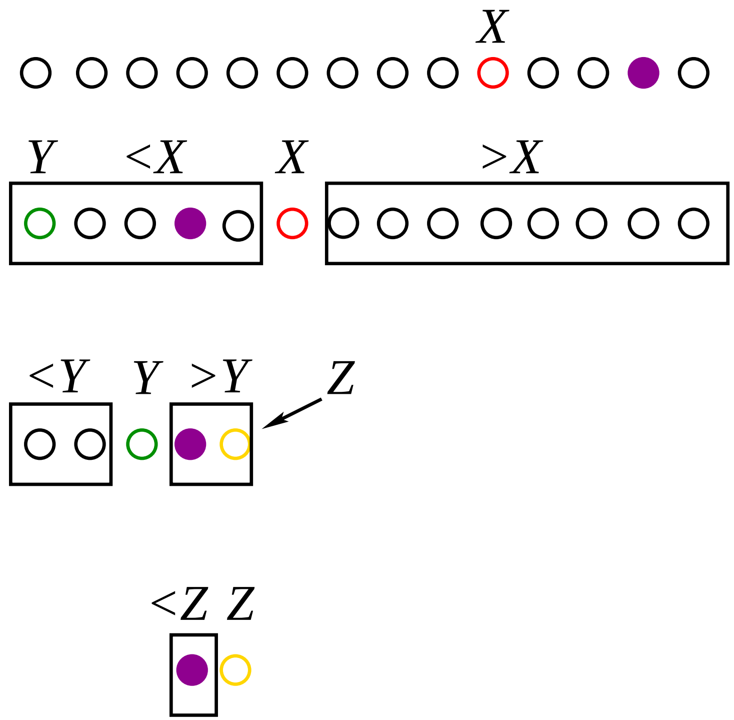

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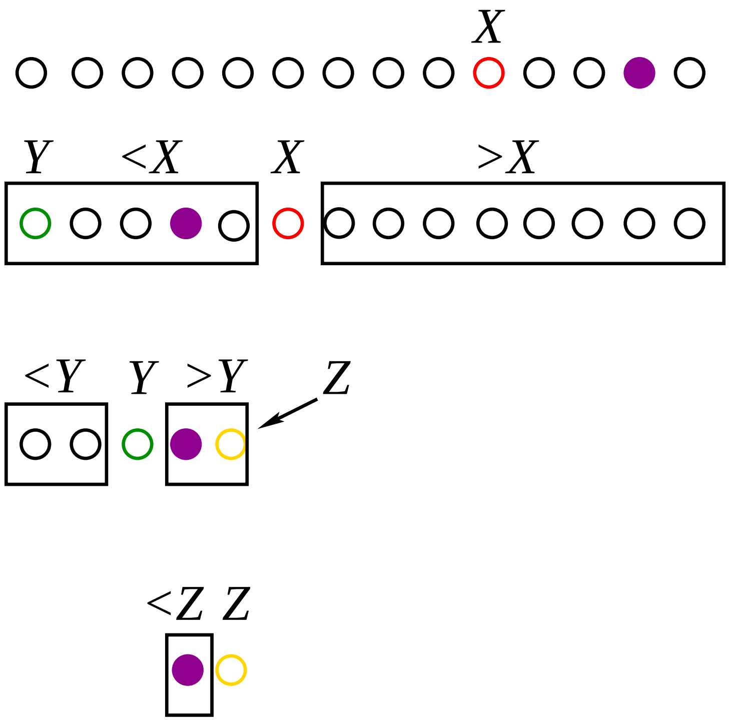

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Given numbers: $6, 1, 8, 7, 5, 3, 10, 2, 11, 4, 9.$

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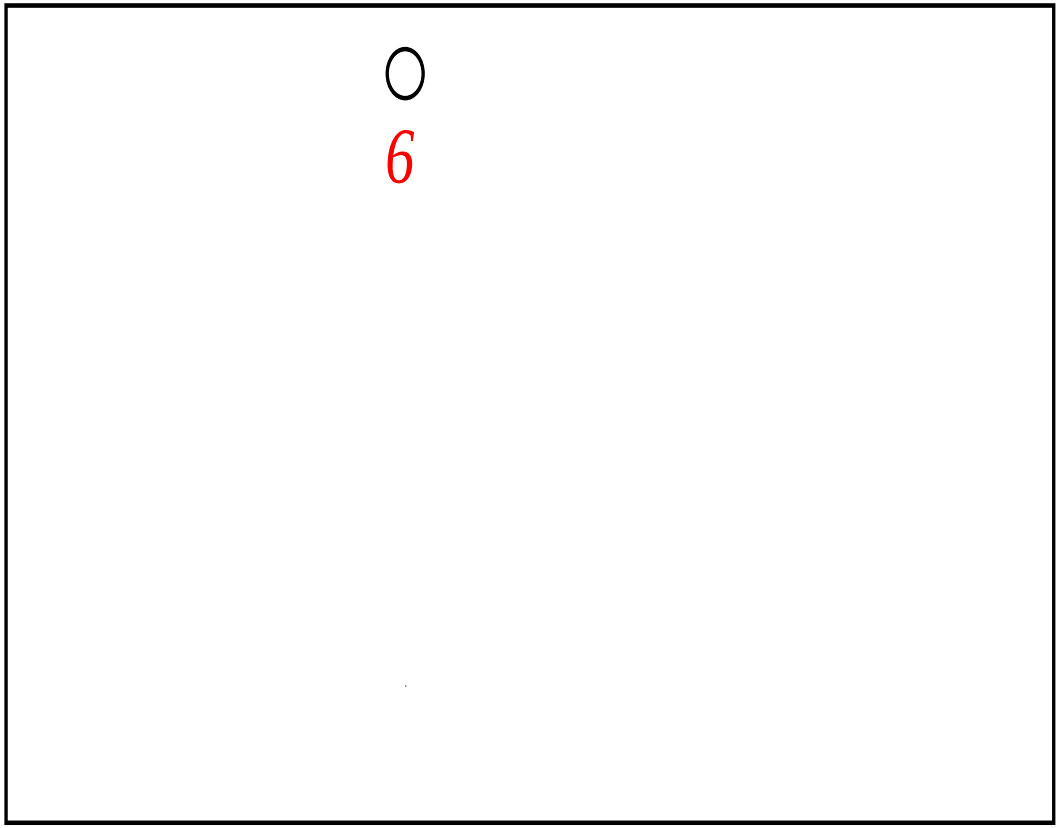

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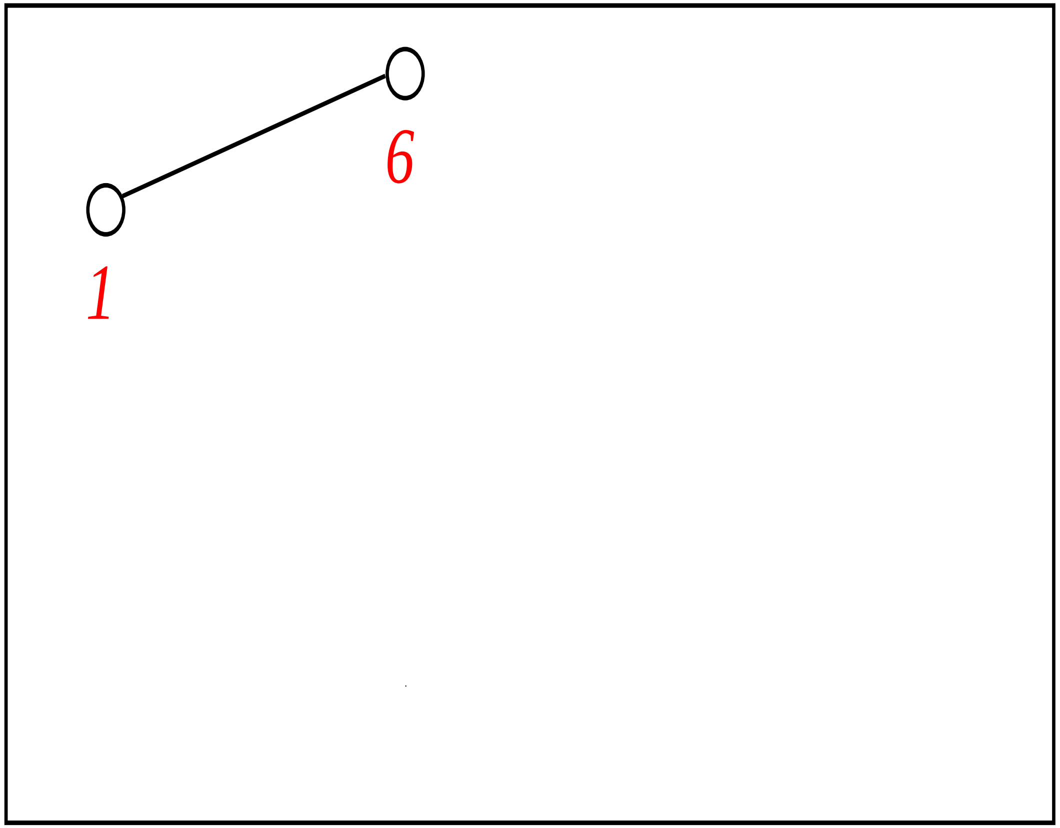

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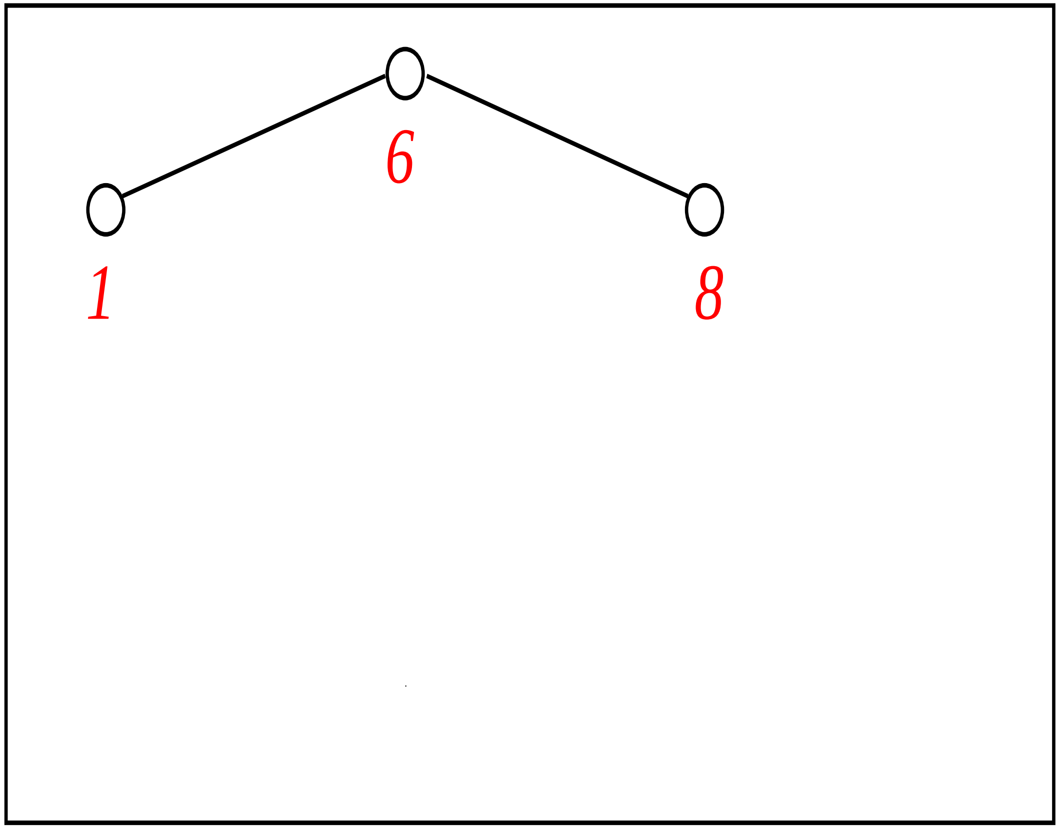

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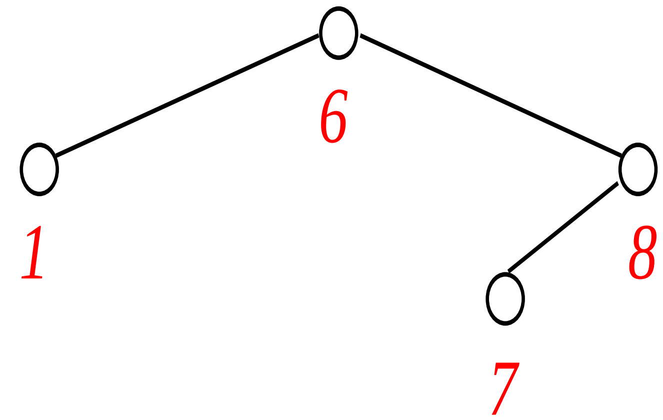

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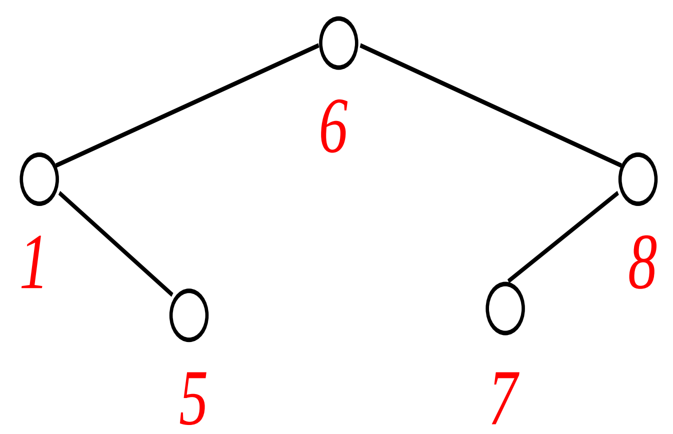

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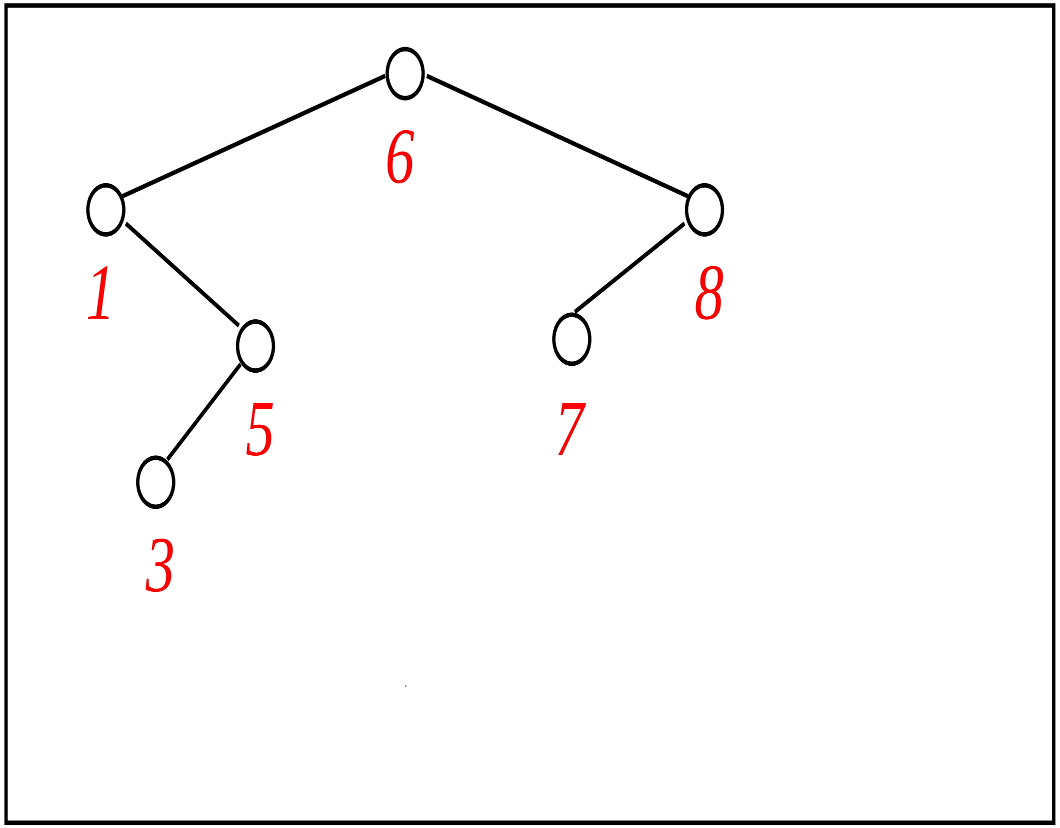

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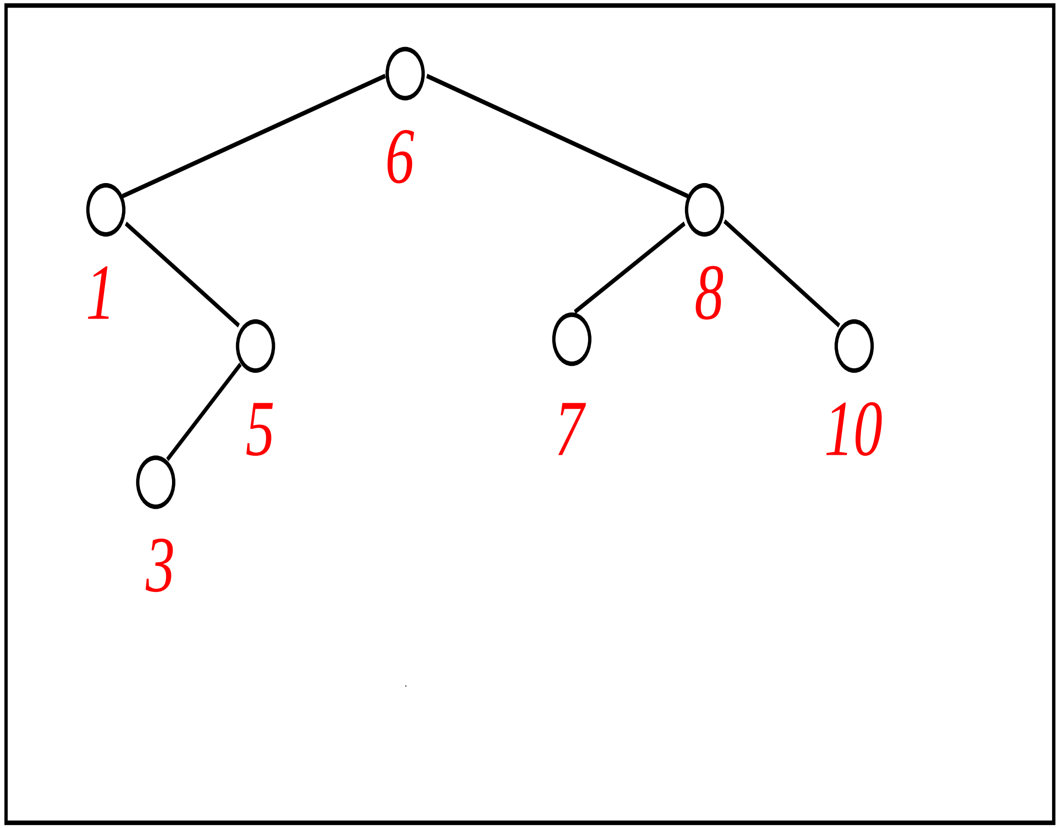

