Some Remarks on Depth of Dead Ends in Groups

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Abstract

It is known, that the existence of dead ends (of arbitrary depth) in the Cayley graph of a group depends on the chosen set of generators. Nevertheless there exist many groups, which do not have dead ends of arbitrary depth with respect to any set of generators. Partial results in this direction were obtained by Šunić and by Warshall. We improve these results by showing that abelian groups have only finitely many dead ends and that groups with more than one end (in the sense of Hopf and Freudenthal) have only dead ends of bounded depth. Only few examples of groups with unbounded dead end depth are known. We show that the Houghton group H_2 with respect to the standard generating set is a further example. In addition we introduce a stronger notion of depth of a dead end, called strong depth. The Houghton group H_2 has unbounded strong depth with respect to the same standard generating set.

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1 Introduction

Let G be a group and X a finite set of generators. The (unoriented) Cayley graph $\Gamma = \Gamma(G, X)$ is the graph with vertex set G whose edges are pairs $(g_1, g_2) \in G \times G$ with $g_1^{-1} g_2 \in X^{\pm 1}$. Giving all edges the length 1 we obtain a metric structure on Γ. We denote this metric $d_X(\cdot, \cdot)$.

Many results on groups rely on the structure of geodesics in the Cayley graph. Because of the transitive action of G on the vertex set it suffices to consider geodesics from 1 to each vertex $g \in G$. For some $g \in G$ there might be no geodesic, that can be extended to a geodesic from 1 to a $g' \in G$ further away. Such elements g are called *dead ends* of G. More precisely: Let $n = d(1, g)$; g is called a dead end of G if the ball $B_g(1)$ of radius 1 and center g is contained in the ball $B_1(n)$ of radius n with center 1.

Let $k = \max\{l | B_g(l) \subseteq B_1(n)\}$; then k is called the *depth* of the dead end g. Dead ends and their depth were first considered by Bogopolski in [1], who proved that the depth of dead ends in a given non-elementary hyperbolic group with a given set of generators is uniformly bounded and [1] and [3] give examples of groups with infinitely many dead ends, all of depth one or two. On the other hand [4] gives an example of a group with arbitrary deep dead ends.

It is easy to see that the property of having dead ends is not an invariant of a group. For example $\mathbb{Z}$ generated by $\{2, 3\}$ has the dead ends 1 and -1. In fact Šunić [9] proves that for each infinite group G exists a generating set X, such that G has dead ends with respect to d_X. Unfortunately even the property of having only dead ends of bounded depth is not a group invariant, as Riley and Warshall show in [8][a].

But there are some results which do not depend on the set of generators. The result of Bogopolski concerning non-elementary hyperbolic groups was already mentioned above. In [9] Šunić shows that $\mathbb{Z}$ has only finitely many dead ends with respect to any generating set and Warshall [10] shows that for all weakly geodesically automatic groups, and hence all abelian groups, there exists a uniform bound on the depth of dead ends depending on the set of generators.

We will generalize the result of Šunić and a part of the result of Warshall by showing in Section 2:

Theorem 1. *Let G be an abelian group, generated by the finite set X. Then there exist only finitely many dead ends in G with respect to X.*

The notion of ends of a graph goes back to Hopf [7] and Freudenthal [6]. Although the space of ends of a graph can be defined explicitly for our purposes it is sufficient to define the number of ends of a graph. Given a graph Γ and a finite subgraph C let $n(C)$ denote the number of infinite connected components of $\Gamma - C$. The number of ends of Γ is $e(\Gamma) := sup_C(n(\Gamma))$. In the case of Cayley graphs we consider w.l.o.g. only subgraphs containing 1. The number of ends of a Cayley graph is known to be a quasi-isometry invariant. In particular we can speak of the number of ends of G. If G is finite then G has no ends. Hopf's result asserts that a finitely generated infinite group has one, two or infinitely many ends, and that two-ended groups are virtually $\mathbb{Z}$. Stallings' celebrated structure Theorem describes the structure of groups with more than two ends in terms of amalgamated free products and HNN-extensions over finite amalgamated (associated) subgroups.

In Section 2 we will also show the following theorem.

Theorem 2. *Let G be a finitely generated group with more than one end. Then there exists a uniform bound on the depth of dead ends depending only on the set of generators.*

[a] In this article Riley and Warshall give two examples of groups, with unbounded dead end depth with respect to one set of generators and bounded dead end depth with respect to some other set of generators. The proof of the finitely presented example is based on the uncorrected version of [5]. Here we only refer to the correct not finitely presented example.