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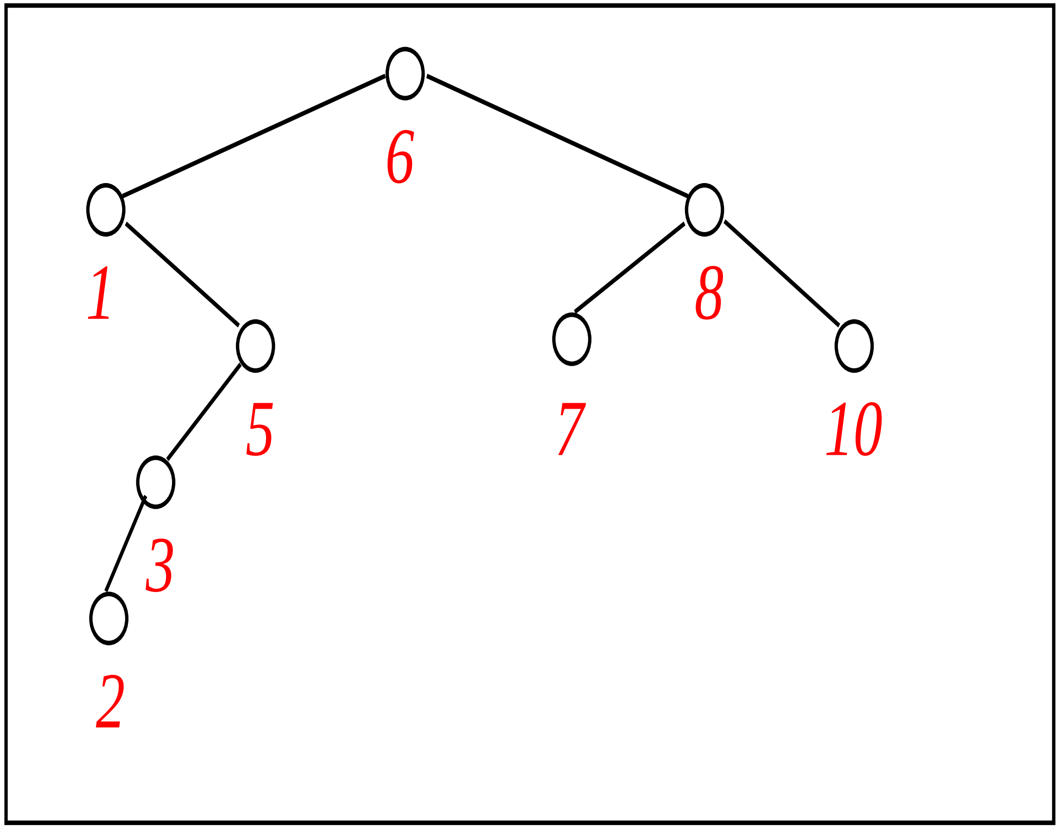

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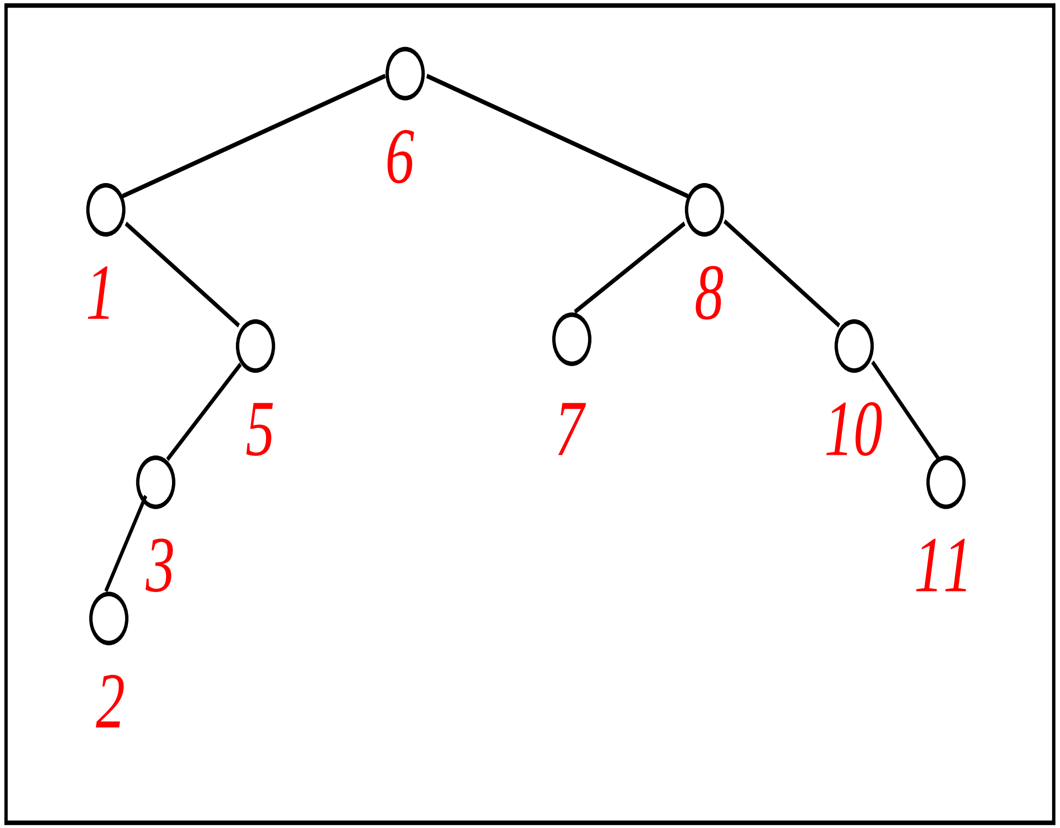

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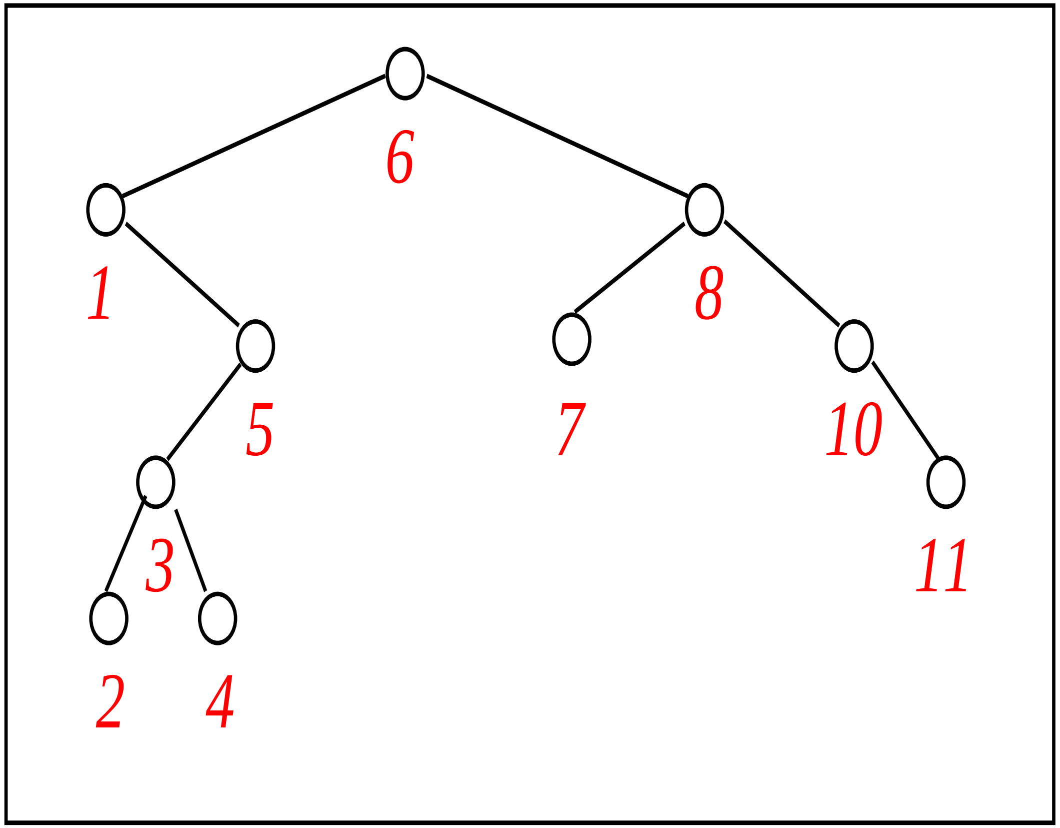

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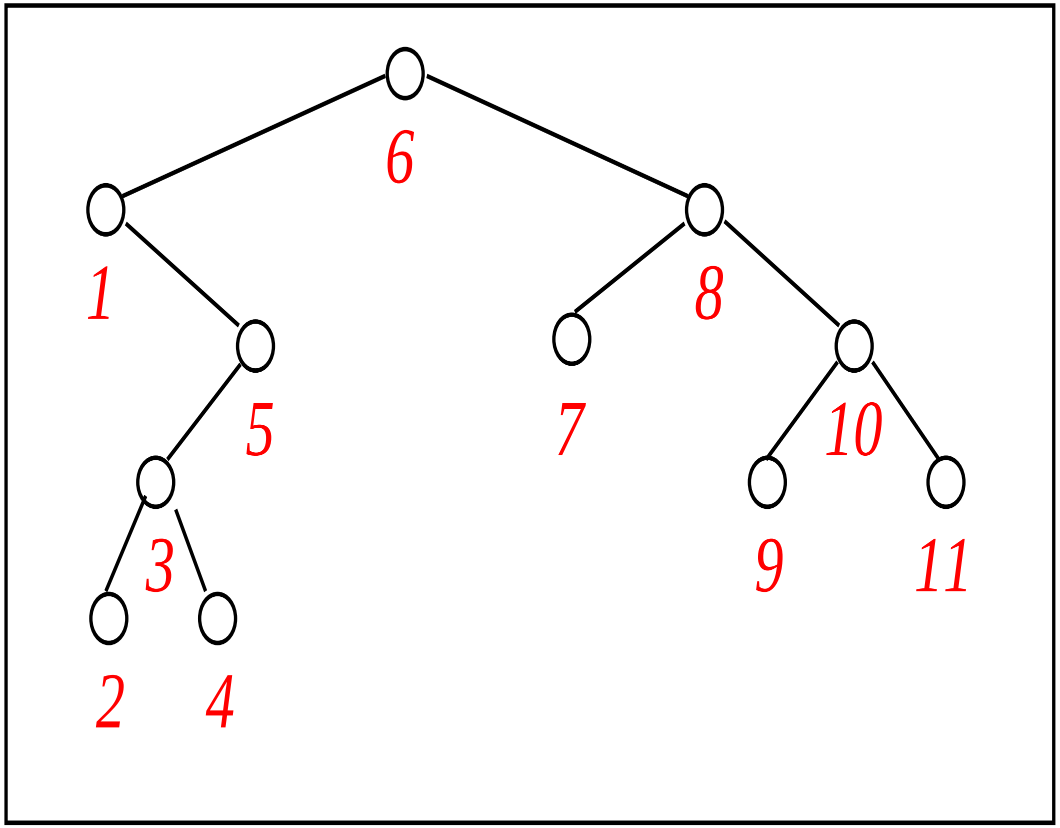
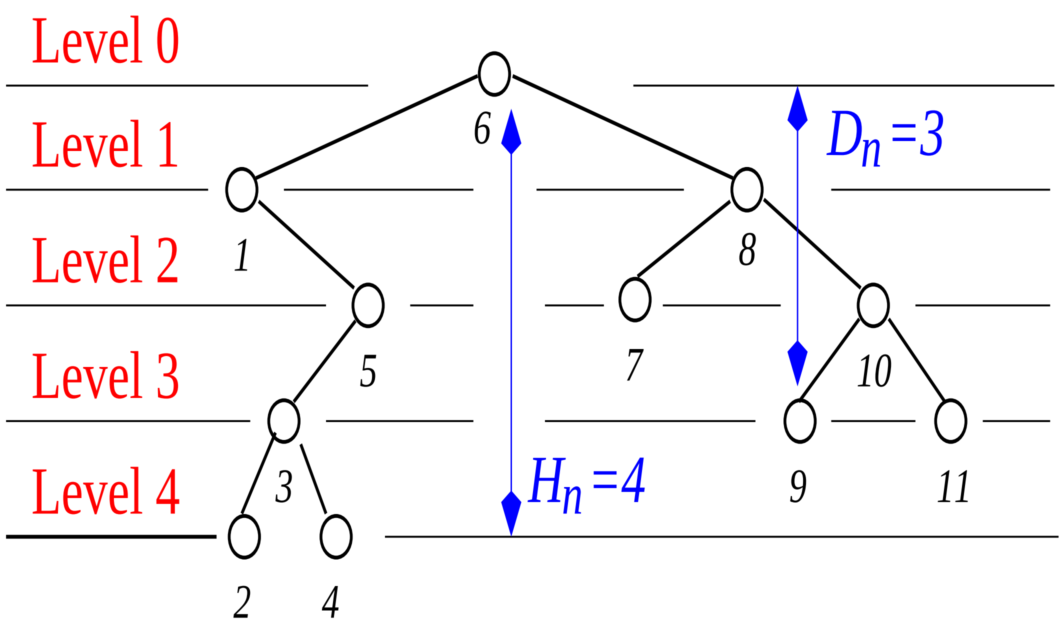


Quantities in BST



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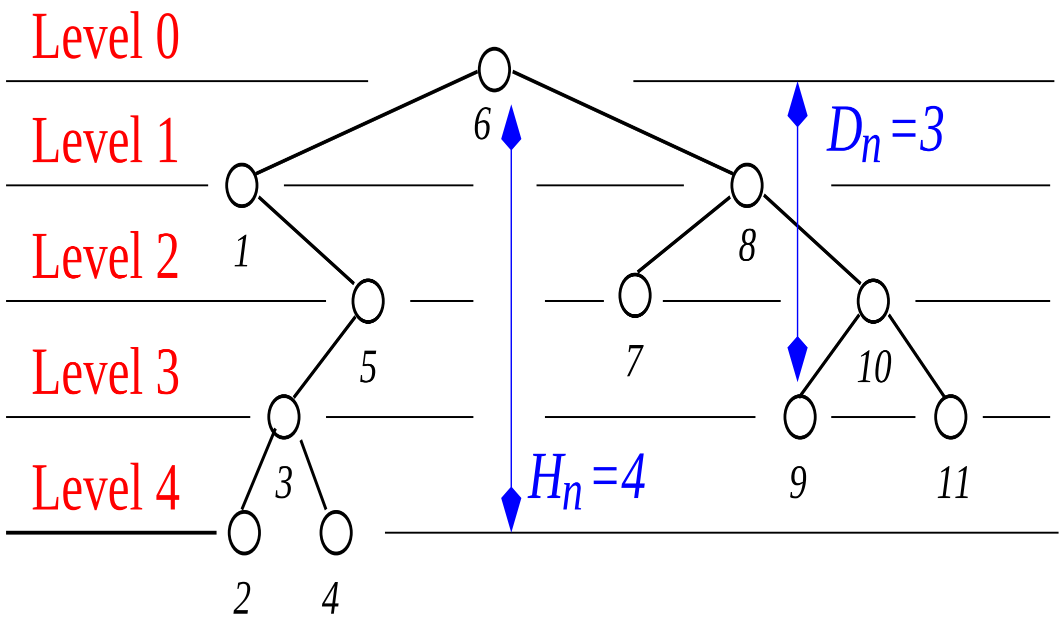
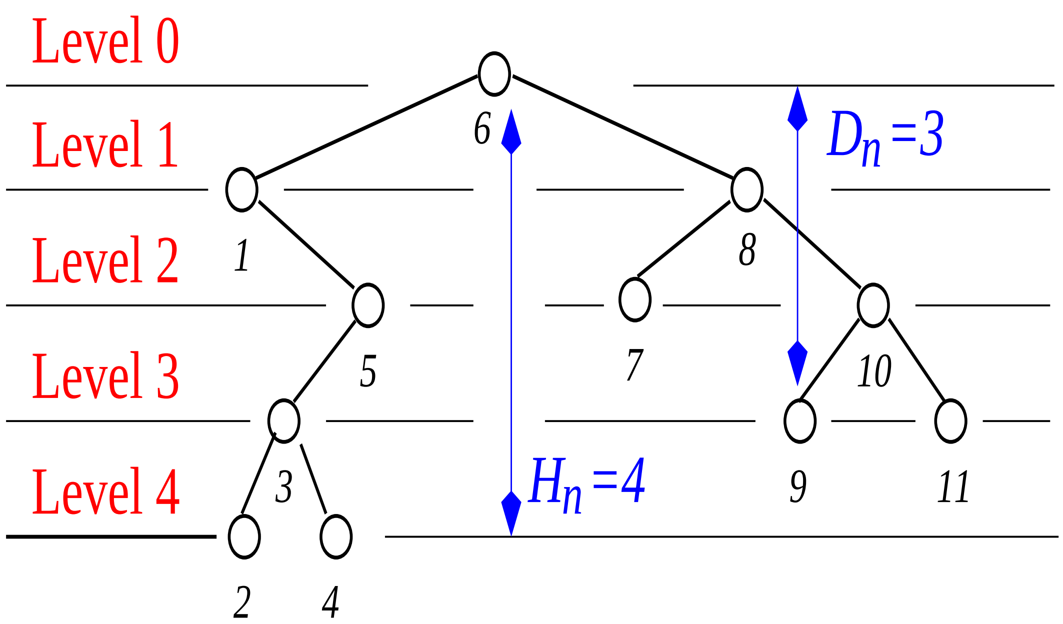


D_n — depth = distance root to n-th inserted node

Quantities in BST



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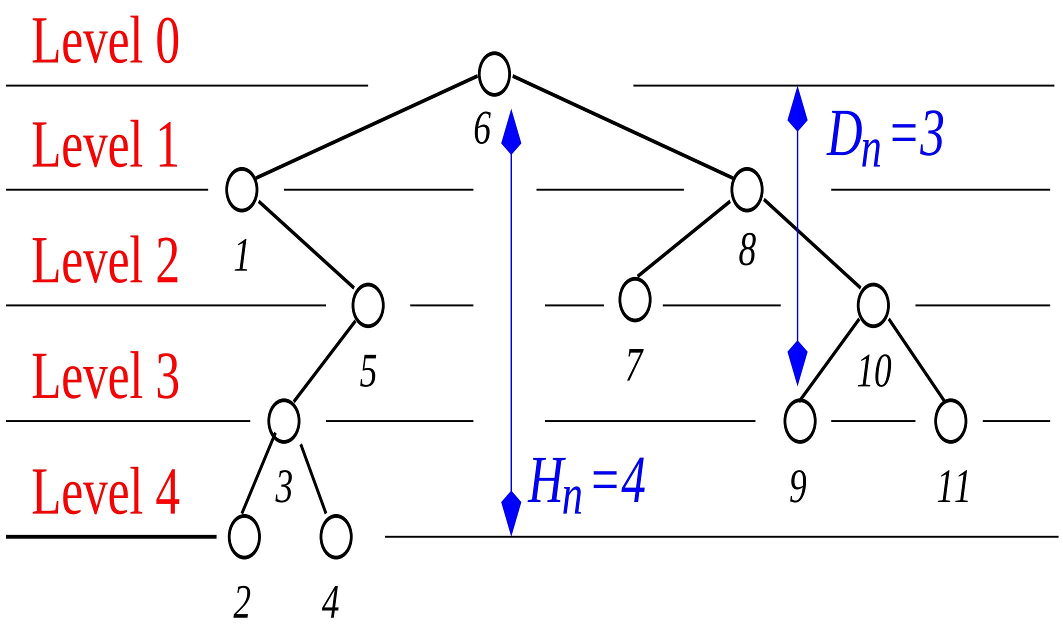
Quantities in BST



D_n — depth = distance root to n-th inserted node

$H_n = \max_{1 \le j \le n} D_j$ — height

$Q_n = \sum_{1 \le j \le n} D_j$ — internal path length

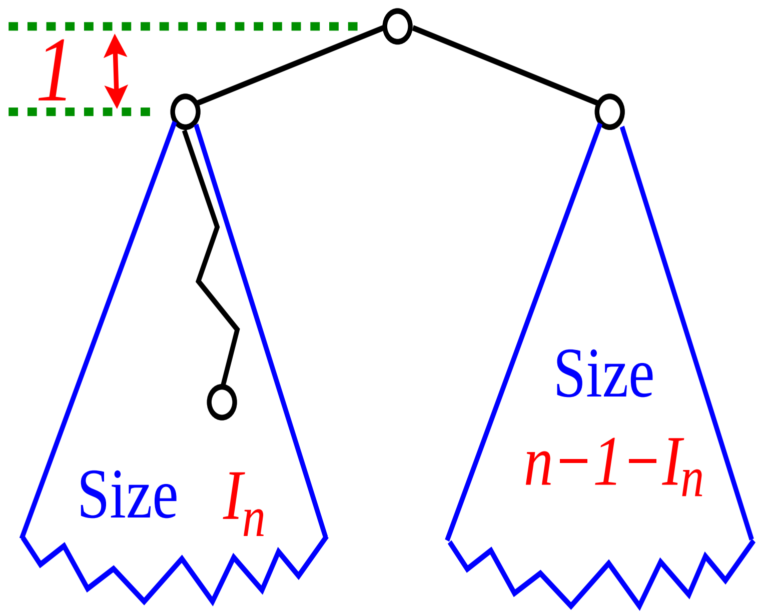
Random binary search tree

Model of randomness:

All permutations of $1, \ldots, n$ equally likely.

Equivalent: Use $U_1, \ldots, U_n$ i.i.d. unif$[0, 1]$.

Internal path length



Internal path length Q_n:

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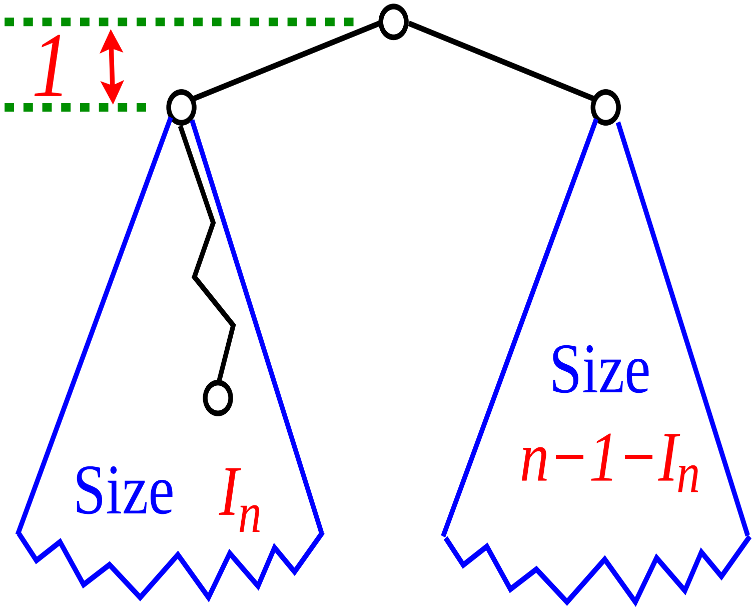
Internal path length

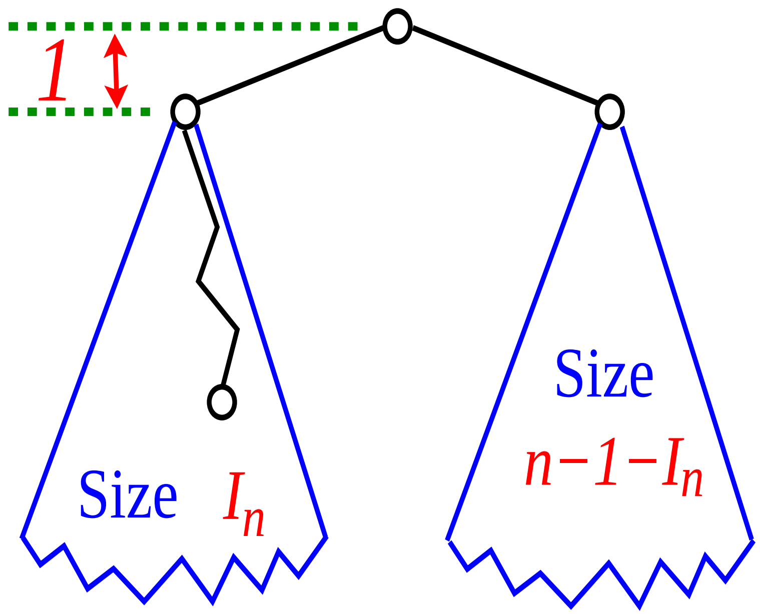


Internal path length Q_n:

$$Q_n \stackrel{d}{=} Q^*_{I_n} + Q^{**}_{n-1-I_n} + n - 1$$

.

Internal path length



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.

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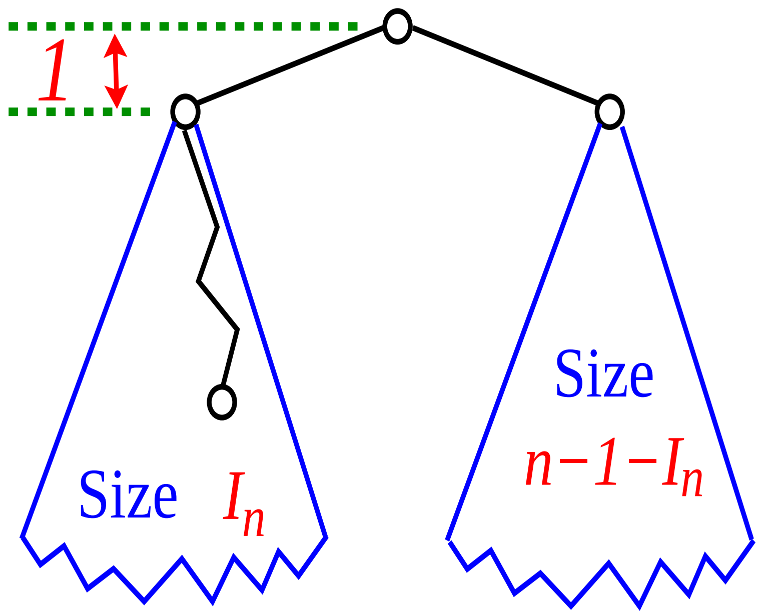

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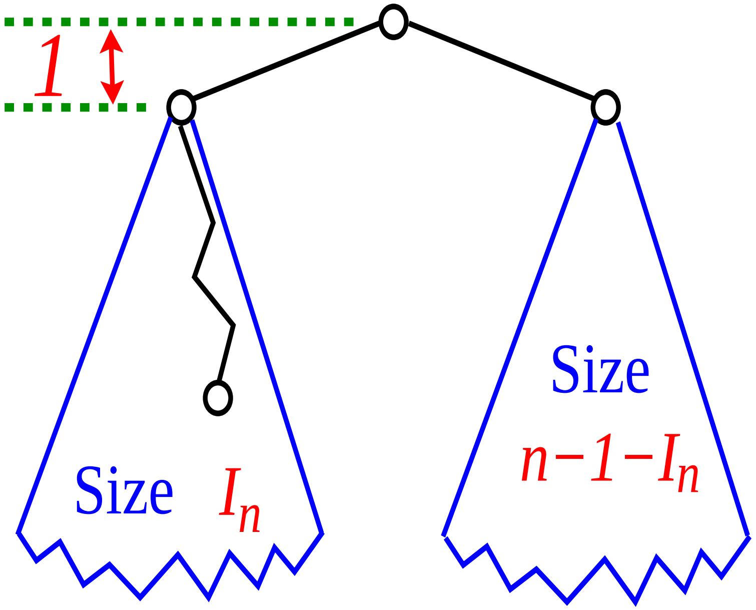

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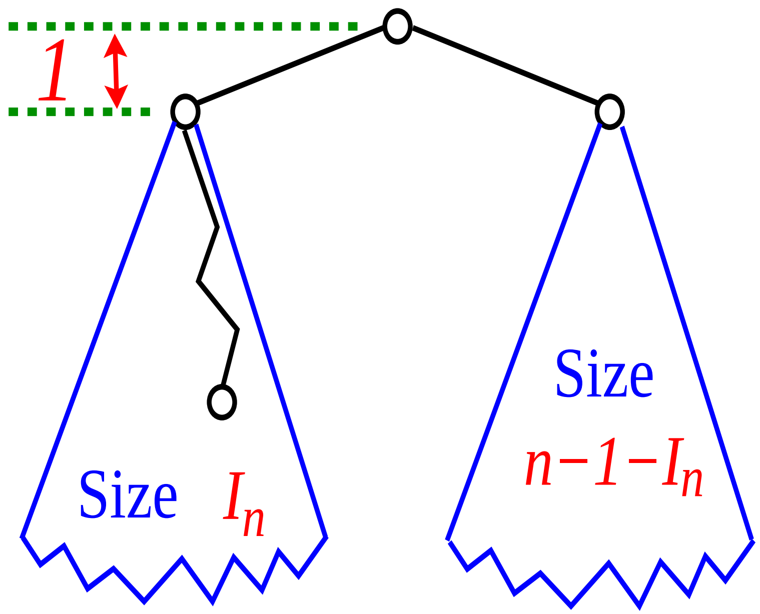

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This needs a proof!

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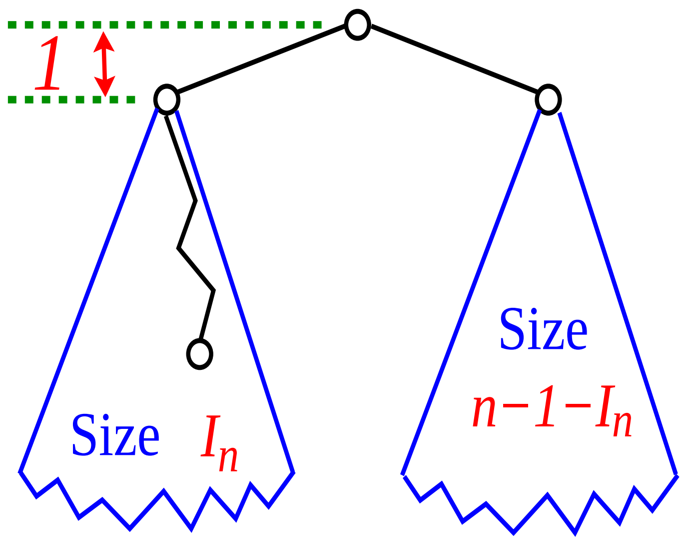

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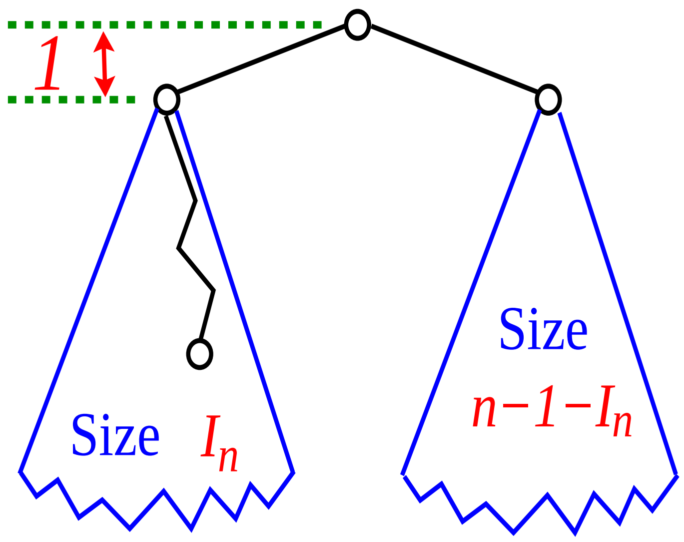

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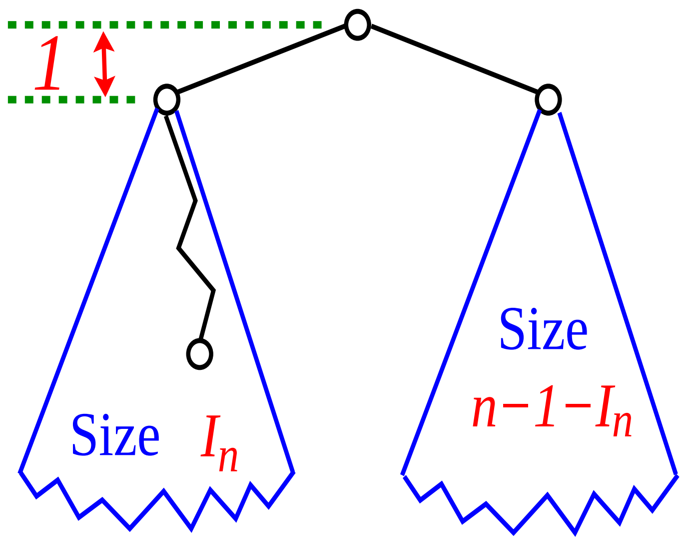

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Sufficient: $\mathbb{P}(Q_n = j) = \mathbb{P}(Z_n = j)$ for all $j \in \mathbb{N}$.

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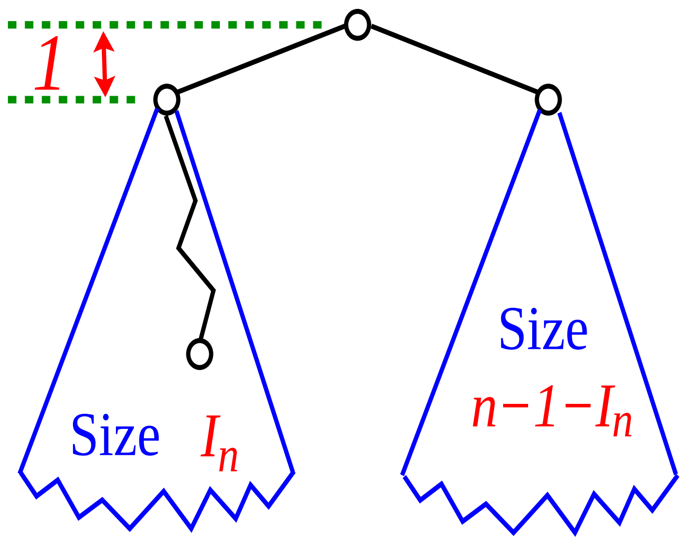

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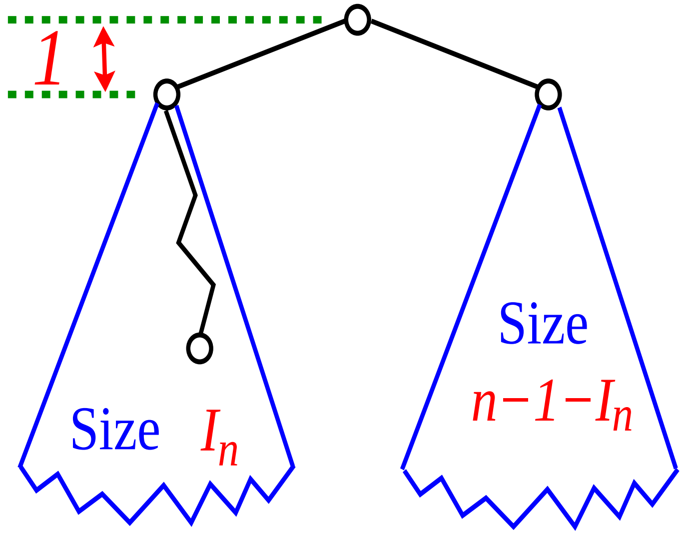

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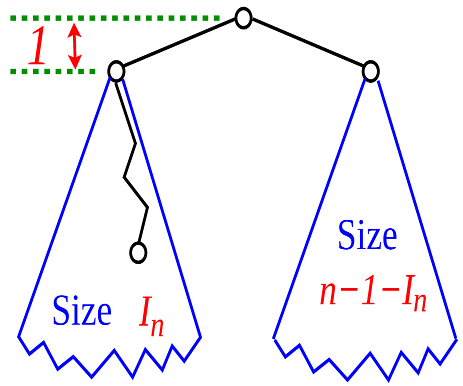
[Total probability theorem yields:

$$\begin{aligned} \mathbb{P}(Q_n = j) &= \sum_k \mathbb{P}(I_n = k)\, \mathbb{P}(Q_n = j \,|\, I_n = k) \\ &= \sum_k \mathbb{P}(I_n = k)\, \mathbb{P}(Z_n = j \,|\, I_n = k) = \mathbb{P}(Z_n = j).] \end{aligned}$$

Proof of the recurrence



To prove:

$$\mathbb{P}(Q_n = j \,|\, I_n = k)$$

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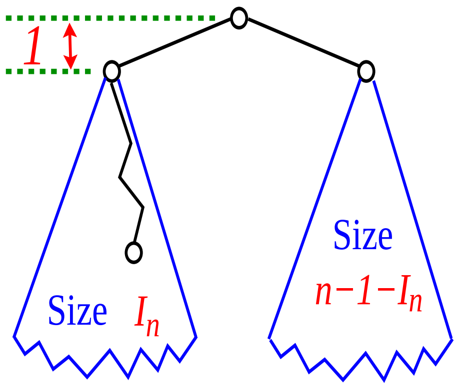

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We show: Given $\{I_n = k\}$ we have:

a) $\mathcal{T}_1$ and $\mathcal{T}_2$ independent, b) $\mathcal{T}_1$ and $\mathcal{T}_2$ random BSB.