As mentioned above in [8] Riley and Warshall have shown that having dead ends of arbitrary depth is not a group invariant. One of their examples is the group with presentation

$$G = \langle a, t, u | a^2, [t, u], a^{-u} a^t; \forall i \in \mathbb{Z}, [a, a^{t^i}] \rangle$$

which has unbounded depth with respect to

$$X = \{a, t, u, at, ta, ata, au, ua, aua\}$$

and depth bounded above by 2 with respect to another set of generators. Their examples of dead ends of depth k have the following interesting property: Let g be one of these dead ends and $n = d(1, g)$. There exists a geodesic from g to an element g' with distance $d(1, g') = n + 1$ which never gets closer than $n - 1$ to the identity. In other words: g and g' can be connected in $\Gamma(G, X) \setminus B_1(n - 2)$. This yields to a new definition of depth of a dead end. To distinguish between them we will call the new one strong depth.

Definition 3. Let Γ be the Cayley graph of a group G with respect to a generating set X and $g \in G$ a dead end with $d(1, g) = n$. The strong depth of g is defined as the minimal number k such that g can be connected to a point of $\Gamma \setminus B_1(n)$ inside $\Gamma \setminus B_1(n - k)$. In other words: The strong depth of g measures how far back towards the identity a geodesic starting in g has to go in order to leave the ball of radius n.

The strong depth of a dead end is obviously no greater than its depth. The dead ends in [8] mentioned above are all of strong depth 2.

In Section 3 we recall the definition of the second Houghton group H_2, which is an extension of $\mathbb{Z}$ by S_∞, the group of permutations of finite support on a countable set. The main result of this Section is the following theorem.

Theorem 4. *The strong depth of dead ends of elements of the Houghton group* H_2 *with respect to the standard generating set is unbounded.*

2 Dead ends in abelian groups and groups with more than one end

Let G be a group and X a finite set of generators for G. Without loss of generality let X be closed under inversion. Throughout this section we make frequent use of the canonical homomorphism π which projects the free monoid X^* over X onto G by sending each $x \in X$ to $x \in G$. We refer to the elements of the free monoid as words over the alphabet X. We call a word w, representing an element g (i.e. $\pi(w) = g$), a geodesic word, if the length of w (i.e. number of letters) equals $d_X(1, g)$. Then w describes a geodesic path in $\Gamma = \Gamma(G, X)$ from 1 to g.

The main step in the proof of Theorem 1 will be the following observation: Assume that a word w represents a dead end of a group. Then a word $w' \subset wX^*$ can be geodesic only if $w = w'$.

Let $\mathbb{N} = \{0, 1, 2 \ldots\}$ denote the set of non-negative integers. First of all we need the following observation concerning points in $\mathbb{N}^n$.

Definition 5. Let $p = (p_1, p_2, \ldots, p_n)$, $q = (q_1, q_2, \ldots, q_n)$ be points in $\mathbb{N}^n$. We call p and q crossrelated ($p \lesseqgtr q$), if there exist i, j, such that $p_i < q_i$ and $p_j > q_j$.

Lemma 6. *Any set C of pairwise crossrelated points in $\mathbb{N}^n$ has to be finite.*

Proof. We prove this by induction on n. Let $n = 1$. By definition, the only crossrelated subsets of $\mathbb{N}$ are singeltons and the empty set. All of them are finite.

Let $k \geq 1$ and assume that any set of pairwise crossrelated points in $\mathbb{N}^k$ is finite. Let $n = k+1$ and let $(d_1, d_2, \ldots d_n) \in C$. Define $C_j := \{(c_1, c_2, \ldots, c_n) \in C | c_j < d_j\}$. Then $C = \bigcup_{j=1}^{n} C_j$. We claim that for all $j = 1 \ldots n$ the set C_j is finite. We only show that C_1 is a finite set, the finiteness of C_j, $j = 2, 3, \ldots, n$ follows analogously. For $0 \leq l < d_1$ define

$$D_l := \{(c_2, c_3, \ldots, c_n) | (l, c_2, c_3, \ldots, c_n) \in C_1\}.$$

Because C is a set of pairwise crossrelated points, C_1 is a set of crossrelated points, hence all the D_j are sets of pairwise crossrelated points in $\mathbb{N}^k$. So C_1 is a finite union of finite sets and therefore finite. Hence C is a finite union of finite sets and thus finite. □