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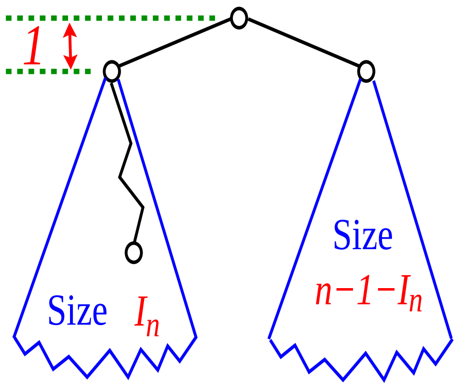

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$\pi = (5, 7, 3, 2, 6, 8, 1, 4)$

π equiprobable permutation in S_n.

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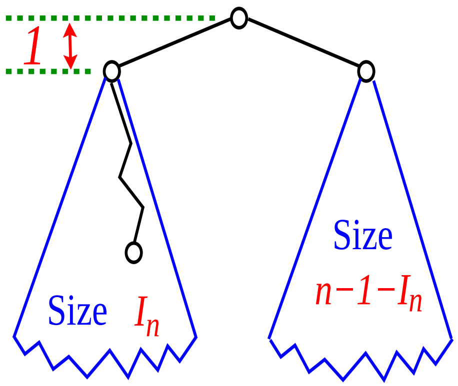

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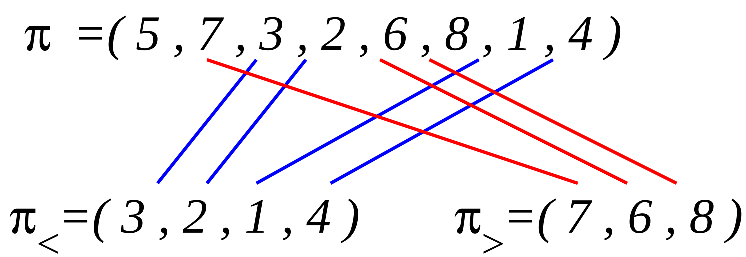

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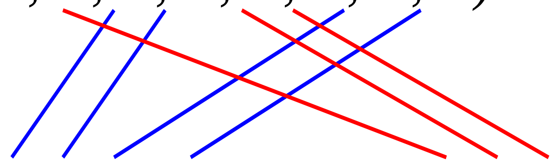
Construct $\pi_<$ and $\pi_>$

Proof of the recurrence II

π equiprobable in S_n.

$$\pi = (5, 7, 3, 2, 6, 8, 1, 4)$$



$$\pi_< = (3, 2, 1, 4) \qquad \pi_> = (7, 6, 8)$$

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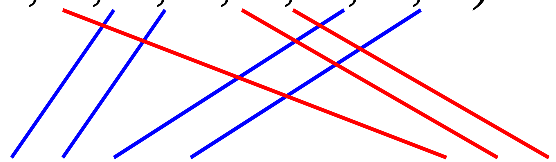

$$\pi_< = (3, 2, 1, 4) \qquad \pi_> = (7, 6, 8)$$

We show: Given $\pi_1 = k + 1$:

$\pi_<$ and $\pi_>$ independent and equiprobable on S_k and S_{n-1-k}. resp.

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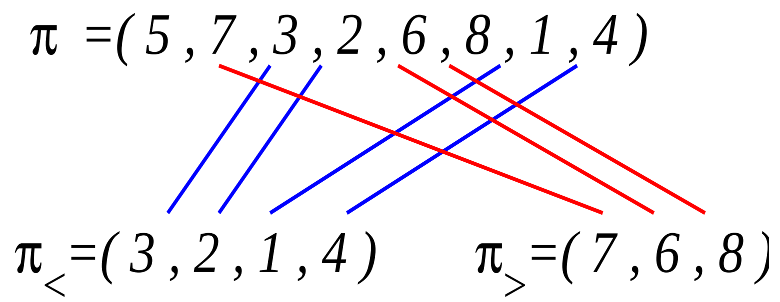

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For arbitrary $\sigma \in S_k$:

$$\mathbb{P}(\pi_< = \sigma \,|\, \pi_1 = k + 1) = \frac{1}{1/n} \frac{\binom{n-1}{k}(n - 1 - k)!}{n!} = \frac{1}{k!}$$

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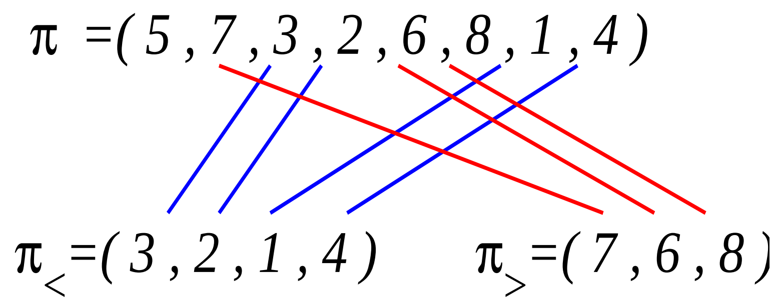

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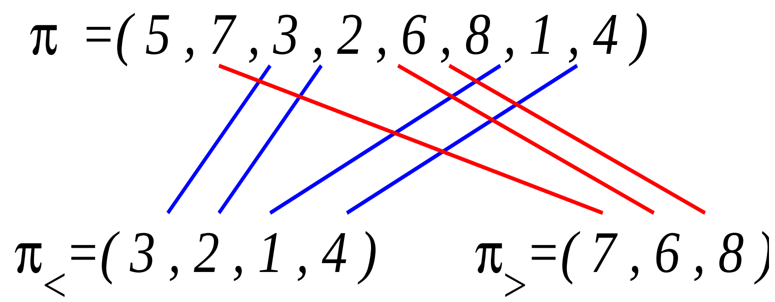

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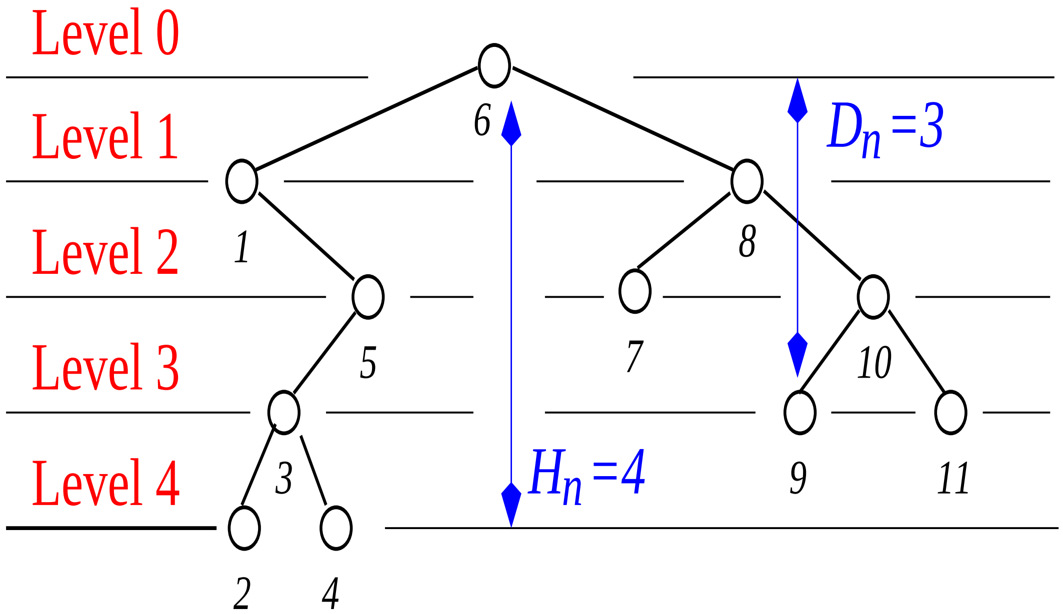
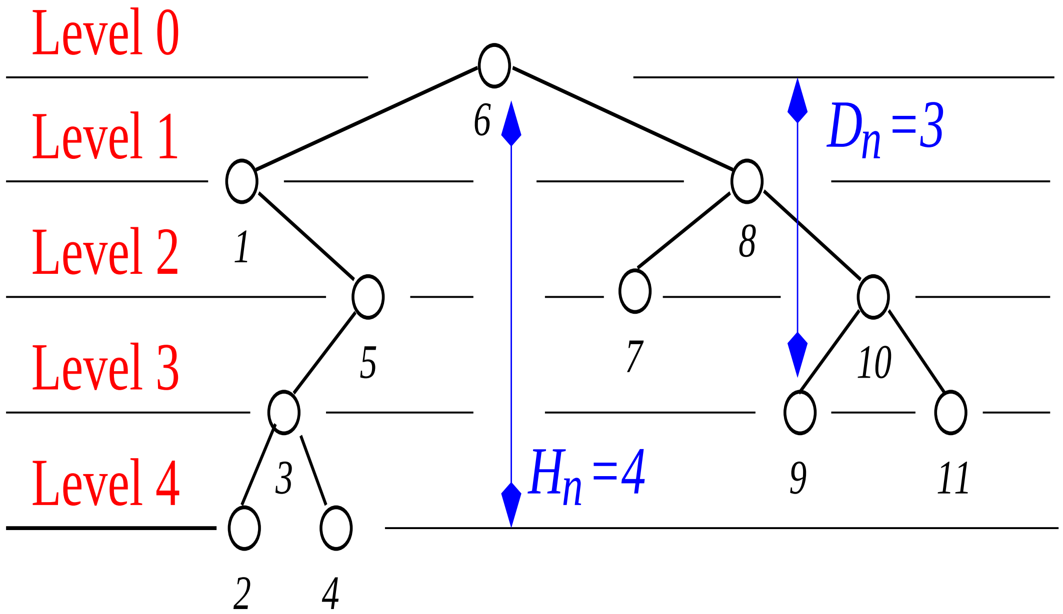
Second assertion similar.

Other BST recurrences



$$Q_n \stackrel{d}{=} Q^{(1)}_{I_n} + Q^{(2)}_{n-1-I_n} + n - 1$$

Other BST recurrences



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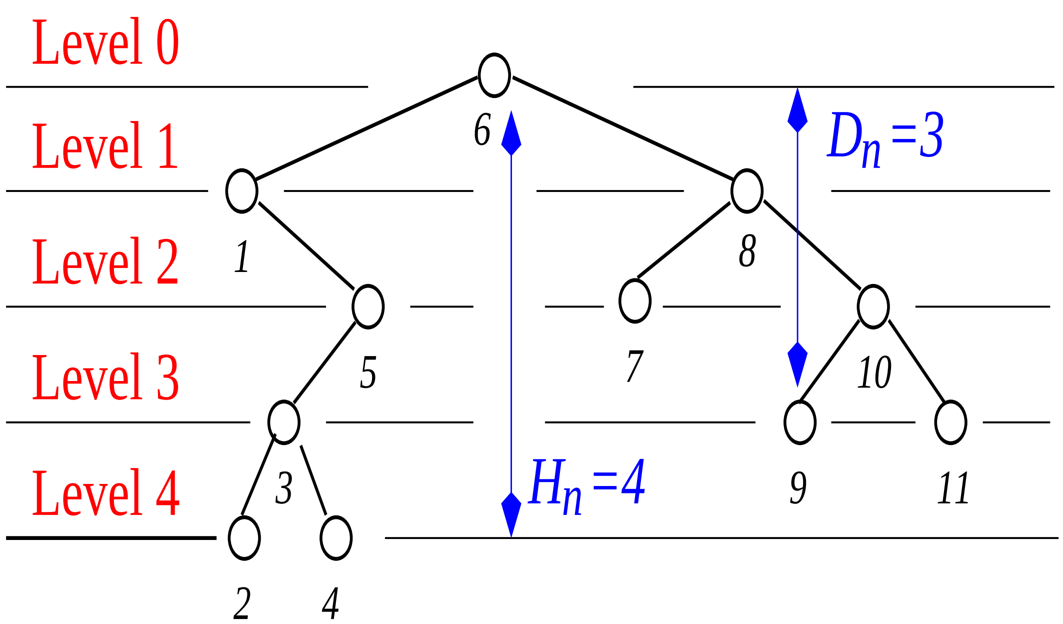

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$$D_n \stackrel{d}{=} \mathbf{1}_{A_n} D_{I_n} + \mathbf{1}_{A_n^c} D_{n-1-I_n} + 1$$

Expected internal path length

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$$Q_n \stackrel{d}{=} Q^*_{I_n} + Q^{**}_{n-1-I_n} + n - 1, \quad I_n \stackrel{d}{=} \mathrm{unif}\{0, \ldots, n-1\}.$$

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$$q_n := \mathbb{E}\, Q_n = \mathbb{E}\left[Q^*_{I_n} + Q^{**}_{n-1-I_n} + n - 1\right]$$

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Solves easily:

$$q_n = 2(n+1)\mathcal{H}_n - 4n = 2n\log(n) + (2\gamma - 4)n + o(n).$$

[$\mathcal{H}_n := \sum_{i=1}^n 1/i$ harmonic numbers.]

Rescaling

$$Q_n \stackrel{d}{=} Q^*_{I_n} + Q^{**}_{n-1-I_n} + n - 1, \quad q_n = 2n\log(n) + cn + o(n).$$

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$$Y_n := \frac{Q_n - q_n}{n}.$$

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Then

$$Y_n \stackrel{d}{=} \frac{1}{n}\left(Q^*_{I_n} + Q^{**}_{n-1-I_n} + n - 1 - q_n\right)$$

Rescaling

$$Q_n \stackrel{d}{=} Q^*_{I_n} + Q^{**}_{n-1-I_n} + n - 1, \qquad q_n = 2n\log(n) + cn + o(n).$$

Scaling

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Then

$$\begin{aligned} Y_n &\stackrel{d}{=} \frac{1}{n}\left(Q^*_{I_n} + Q^{**}_{n-1-I_n} + n - 1 - q_n\right) \\ &= \frac{1}{n}\left(I_n \frac{Q^*_{I_n} \pm q_{I_n}}{I_n} + (n-1-I_n)\frac{Q^{**}_{n-1-I_n} \pm q_{n-1-I_n}}{n-1-I_n} + n - 1 - q_n\right) \end{aligned}$$

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Rescaling II

$$q_n = 2n\log(n) + cn + o(n).$$

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&\to 2U\log(U) + 2(1-U)\log(1-U) + 1 =: g(U)
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Rescaling: Summary

$$Q_n \stackrel{d}{=} Q^*_{I_n} + Q^{**}_{n-1-I_n} + n - 1, \qquad q_n = 2n\log(n) + cn + o(n).$$

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$$Y_n \to Y$$

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Hence, this suggests

$$Y_n \to Y \stackrel{d}{=} UY^* + (1-U)Y^{**} + g(U),$$

with Y^*, Y^{**}, U independent, $Y \stackrel{d}{=} Y^* \stackrel{d}{=} Y^{**}$.

m-ary search trees

Example: $m = 4$

List of data: $8, 3, 9, 6, 2, 1, 11, 7, 10, 4, 5.$

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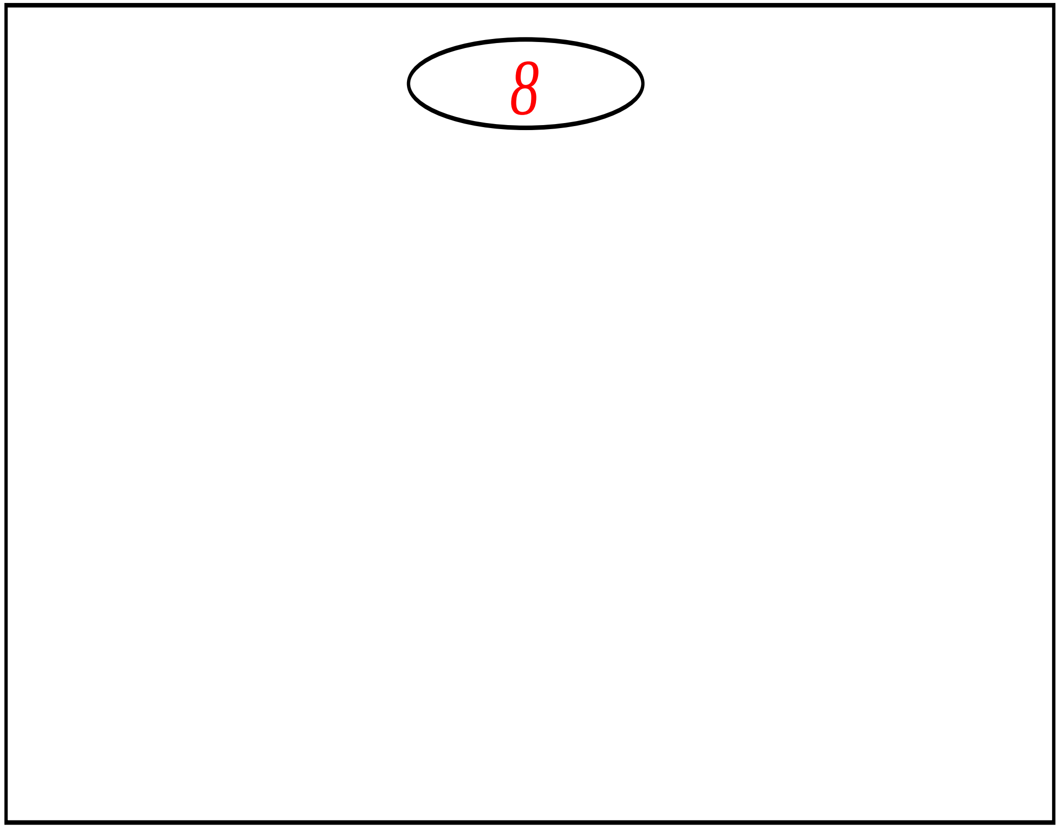

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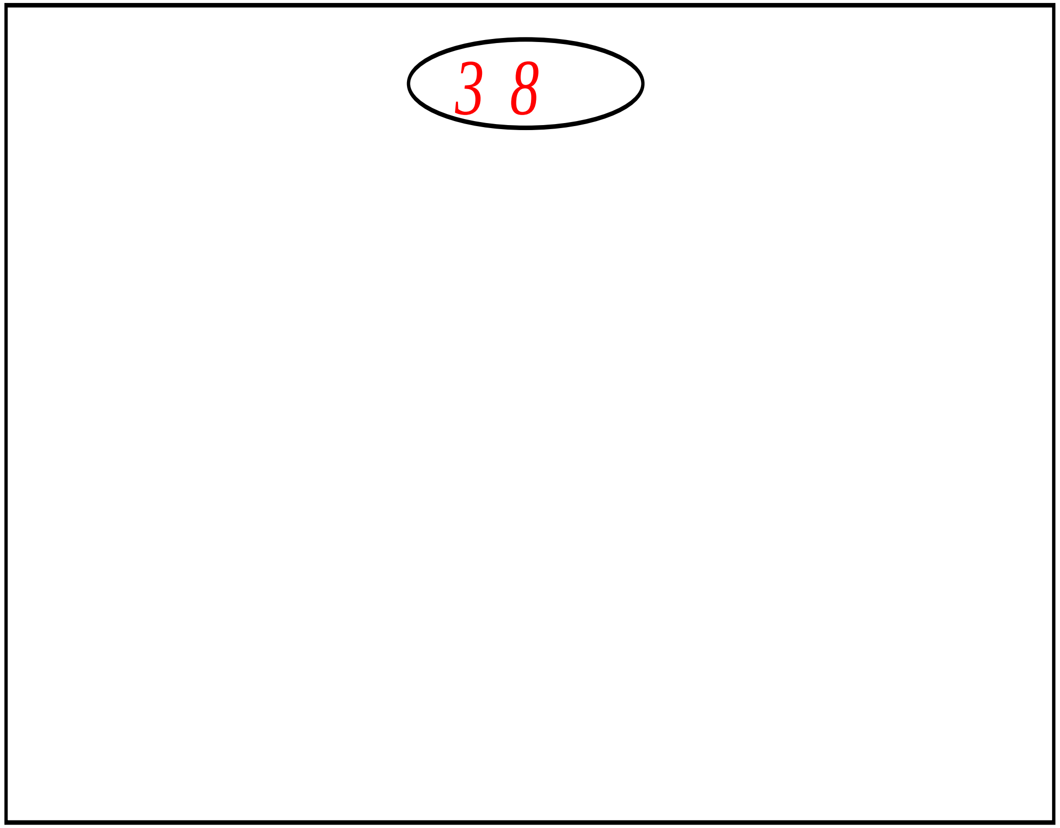

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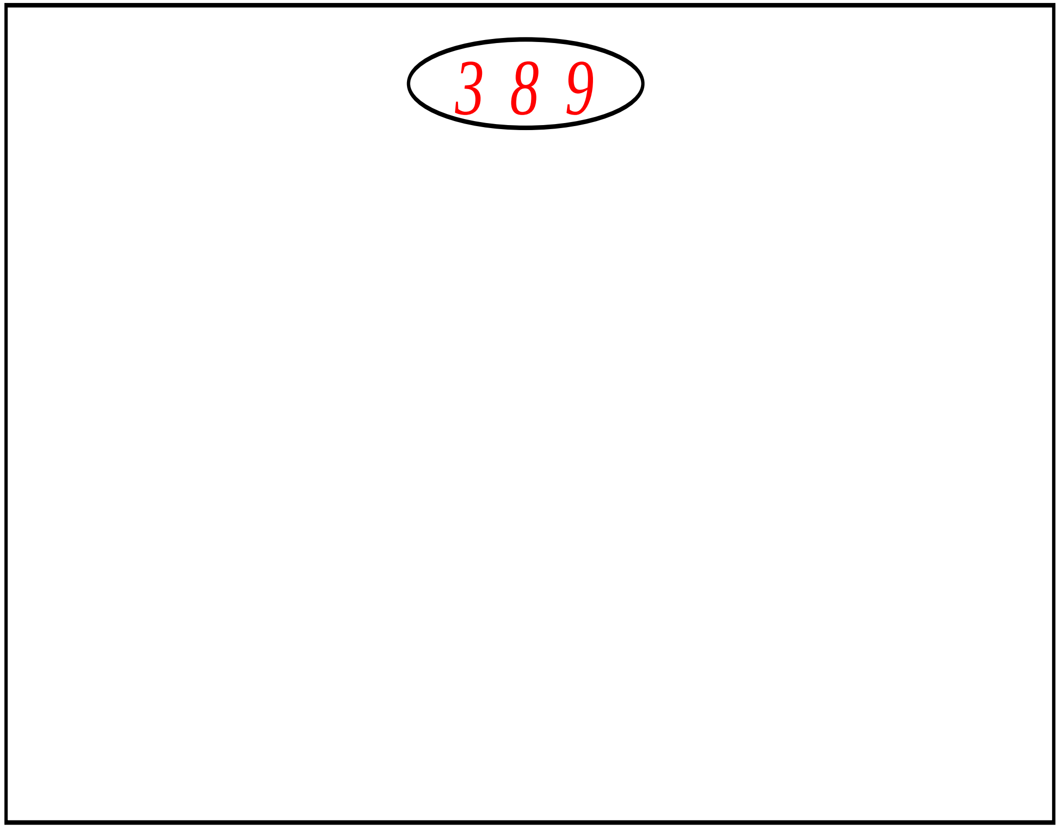

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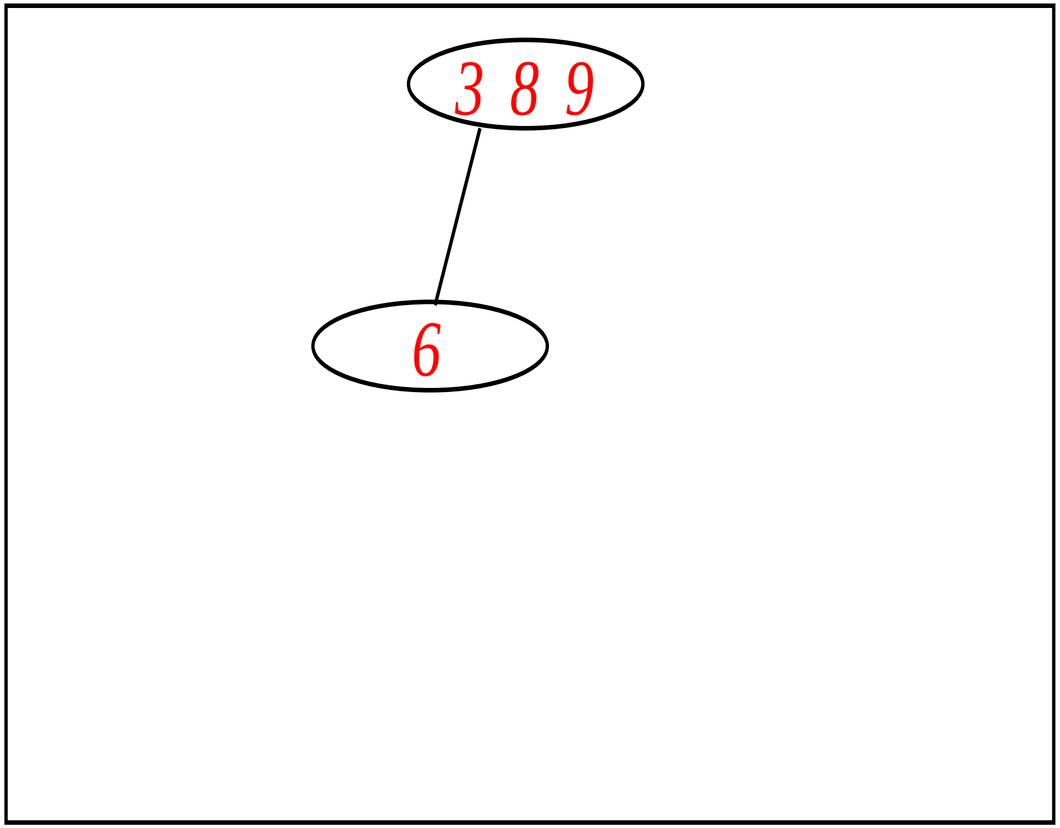

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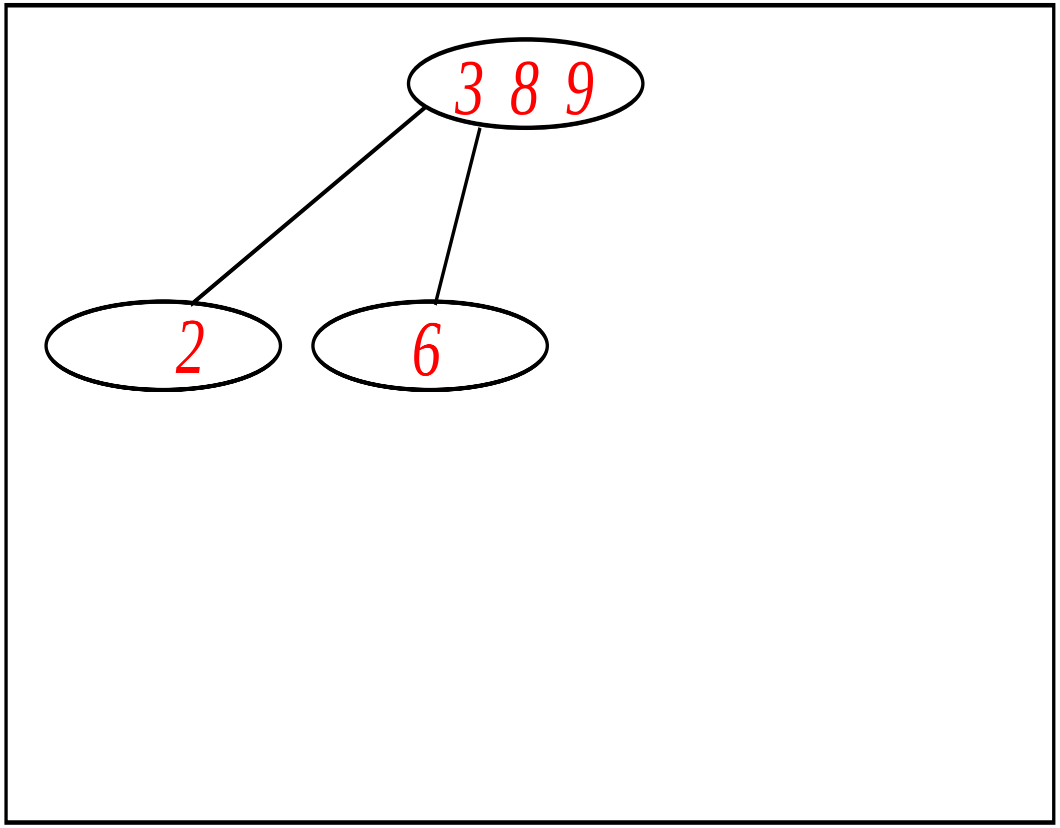

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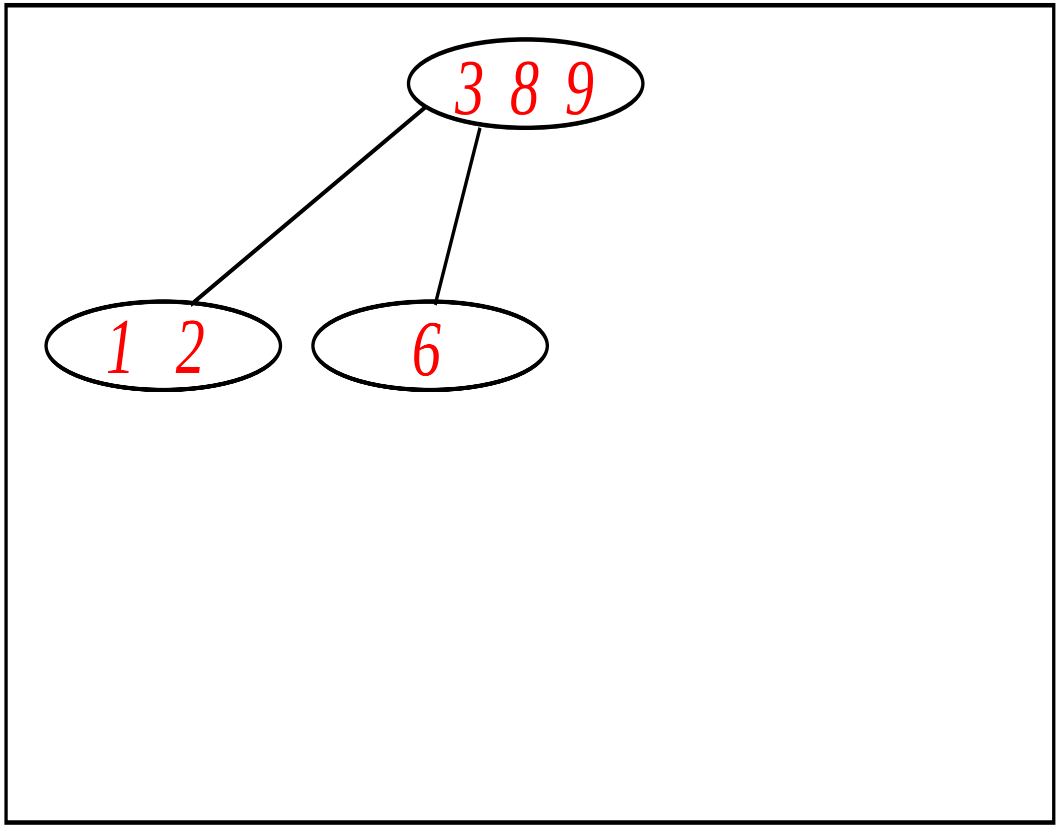

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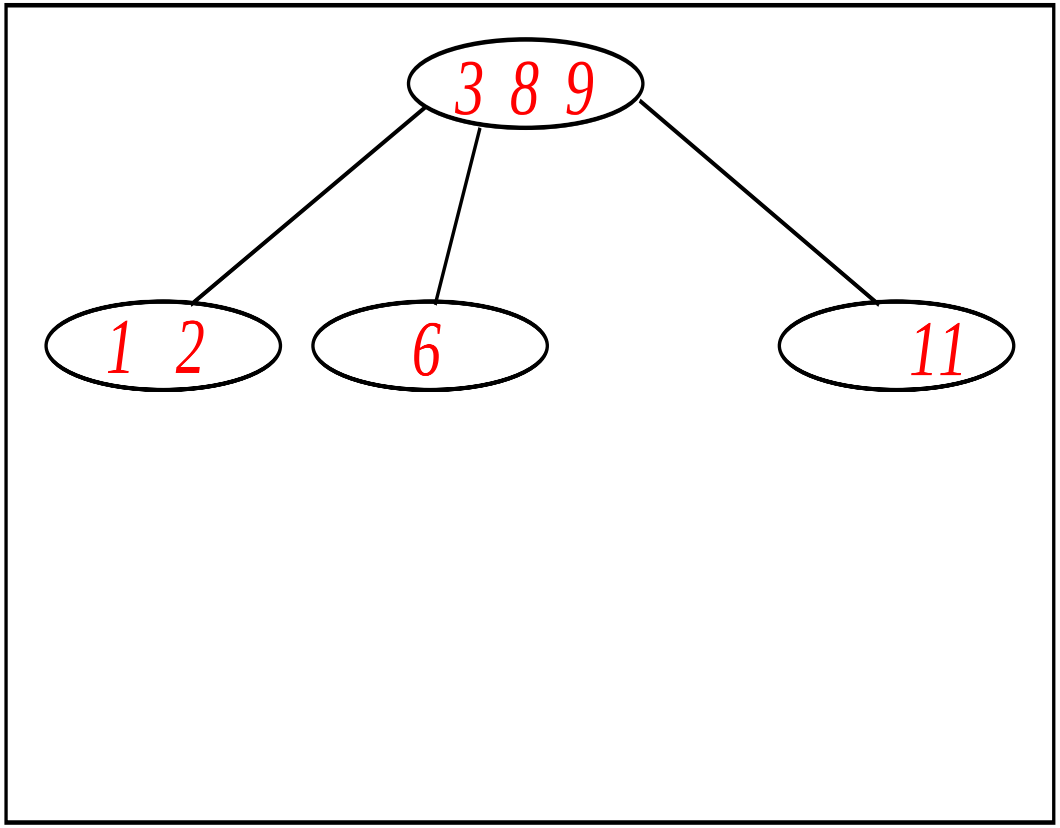