Proof of Theorem 1. Let G be an infinite abelian group generated by the set $X = \{x_1, x_2, \ldots, x_{2n}\}$, where $x_i = x_{n+i}^{-1}$. Let w be a (geodesic) word representing an element $g \in G$. Then any permutation of the word w is again a (geodesic) word representing the same element g. Hence it is sufficient to count the number of occurrences of the single letters and we can forget about the ordering of the letters. We obtain a surjective monoid homomorphism $\phi = \pi \circ \psi$ from $\mathbb{N}^{2n}$ onto G by defining the map (which is not a homomorphism) $\psi : \mathbb{N}^{2n} \to X^*$,

$$\psi\left((i_1, i_2, \ldots, i_{2n})\right) := x_1^{i_1} x_2^{i_2} \ldots x_{2n}^{i_{2n}}.$$

Let $D = \{g_1, g_2, g_3, \ldots\}$ be the set of dead ends in G and $w_j \in \phi^{-1}(g_j)$ such that $\psi(w_j)$ is a geodesic word. According to the observation at the beginning of this section, no π-preimage of a dead end can be a subword of the π-preimage of another dead end. Therefore the w_j have to be pairwise crossrelated and Lemma 6 implies, that D is a finite set. □

Proof of Theorem 2. Let X be a finite set of generators for the group G, and $\Gamma = \Gamma(G, X)$ be the Cayley graph. Let $g \in G$ be a dead end. Because G has more than one end, there exists $k \in \mathbb{N}$ such that $\Gamma \setminus B_1(k)$ has more than one component. By transitivity of Γ, $\Gamma \setminus B_g(k)$ has more than one component, too. If $d(1, g) \leq k$ the depth of the dead end is $\leq 2k$. Otherwise 1 lies in one component of $\Gamma \setminus B_g(k)$. We choose a geodesic ray starting in 1 whose intersection with the component of 1 is finite. (See Figure 1.) This ray hits the ball $B_g(k)$ at least once. Let v_1 denote the first intersection vertex of the ray and the ball and v_2

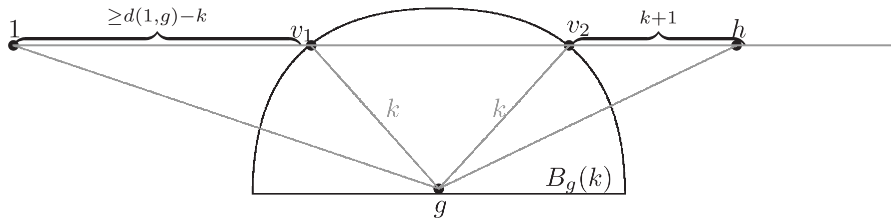

Figure 1: The depth of the dead end g is bounded above by $2k$: The ray starting in 1 and ending in a different component of $\Gamma \setminus B_g(k)$ has to hit the ball $B_g(k)$. The distance $d(1,h) \geq d(1,v_1) + d(v_2,h) \geq d(1,g) + 1$ and $d(g,h) \leq 2k+1$.

the last intersection vertex. Let h be the vertex on the chosen ray and in the component of the end which has distance $k+1$ to v_2. Then $d(g,h) \leq 2k+1$ and

$$d(1,h) \geq d(1,v_1) + d(v_2,h) \geq (d(1,g) - k) + (k+1) = d(1,g) + 1.$$

Hence the depth of dead ends is bounded above by $2k$. □

3 Dead ends in Houghton groups

The Houghton groups H_n were introduced by C.H. Houghton in the late 70s (see [2] for a more detailed discussion of Houghton groups). They are defined as groups of permutations of n copies of the set of non-negative integers. Here we only discuss H_2.

Definition 7. Let $\mathbb{Z}^{\pm}$ be the set of all non-zero integers. The group H_2 is the subgroup of all permutations of $\mathbb{Z}^{\pm}$ consisting of all permutations σ for which there exist an integer $S(\sigma)$, the shift of σ, such that σ maps all but finitely many $k \in \mathbb{Z}^{\pm}$ to $k + S(\sigma)$.

Convention. We consider the action of H_2 on $\mathbb{Z}^{\pm}$ from the right, so that a word in the generators can be read from left to right. We define for $f, g \in \mathrm{H}_2$ that $[f,g] := f^{-1}g^{-1}fg$ and $f^g := g^{-1}fg$. In this section σ or σ_i always denotes a generator which represents an element of order 2. Also σ_i and σ_i^{-1} (or σ and σ^{-1}) are different elements in the free monoid over the set of generators, we will not distinguish between them, because they represent the same group elements.

The set of all finitary permutations of $\mathbb{Z}^{\pm}$ forms a normal subgroup of H_2 isomorphic to S_∞ and it can easily be seen that the group H_2 is isomorpic to $S_\infty \rtimes \mathbb{Z}$, where the projection $\mathrm{H}_2 \to \mathrm{H}_2/S_\infty$ maps each σ to $S(\sigma)$. The group H_2 is generated by the pure shift

$$s(k) \quad := \quad \begin{cases} 1 & k = -1 \\ k+1 & k \neq -1 \end{cases}$$

and the transposition $\sigma = (0,0) \leftrightarrow (0,1)$. From now on we fix this set of generators $X := \{s, \sigma\}$ for H_2 and with respect to X the group H_2 has the presentation

$$\mathrm{H}_2 \quad = \quad \left\langle s, \sigma \,\middle|\, \sigma^2 = (\sigma\sigma^s)^3 = [\sigma, \sigma^{s^n}] = 1, n \geq 2 \right\rangle .$$

As usual it is difficult to use directly the presentation of H_2 to make statements about geodesics. Therefore we now develop a model for the group elements which will help to do calculations during this section. In order to obtain this vivid description of H_2 and to distinguish between the integers the group H_2 acts on and the position of these integers we interpret H_2 as a group of permutations of pearls labeled with a number $k \in \mathbb{Z} \setminus \{0\}$. One standard notation for such a permutation is

$$g = \begin{pmatrix} \dots & g^{-1}(-k) & \dots & g^{-1}(-2) & g^{-1}(-1) & g^{-1}(1) & g^{-1}(2) & \dots & g^{-1}(k) & \dots \\ \dots & -k & \dots & -2 & -1 & 1 & 2 & \dots & k & \dots \end{pmatrix}$$

In this notation the lower row always lists all non-zero pearls in increasing order. So it is sufficient to know the upper row of pearls and the position of e.g the pearl -1 in the lower row. We mark this position by putting an arrow to the right of the preimage of -1. Then g is represented by

$$\dots\ g^{-1}(-k)\ \dots\ g^{-1}(-2)\ g^{-1}(-1) \downarrow g^{-1}(1)\ g^{-1}(2)\ \dots\ g^{-1}(k)\ \dots$$

In this model the trivial element of H_2 is given by the configuration

$$id = \dots\ -k\ -(k-1)\ \dots -2\ \ -1 \downarrow 1\ 2\ \dots\ (k-1)\ k\ \dots$$

Now it is very easy calculate the products $g \cdot \sigma$ and $g \cdot s$ (which can be seen as a construction of the Cayley graph $\Gamma = \Gamma(\mathrm{H}_2, X)$). The multiplication with s moves the arrow one position to the left, s^{-1} one position to the right, the multiplication with σ interchanges the pearls left and right of the arrow. From now on we will call this arrow the cursor, because in our vivid model it stands for a cursor indicating the position where the next σ in a word acts.

By describing the action of s as a movement of the cursor in stead of the movement of the pearl-string we have changed the frame of reference. We now define the cursor position of an element $g \in \mathrm{H}_2$. Let $w = x_1 x_2 \dots x_n$ be a word representing g and $c(g, w) := \#\{i | x_i = s^{-1}\} - \#\{i | x_i = s\}$. It is easy to check that $c(g, w)$ equals the difference of the number of positive pearls left of the cursor and the number of negative pearls right of the cursor. Hence $c(g, w)$ depends only on g. Define $c(g) := c(g, w)$. If $c(g) = i$ we say g *has cursor position* c_i.

We now have seen that an element of H_2 is given by a finitary permutation of the biinfinite string of numbered pearls and an integer denoting the cursor position inside this string. We introduce the following notation for such a configuration. Write the finitary permutation of pearls as a product of cycles and write the cursor position as index of this product. For example $s = (1)_{-1}$,

$\sigma = (-1,1)_0$ or $g = (-3,-1,2,4)_{-3}$ (which means g equals the configuration $\ldots -5\ -4 \downarrow 4\ -2\ -3\ 1\ -1\ 3\ 2\ 5\ 6\ldots$).

As usual the distance $d_X(1,g)$ of a group element to the identity is defined as the minimum of the length of the words representing g and a word is a sequence of commands to move the cursor one step or to interchange the pearls next to the cursor. The length of such a word equals the number of commands.