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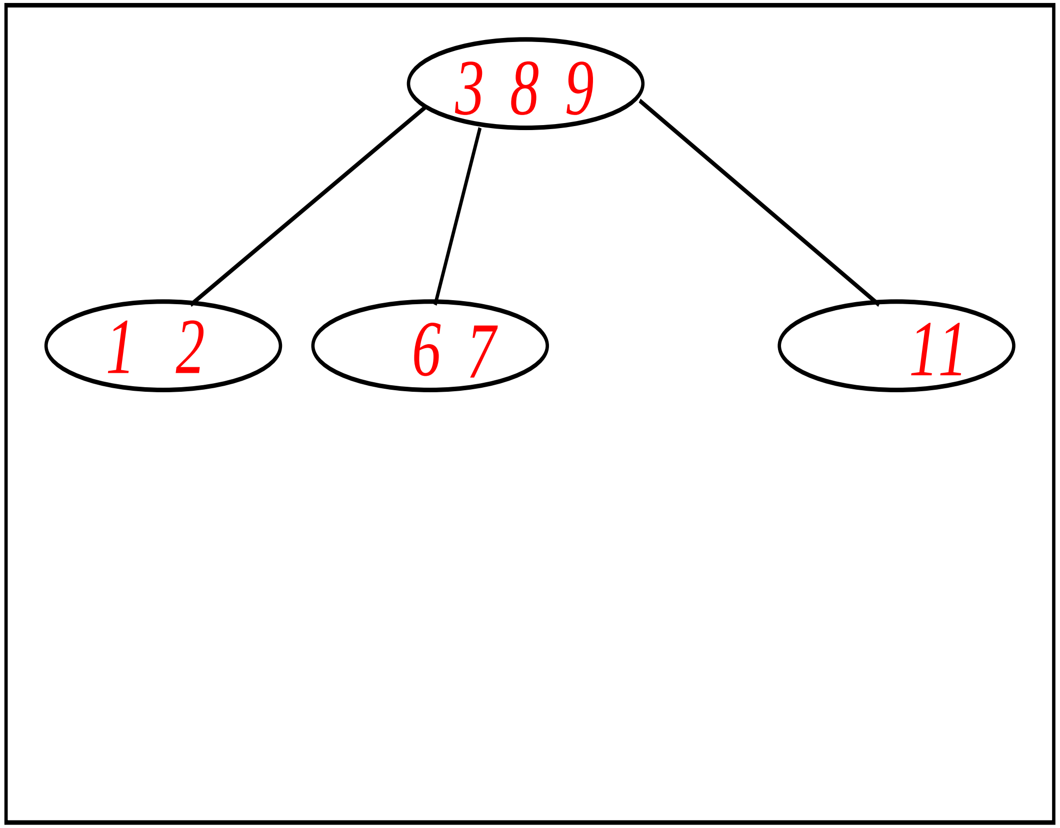

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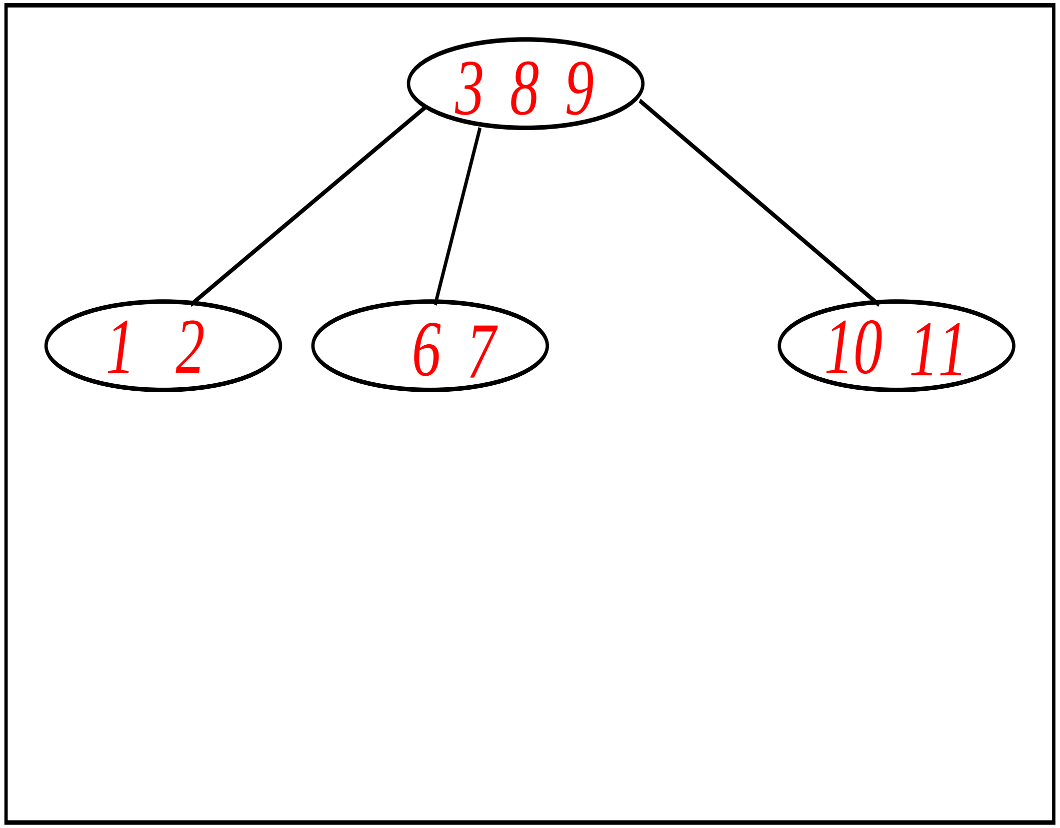

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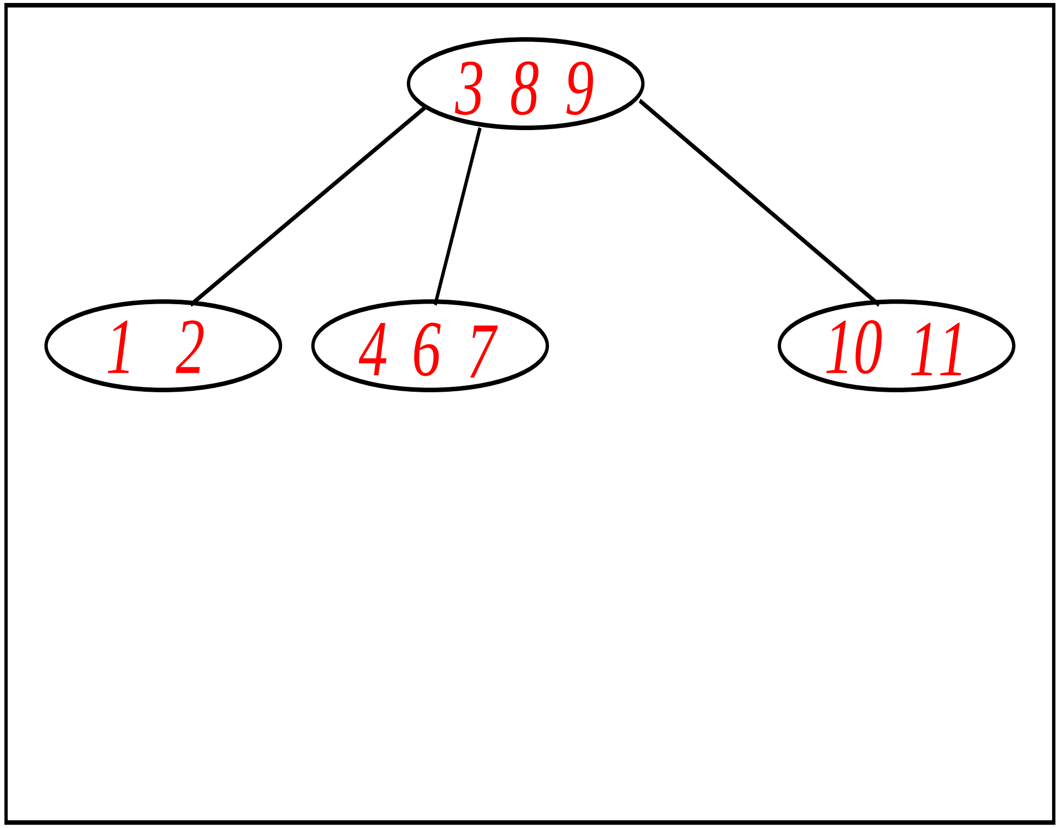

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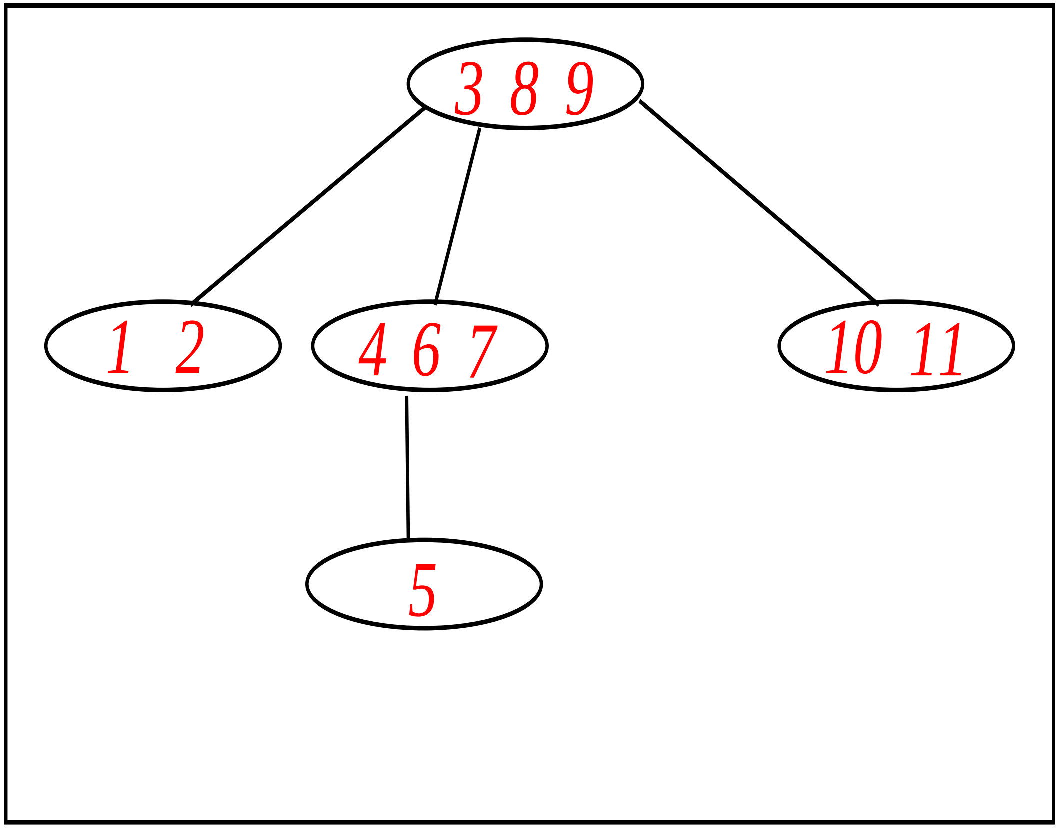

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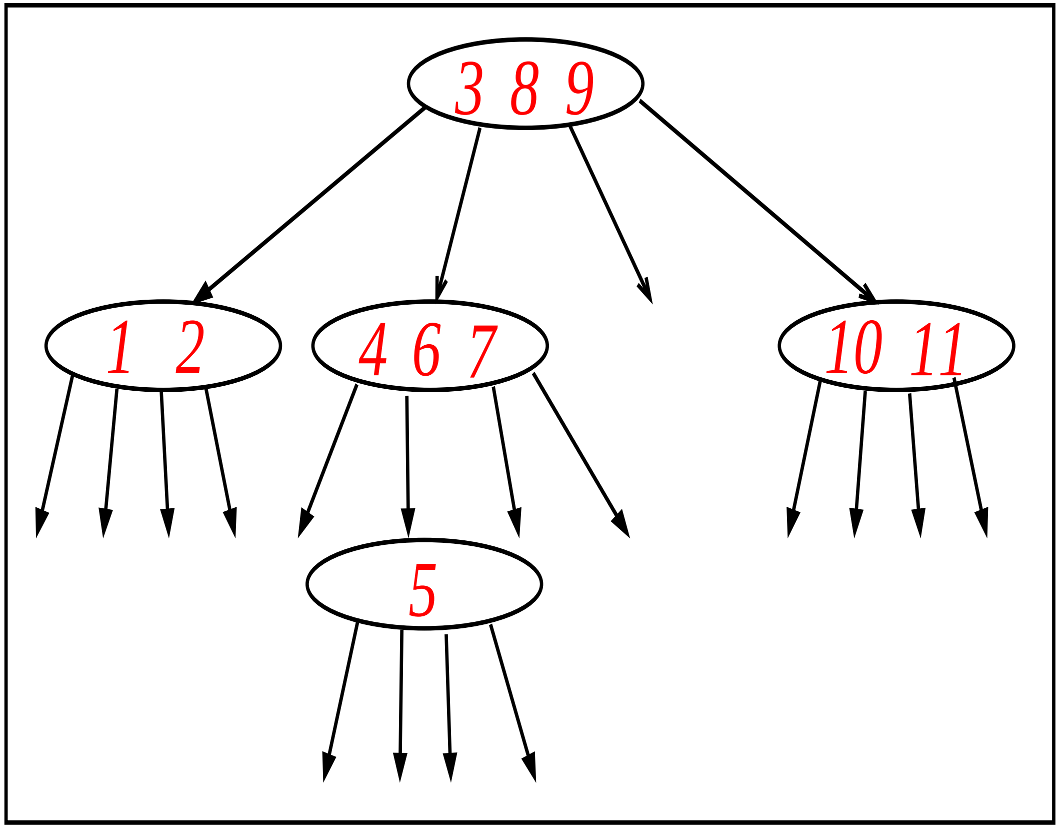
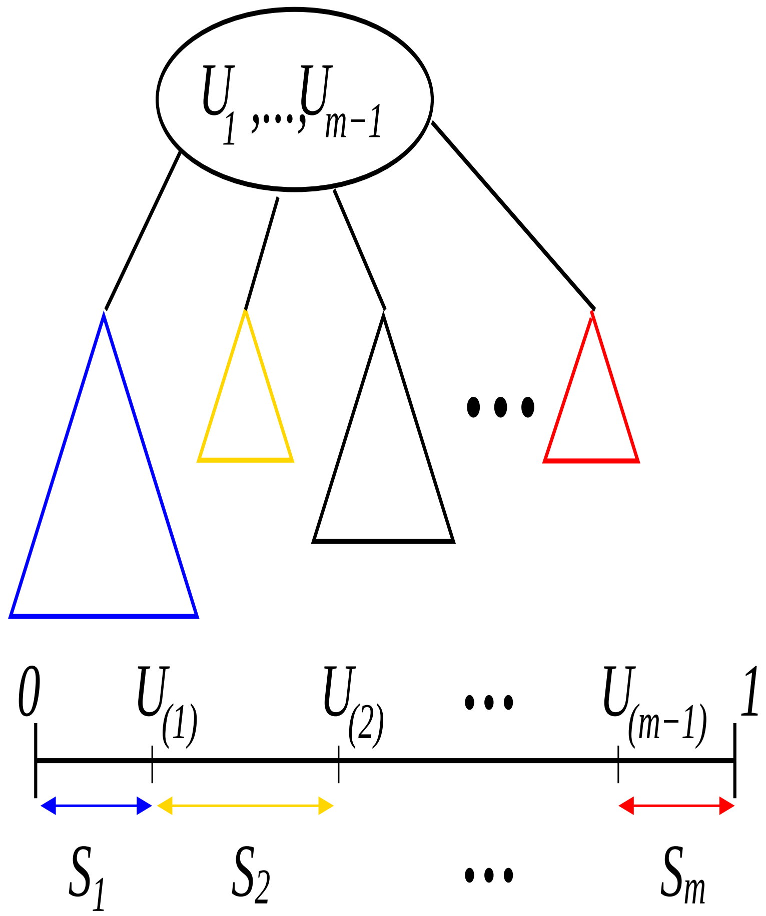


m-ary search tree

Data: $U_1, \ldots, U_n$ i.i.d. unif$[0,1]$
(or uniform permutation)



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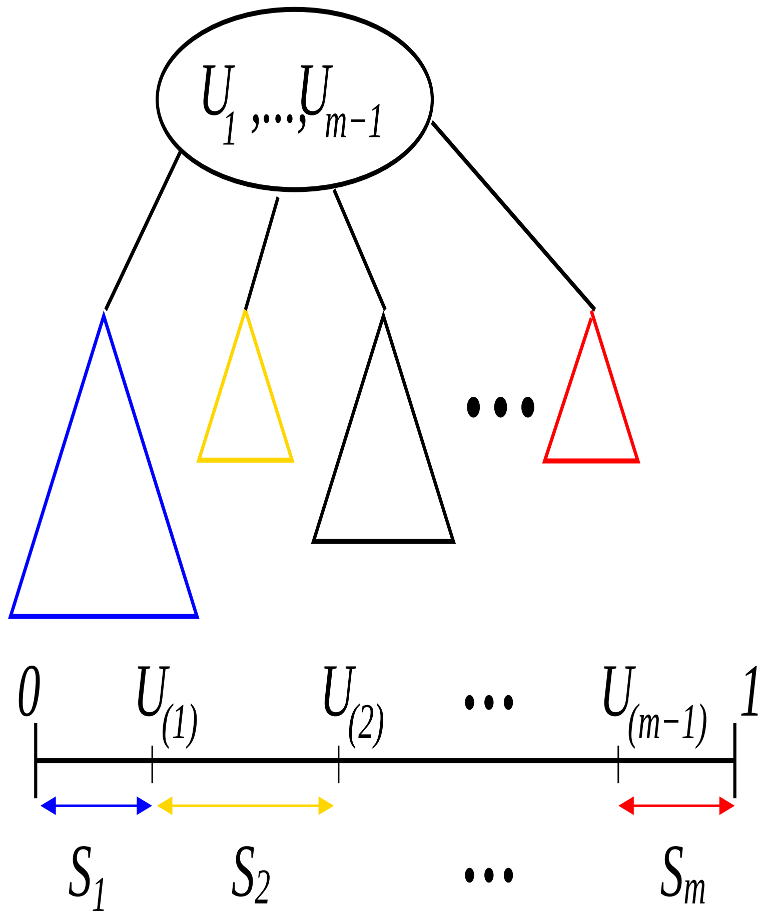