The elements with cursor in position c_0 form a subgroup isomorphic to S_∞ which is generated by $Y = \{\sigma_t = \sigma^{s^t} | t \in \mathbb{Z}\}$. We now turn our attention to some special elements in this subgroup. Let $g_k = ((-k,k)(-(k-1),k-1)\ldots(-3,3)(-2,2)(-1,1))_0$. In order to change the configuration from the identity to g_k one has to flip a string of length $2k$ centered around the origin. This seems to be the worst thing one can do and in fact the following theorem holds.

Theorem 8. *The element g_k is a dead end of depth at least k in H_2 with respect to the generating set X.*

The proof of this statement consists of several steps. We have to calculate $d_X(1,g_k)$ and to find geodesic words representing g_k. This will be done in Lemma 10. In addition we have to show, that all elements in $B_{g_k}(k)$ are closer to the identity. The proof of this statement is prepared by Lemma 9, which considers the generating set Y in stead of X. These ingredients will then be mixed to finish the proof.

We first calculate the distance $d_{Y_k}(1,g_k)$ in the subgroup S_{2k} generated by $Y_k = \{\sigma_t | -k < t < k\}$. Let M be the support of g_k, $M = \{-k\ldots-1,1,\ldots k\}$. For each $g \in S_{2k}$ define the inversion number $\mathrm{inv}(g) := \#\{(i,j) | i < j \in M,\ g(i) > g(j)\}$. Then $\mathrm{inv}(g\cdot\sigma_t) = \mathrm{inv}(g) \pm 1$. Hence $d_{Y_k}(1,g) \geq \mathrm{inv}(g)$. The inversion number of g_k equals $\mathrm{inv}(g_k) = \sum_{i=1}^{2k} i - 1 = k(2k-1)$. If we find a word w_k of length $k(2k-1)$ representing g_k then w_k is geodesic.

First of all we need for all $l \in \mathbb{N}$ geodesic words representing $(-l,l)_0$. For $l = 1$ this is the word $u_1 = \sigma_0$. Let $l > 1$. The word $v_l := \sigma_{l-2}\sigma_{l-3}\ldots\sigma_{-(l-1)}$ represents the group element $(l-1,-l,-(l-1),\ldots,l-2)_0$ and hence $u_l := (\sigma_{l-1})^{v_l}$ represents $(-l,l)_0$. The length of $u_l = 1 + 4(l-1) = \mathrm{inv}(u_l)$. Hence u_l is a geodesic word.

Because $\pi(u_l)$ and $\pi(u_{l'})$ have disjoint support for $l \neq l'$ the concatenation of u_l for all $0 < l \leq k$ in any order represents g_k. Let $w_k = u_k u_{k-1}\ldots u_1$. Then the length of $w_k = \sum_{l=1}^{k} 1 + 4(l-1) = k(2k-1)$. This word w_k has a property which will be important later on. For each group element $g \in S_{2k}$ one can obtain a word representing g by deleting some letters of w_k. From a Coxeter point of view this might not be surprising (S_{2k} is a Coxeter group) but can be easily seen without such arguments.

Lemma 9. *For any element $g \in S_{2k}$ one can construct a word representing g out of the word w_k by deleting some of its letters.*

Proof. We do this by induction on k. The case $k = 1$ is trivial. Let $n \geq 1$ and assume that the statement is correct for n. Let $k = n+1$. The word w_k

equals $u_k w_{k-1}$. We construct a subword x of u_k which represents an element h such that $h^{-1}(k) = g^{-1}(k)$ and $h^{-1}(-k) = g^{-1}(-k)$. Then $h^{-1}g$ fixes k and $-k$ and by hypothesis there exist a subword y of w_{k-1} such that y represents $h^{-1}g$. Hence xy represents g.

How to obtain the word x from u_k? Let $a = g^{-1}(k)$ and $b = g^{-1}(-k)$. If $-k < a < 0$ then delete the first σ_a, if $0 < a \leq k$ then delete the first σ_{a-1}, if $-k \leq b < 0$ then delete the last σ_{b+1} and if $0 < b < k$ then delete the last σ_b. One easily checks that this yields a word representing g. □

We are now going to compute $d_X(1, g_k)$. For each word w in Y (or in Y_k) let w' be the word in X, which one obtains by replacing all σ_t by $s^{-t}\sigma s^t$ and let $\tilde{w}$ be the reduced form of w'. Then $\tilde{u}_l = s^{l-1}(\sigma s^{-1})^{2(l-1)}(\sigma s)^{2(l-1)}\sigma s^{-(l-1)}$.