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Space needed
(number of int. nodes):

$$X_n \stackrel{d}{=} \sum_{r=1}^{m} X^{(r)}_{I_r^{(n)}} + 1, \quad n \geq m.$$

$$X_0 = 0, X_1 = \cdots = X_{m-1} = 1.$$

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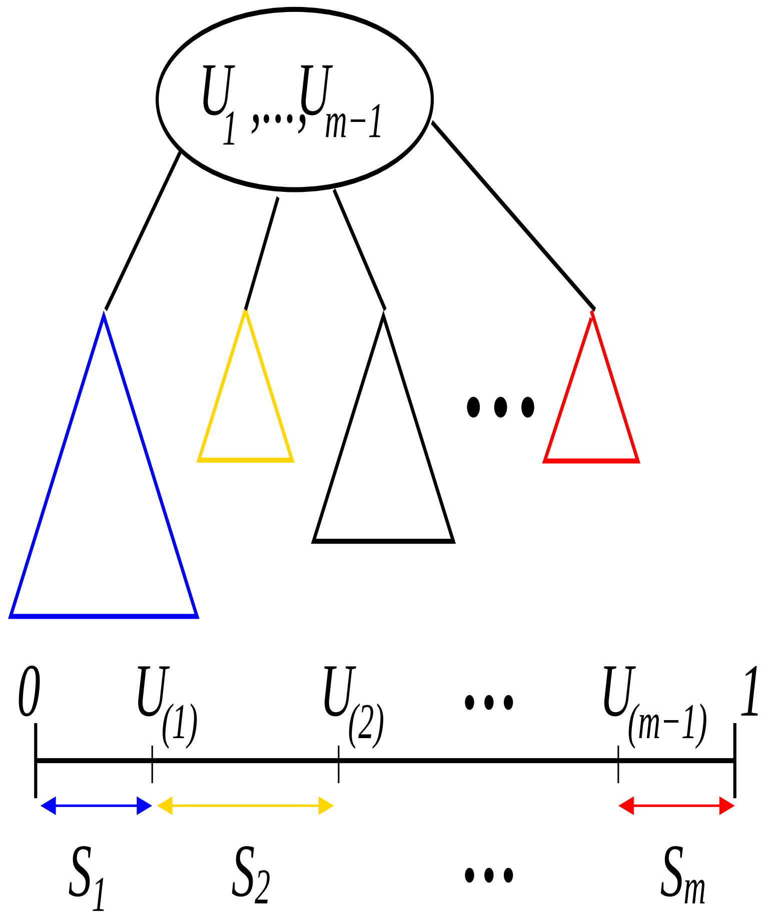

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Sizes of subtrees:

$$I^{(n)} \stackrel{d}{=} M(n-m+1, S_1,\dots,S_m),$$

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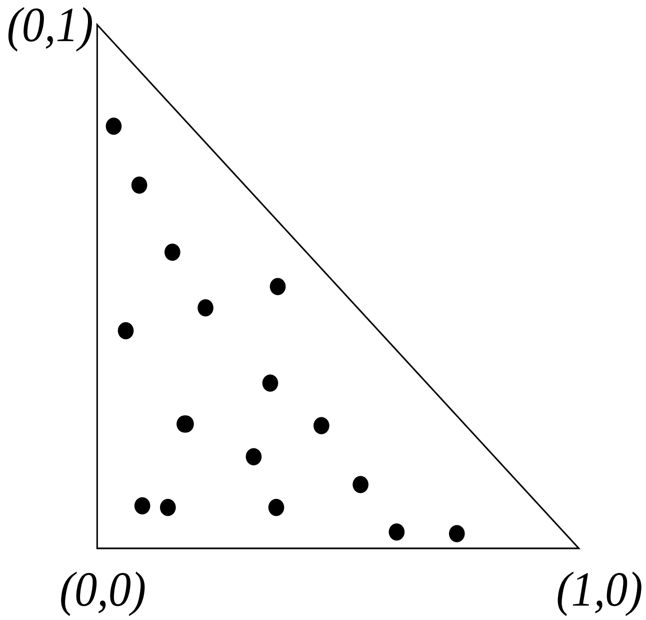
Maxima in right triangles

Data: $u_1, \ldots, u_n$ indep. unif. in right triangle



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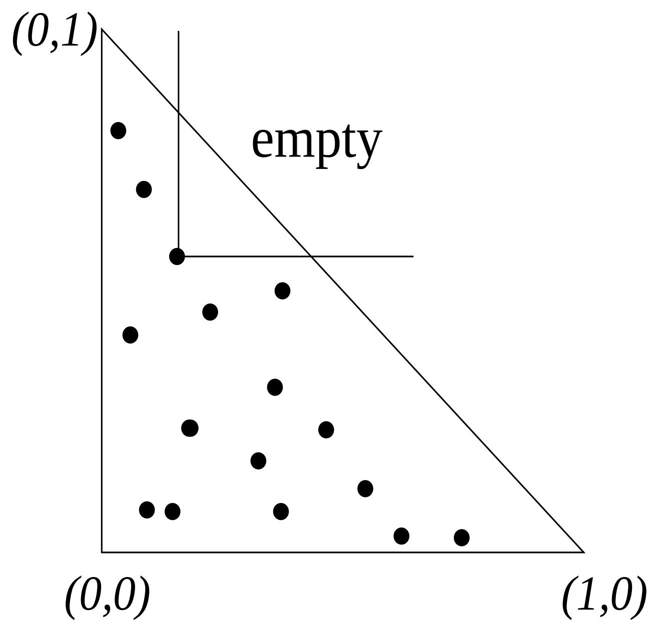

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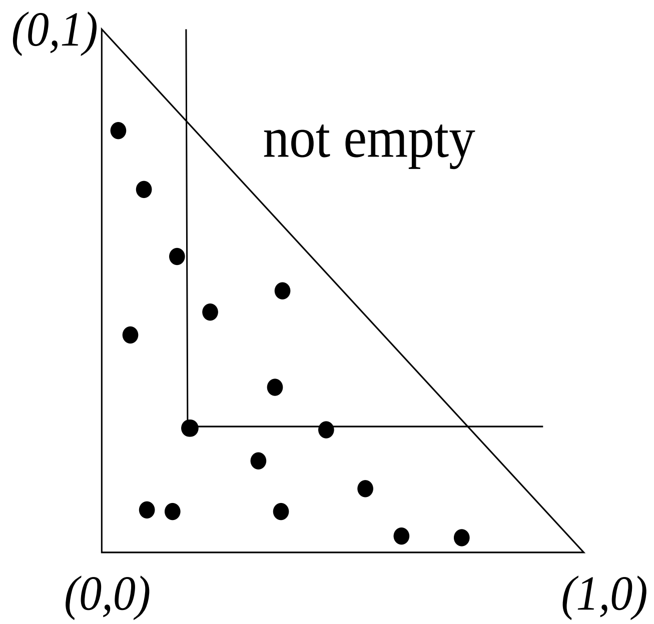

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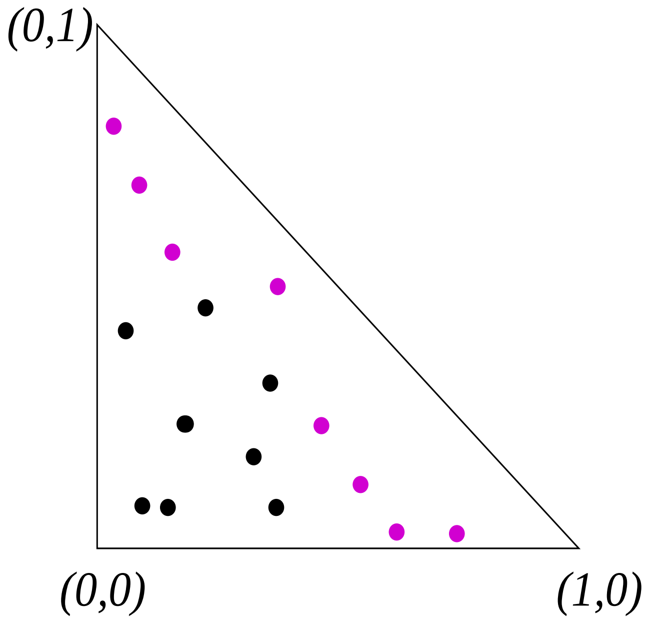

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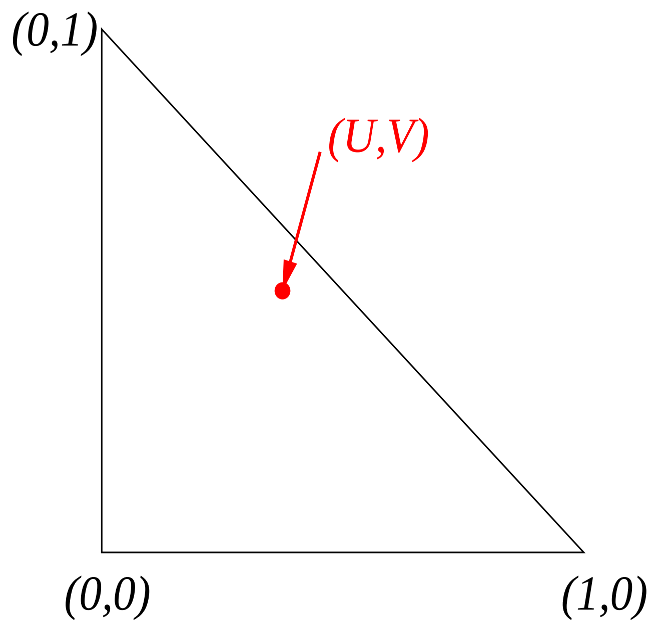

(U, V) has maximal sum of coordinates.

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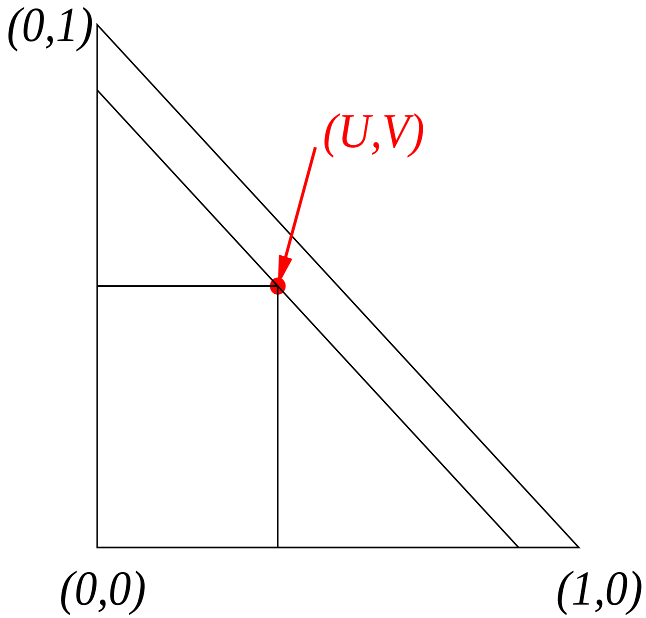

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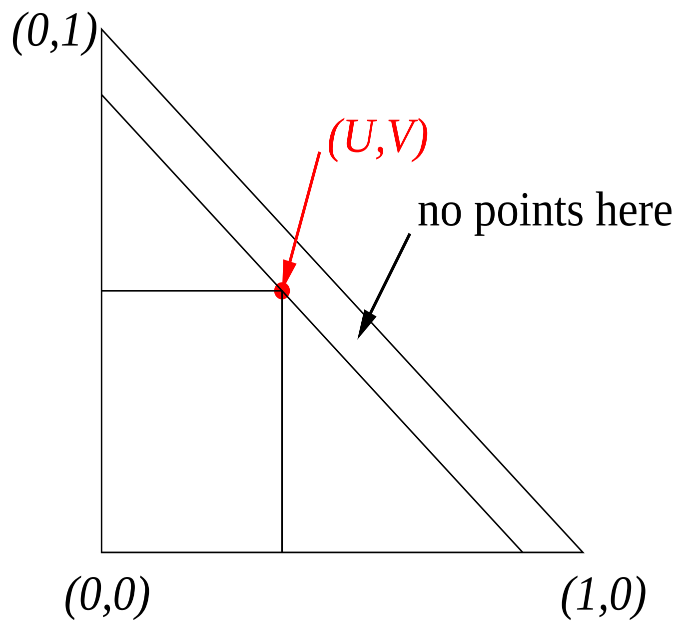

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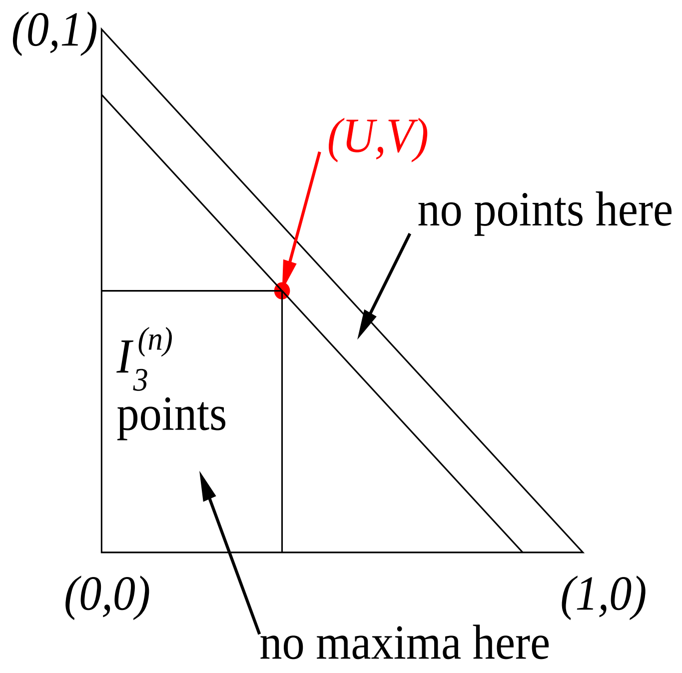

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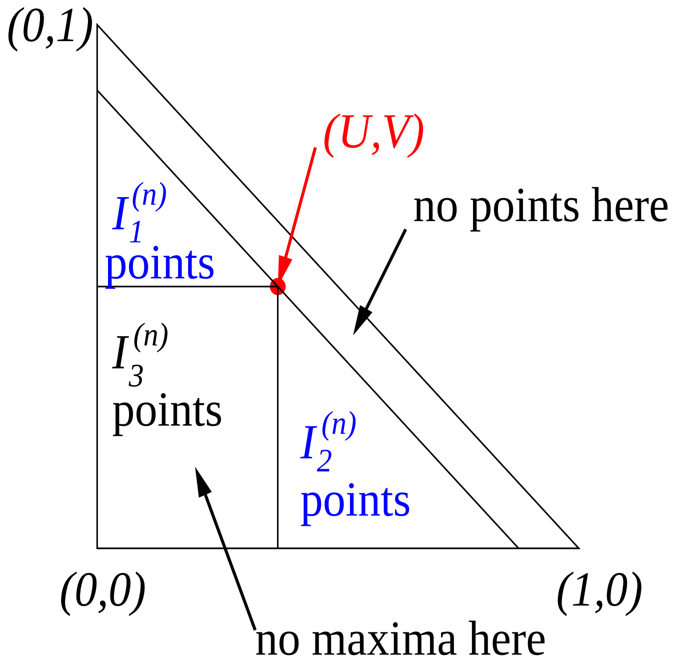

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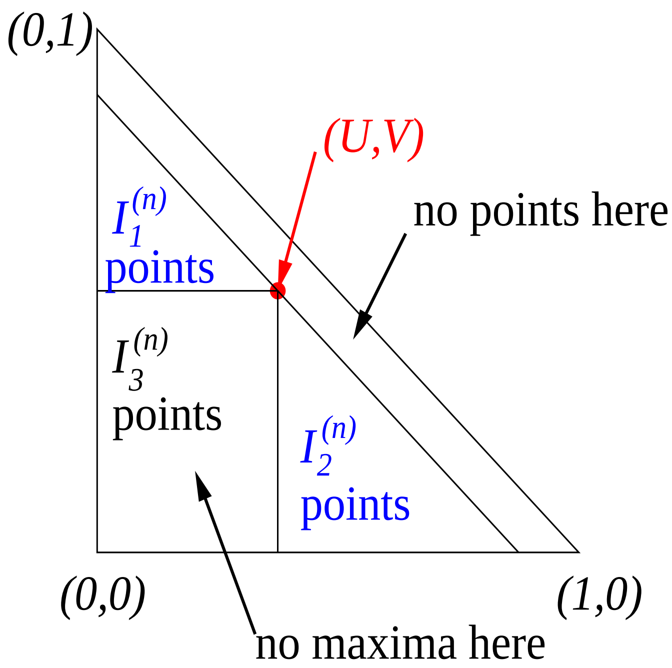

(U, V) has maximal sum of coordinates.

$$X_n \stackrel{d}{=} X^{(1)}_{I_1^{(n)}} + X^{(2)}_{I_2^{(n)}} + 1, \quad n \geq 2.$$

General recursion

$$X_n \stackrel{d}{=} \sum_{r=1}^{K} A_r(n) X^{(r)}_{I_r^{(n)}} + b_n, \qquad n > n_0.$$

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- $K \geq 1$ Number of subproblems (also $K = K_n$).
- $X_n^{(r)} \stackrel{d}{=} X_n$ (recursive).
- $I^{(n)} = (I_1^{(n)}, \ldots, I_K^{(n)})$ Sizes of subproblems.
- $\left(X_n^{(1)}\right), \ldots, \left(X_n^{(K)}\right), (A_1(n), \ldots, A_K(n), b_n, I^{(n)})$ independent.

Contraction method

Rösler (1991, 1992)
Rachev and Rüschendorf (1995)

$$X_n \stackrel{d}{=} \sum_{r=1}^{K} A_r(n) X^{(r)}_{I_r^{(n)}} + b_n, \quad n > n_0.$$

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$$Y_n := \frac{X_n - \mu(n)}{\sigma(n)}.$$

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Rösler (1991, 1992)
Rachev and Rüschendorf (1995)

$$X_n \stackrel{d}{=} \sum_{r=1}^{K} A_r(n) X^{(r)}_{I_r^{(n)}} + b_n, \qquad n > n_0.$$

Scaling

$$Y_n := \frac{X_n - \mu(n)}{\sigma(n)}.$$

Then

$$Y_n \stackrel{d}{=} \sum_{r=1}^{K} A_r^{(n)} Y^{(r)}_{I_r^{(n)}} + b^{(n)}, \qquad n > n_0,$$

with

$$A_r^{(n)} = \frac{\sigma(I_r^{(n)})}{\sigma(n)} A_r(n),$$

$$b^{(n)} = \frac{1}{\sigma(n)} \Big(b_n - \mu(n) + \sum_{r=1}^{K} A_r(n) \mu(I_r^{(n)}) \Big).$$

Convergence

Idea:

$$\begin{array}{ccccccc} Y_n & \stackrel{d}{=} & \sum_{r=1}^{K} & A_r^{(n)} & Y_{I_r^{(n)}}^{(r)} & + & b^{(n)} \\ \downarrow & & & \downarrow & \downarrow & & \downarrow \\ Y & \stackrel{d}{=} & \sum_{r=1}^{K} & A_r^{*} & Y^{(r)} & + & b^{*} \end{array}$$

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$$\left.\begin{array}{l} A_r^{(n)} \longrightarrow A_r^{*} \\ b^{(n)} \longrightarrow b^{*} \end{array}\right\} \Longrightarrow Y_n \longrightarrow Y.$$

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Limit map:

$$\begin{aligned} T : \mathcal{M} &\to \mathcal{M} \\ \nu &\mapsto \mathcal{L}\Big(\sum_{r=1}^{K} A_r^* Z^{(r)} + b^* \Big) \end{aligned}$$

with $(A_1^*, \ldots, A_K^*, b^*)$, $Z^{(1)}, \ldots, Z^{(K)}$ independent, $Z^{(r)} \stackrel{d}{=} \nu$.

The minimal ℓ_p metric

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Definition: The minimal ℓ_p metric $(p \geq 1$ fixed) is given by

$$\begin{aligned}
\ell_p : \mathcal{M}_p \times \mathcal{M}_p &\to [0, \infty) \\
(\mu, \nu) &\mapsto \inf\{\|X - Y\|_p : \mathcal{L}(X) = \mu, \mathcal{L}(Y) = \nu\}
\end{aligned}$$

The minimal ℓ_p metric II

$$\ell_p(\mu, \nu) = \inf\{\|X - Y\|_p : \mathcal{L}(X) = \mu, \mathcal{L}(Y) = \nu\}$$

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1st Step: Construct X with $\mathcal{L}(X) = \mu$: Define distribution function of X and its (generalized) inverse by

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Well-known fact: For a unif$[0,1]$ r.v. U we have

$$\mathcal{L}\left(F_X^{-1}(U)\right) = \mathcal{L}(X).$$

The minimal ℓ_p metric — optimal couplings

2^{nd} step: Use the same unif$[0,1]$ r.v. U for both, i.e.

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Theorem: Optimal coupling do always exist. For $\mu, \nu \in \mathcal{M}_p$ optimal couplings are given by (**??**).

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Theorem: Optimal coupling do always exist. For $\mu, \nu \in \mathcal{M}_p$ optimal couplings are given by (**??**).

Corollary: We have

$$\ell_p(\mu,\nu) = \left(\int_0^1 \left|F_\mu^{-1}(u) - F_\nu^{-1}(u)\right|^p du\right)^{1/p}.$$

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$\Rightarrow \mu_n \xrightarrow{\ell_p} \mathcal{L}(X) \in \mathcal{M}_p$. ♣.

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Corollary: For $\mu_n, \mu \in \mathcal{M}_p$ with $\ell_p(\mu_n, \mu) \to 0$:

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L^p convergence implies convergence in distribution.

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Moreover

$$\Big| \|\mu_n\|_p - \|\mu\|_p \Big| = \Big| \|X_n\|_p - \|X\|_p \Big| \leq \|X_n - X\|_p \to 0 \quad \clubsuit$$

Lipschitz continuity on $(\mathcal{M}_\mathrm{p}, \ell_\mathrm{p})$

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Theorem: Assume that $(A_1, \ldots, A_k, b)$ are L^p-integrable r.v.,

$$\begin{aligned} T : \mathcal{M}_p &\longrightarrow \mathcal{M}_p \\ \mu &\longmapsto \mathcal{L}\left(\sum_{r=1}^{K} A_r Z^{(r)} + b\right), \end{aligned}$$

where $(A_1, \ldots, A_k, b), Z^{(1)}, \ldots, Z^{(K)}$ are indep. and $\mathcal{L}(Z^{(r)}) = \mu$.

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Then, for all $\mu, \nu \in \mathcal{M}_p$,

$$\ell_p(T(\mu), T(\nu)) \leq \left(\sum_{r=1}^{K} \|A_r\|_p\right) \ell_p(\mu, \nu).$$

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Then, for all $\mu, \nu \in \mathcal{M}_p$,

$$\ell_p(T(\mu), T(\nu)) \leq \left(\sum_{r=1}^{K} \|A_r\|_p \right) \ell_p(\mu, \nu).$$

If $\sum_{r=1}^{K} \|A_r\|_p < 1$ then T is a contraction on $(\mathcal{M}_p, \ell_p)$.

Proof

$$T(\mu) = \mathcal{L}\left(\sum_{r=1}^{K} A_r Z^{(r)} + b\right), \quad \mu \in \mathcal{M}_p$$

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There exists opt. coupl. $\left(Z^{(1)}, W^{(1)}\right)$ of μ and ν.

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Let $\left(Z^{(1)}, W^{(1)}\right), \ldots, \left(Z^{(K)}, W^{(K)}\right)$ be i.i.d.
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&= \left\| \sum_{r=1}^{K} A_r (Z^{(r)} - W^{(r)}) \right\|_p \leq \sum_{r=1}^{K} \|A_r\|_p \left\| Z^{(r)} - W^{(r)} \right\|_p
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&\leq \left\| \sum_{r=1}^{K} A_r Z^{(r)} + b - \left(\sum_{r=1}^{K} A_r W^{(r)} + b\right) \right\|_p \\
&= \left\| \sum_{r=1}^{K} A_r (Z^{(r)} - W^{(r)}) \right\|_p \leq \sum_{r=1}^{K} \|A_r\|_p \left\| Z^{(r)} - W^{(r)} \right\|_p \\
&= \left(\sum_{r=1}^{K} \|A_r\|_p \right) \left\| Z^{(1)} - W^{(1)} \right\|_p
\end{aligned}$$

Proof

$$T(\mu) = \mathcal{L}\left(\sum_{r=1}^{K} A_r Z^{(r)} + b\right), \quad T(\nu) = \mathcal{L}\left(\sum_{r=1}^{K} A_r W^{(r)} + b\right).$$

Hence

$$\begin{aligned}
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&= \left(\sum_{r=1}^{K} \|A_r\|_p\right) \left\| Z^{(1)} - W^{(1)} \right\|_p \\
&= \left(\sum_{r=1}^{K} \|A_r\|_p\right) \ell_p(\mu, \nu). \quad \clubsuit
\end{aligned}$$

Lipschitz on $(\mathcal{M}_2(0), \ell_2)$

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Theorem: Assume $(A_1, \ldots, A_k, b)$ are L^2-integr. r.v. with $\mathbb{E}\, b = 0$,

$$\begin{aligned} T : \mathcal{M}_2(0) &\to \mathcal{M}_2(0) \\ \mu &\mapsto \mathcal{L}\left(\sum_{r=1}^{K} A_r Z^{(r)} + b\right), \end{aligned}$$

where $(A_1, \ldots, A_k, b), Z^{(1)}, \ldots, Z^{(K)}$ are indep. and $\mathcal{L}(Z^{(r)}) = \mu$.

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If $\sum_{r=1}^{K} \mathbb{E}\, A_r^2 < 1$ then T is a contraction on $(\mathcal{M}_2(0), \ell_2)$.

Convergence: Quickselect I

$$Y_n \stackrel{d}{=} \frac{I_n}{n} Y_{I_n} + \frac{n-1}{n}, \qquad \mathcal{L}(I_n) = \mathrm{unif}\{0, \ldots, n-1\}$$

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$$\Delta(n) := \ell_p(\mathcal{L}(Y_n), \mathcal{L}(Y)) \leq \left\| \frac{I_n}{n} Y_{I_n} + \frac{n-1}{n} - (UY + 1) \right\|_p$$

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&\leq \left\| \frac{I_n}{n} \left(Y_{I_n} - Y \right) \right\|_p + \frac{1 + \|Y\|_p}{n}.
\end{aligned}$$

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We obtain $\Delta(n) \to 0$.
(E.g., for $p = 1$ show $\Delta(n) \le (C \log n)/n$ by induction.)

General theorem in $\mathcal{M}_p$

Let $(Y_n)_{n\geq 0}$ be L^p-integrable, $p \geq 1$, with (as before)

$$Y_n \stackrel{d}{=} \sum_{r=1}^{K} A_r^{(n)} Y_{I_r^{(n)}}^{(r)} + b^{(n)}.$$

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Application: Path length in BST

$$\begin{aligned} Y_n &\stackrel{d}{=} \frac{I_n}{n} Y^*_{I_n} + \frac{n-1-I_n}{n} Y^{**}_{n-1-I_n} + b^{(n)}, \\ Y &\stackrel{d}{=} UY^* + (1-U)Y^{**} + g(U). \end{aligned}$$

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Convergence of coefficients and technical condition satisfied.

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In $(\mathcal{M}_2(0), \ell_2)$:

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In $(\mathcal{M}_2(0), \ell_2)$: YES:

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In $(\mathcal{M}_2(0), \ell_2)$: YES:

$$\mathbb{E}\left[(A_1^*)^2\right] + \mathbb{E}\left[(A_2^*)^2\right] = \mathbb{E}\,U^2 + \mathbb{E}\,(1-U)^2 = \frac{2}{3} < 1.$$

Application: Central limit theorem

Let $W_1, W_2, \ldots$ be i.i.d., L^p-integrable, $p \geq 2$, with $\mathbb{E} W_1 = \mu$, $\mathrm{Var}(W_1) = \sigma^2$.

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$$\zeta_s(\mathcal{L}(X), \mathcal{L}(Y)) = \sup_{f \in \mathcal{F}_s} \left| \mathbb{E}\left[f(X) - f(Y) \right] \right|, \qquad s > 0,$$

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Zolotarev metric II

Spaces of probability measures

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Short Notation: $\zeta_s(X,Y) := \zeta_s(\mathcal{L}(X),\mathcal{L}(Y))$.

Zolotarev metric: properties

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c) If $X_1, \ldots, X_K, Y_1, \ldots, Y_K$ independent, then

$$\zeta_s\left(\sum_{r=1}^{K} X_r, \sum_{r=1}^{K} Y_r\right) \leq \sum_{r=1}^{K} \zeta_s(X_r, Y_r).$$

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Second part similar with $x \mapsto |c|^{-s} f(cx)$ in $\mathcal{F}_s$ for $f \in \mathcal{F}_s$. ♣

Zolotarev metric: properties

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c) Exercise (easy). ♣

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$$T : \mathcal{M} \to \mathcal{M}, \quad \mu \mapsto \mathcal{L}\left(\sum_{r=1}^{K} A_r^* Z^{(r)}\right), \quad \text{with } \sum_{r=1}^{K} (A_r^*)^2 = 1 \quad \text{a.s.},$$

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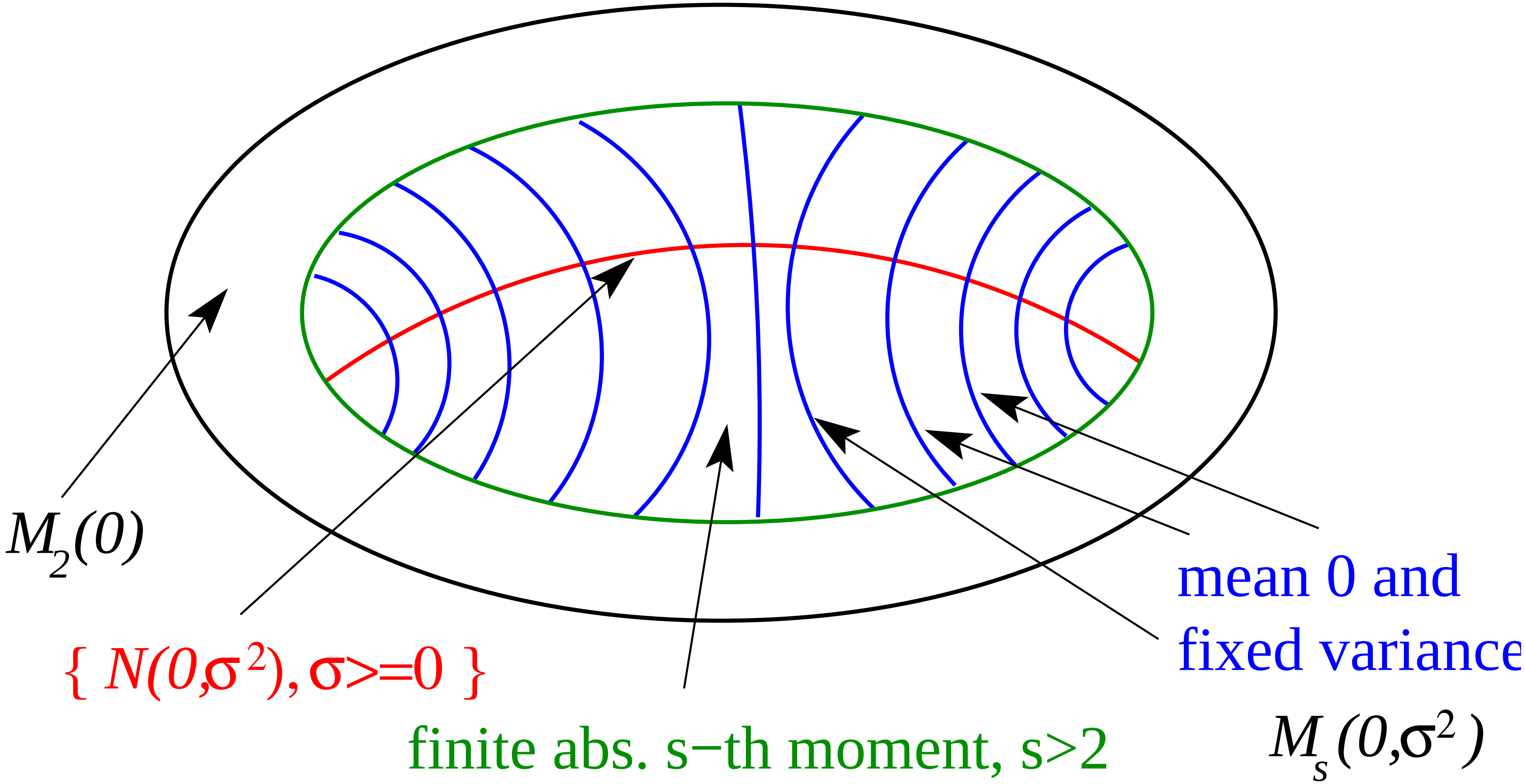
$\Rightarrow \mathcal{N}(0,1)$ unique fixed point of T in $\mathcal{M}_s(0,1)$.

The work space



Central limit theorem II

$W_1, W_2, \ldots$ i.i.d., L^s-integr., $s > 2$, with $\mathbb{E}\, W_1 = \mu$, $\mathrm{Var}(W_1) = \sigma^2$.

$$X_n := \sum_{i=1}^{n} W_i \stackrel{d}{=} X^*_{\lceil n/2 \rceil} + X^{**}_{\lfloor n/2 \rfloor},$$

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$$\Rightarrow \zeta_s(Y_n, N) < \infty \text{ for all } n \geq 1.$$

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$$\zeta_s(Y_n, N) = \zeta_s\left(\sqrt{\frac{\lceil n/2 \rceil}{n}} Y^{*}_{\lceil n/2 \rceil} + \sqrt{\frac{\lfloor n/2 \rfloor}{n}} Y^{**}_{\lfloor n/2 \rfloor}, \sqrt{\frac{\lceil n/2 \rceil}{n}} N^{*} + \sqrt{\frac{\lfloor n/2 \rfloor}{n}} N^{**}\right)$$

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$$\begin{aligned}\zeta_s(Y_n, N) &= \zeta_s\left(\sqrt{\frac{\lceil n/2 \rceil}{n}} Y^*_{\lceil n/2 \rceil} + \sqrt{\frac{\lfloor n/2 \rfloor}{n}} Y^{**}_{\lfloor n/2 \rfloor}, \sqrt{\frac{\lceil n/2 \rceil}{n}} N^* + \sqrt{\frac{\lfloor n/2 \rfloor}{n}} N^{**}\right) \\ &\stackrel{c)}{\leq} \zeta_s\left(\sqrt{\frac{\lceil n/2 \rceil}{n}} Y^*_{\lceil n/2 \rceil}, \sqrt{\frac{\lceil n/2 \rceil}{n}} N^*\right) + \zeta_s\left(\sqrt{\frac{\lfloor n/2 \rfloor}{n}} Y^{**}_{\lfloor n/2 \rfloor}, \sqrt{\frac{\lfloor n/2 \rfloor}{n}} N^{**}\right)\end{aligned}$$

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&\stackrel{a)}{\leq} \left(\frac{\lceil n/2 \rceil}{n}\right)^{s/2} \zeta_s\left(Y^*_{\lceil n/2 \rceil}, N^*\right) + \left(\frac{\lfloor n/2 \rfloor}{n}\right)^{s/2} \zeta_s\left(Y^{**}_{\lfloor n/2 \rfloor}, N^{**}\right)
\end{aligned}$$

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With $\Delta(n) := \zeta_s(Y_n, N)$ we obtain

$$\Delta(n) \leq \left(\frac{\lceil n/2 \rceil}{n}\right)^{s/2} \Delta\left(\lceil n/2 \rceil\right) + \left(\frac{\lfloor n/2 \rfloor}{n}\right)^{s/2} \Delta\left(\lfloor n/2 \rfloor\right).$$

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$$L \le 2^{1-s/2} L, \qquad \text{hence} \qquad L = 0.$$

A useful extension

Theorem:

$$X_n \stackrel{d}{=} \sum_{r=1}^{K} X^{(r)}_{I_r^{(n)}} + b_n, \quad n > n_0$$

with conditions as before and L^s-integrable, $2 < s \leq 3$.

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Then

$$\frac{X_n - \mathbb{E}\, X_n}{\sqrt{\mathrm{Var}(X_n)}} \xrightarrow{d} \mathcal{N}(0,1).$$