Computing the distance $d(1, \pi(\tilde{u}_l))$ we have to count how many commands we need to interchange the pearl $-l$ and l and bring the cursor back to the origin. W.l.o.g. let pearl $-l$ be moved before pearl l. The cursor has to come to $c_{-(l-1)}$ for a first time which needs at least $l-1$ occurrences of s. Afterwards at some stage the cursor has to come c_{l-1} which needs at least $2(l-1)$ occurrences of s^{-1}. (Otherwise one of the elements would be a fixed point.) Finally the pearl l has to be moved to its destination which needs again at least $2(l-1)$ occurrences of s and the cursor has to go back to the origin ($l-1$ times s^{-1}). So we need at least $6(l-1)$ letters s or s^{-1}. In addition we need at least (because of the inversion number) $4l-3$ occurrences of σ. Hence $\tilde{u}_l$ is a geodesic word.

Lemma 10. *The word $\tilde{w}_k$ is a geodesic word representing g_k and $d_X(1, g_k) = 1 + \sum_{l=2}^{k}(8l-5)$.*

Proof. The length of the word $\tilde{w}_k = 1+\sum_{l=2}^{k}(8l-5)$ because $\#\sigma's = \sum_{l=1}^{k}(4l-3)$, $\#s's = \sum_{l=2}^{k}(4l-2)$. We prove the statement by induction. For $k = 1$ the word $\tilde{w} = \sigma$ which is geodesic.

Let $k > 1$ and assume that $\tilde{w}_{k-1}$ is geodesic. For $\tilde{w}_k$ it holds: $\text{length}(\tilde{w}_k) = \text{length}(\tilde{w}_{k-1}) + 8k - 5$. Assume that there exists a word v_k representing g_k such that $\text{length}(v_k) < \text{length}(\tilde{w}_k)$.

Let v'_{k-1} be the word, one obtains by deleting the following letters of v_k: In the first step, we delete all letters s, which move the cursor away from or onto c_0 in v_k. Afterwards we delete all letters σ which interchange two pearls that were already interchanged before.

How many letters s are deleted in this procedure? Each pearl $l > 1$ is moved through c_0 which implies that the following situations have to occur:

1. The cursor is at position c_1 and the pearl l is left next to the cursor and the cursor is moved to c_0.

2. The cursor is at position c_0, the pearl l is left next to the cursor and the cursor is moved to c_{-1}.

Hence for all pearls $l > 1$ there are 2 letters s deleted. Analogously for each pearl $l < -1$ the letter s^{-1} is deleted twice. In addition the cursor has to leave the origin a first time and come back to it a last time. All in all we have to delete at least $2(k-1)+2(k-1)+2=4k-2$ letters $s^{\pm 1}$.

The word v'_{k-1} represents g_{k-1}, because for $l>0$ one can check that $\pi(v'_{k-1})(l)=\pi(v)(l+1)+1=-l$ and for $l<0$ that $\pi(v'_{k-1})(l)=\pi(v)(l-1)-1=-l$. The number of σ's in v'_{k-1} is (by definition) given by the inversion number of $\pi(v'_{k-1})$. Hence the letter σ is deleted $(4k-3)$ times and $\text{length}(v'_{k-1}) \leq \text{length}(v_k)-(4k-2)-(4k-3) < \text{length}(\tilde{w}_k)-(8k-5)=\text{length}(\tilde{w}_k-1)$. But this contradicts the hypothesis. □

Proof of Theorem 8. We have to show that for all elements $g \in B_{g_k}(k)$ the distance $d(1,g) \leq d(1,g_k)$. Let $\omega \in S_\infty$ and $t \in \mathbb{Z}$ such that $g=(\omega)_t$. Now $g \in B_{g_k}(k)$, hence $\omega \in S_{2k}$ and $|t| \leq k$. Lemma 9 implies that $d(1,g) \leq d(1,g_k s^{-t})$. The element g_k is of order 2, so $g_k^{-1}=g_k$ and hence $\tilde{w}_k^{-1}$ is another word representing g_k. This word ends with k occurrences of s and, as a consequence of symmetry there exist a geodesic word which represents g_k and ends with k occurrences of s^{-1}. So in fact $d(1,g) \leq d(1,g_k s^{-t}) \leq d(1,g_k)-|t|$. □

The following Corollary completes the proof of Theorem 4.

Corollary 11. *The element $g_k \in \mathrm{H}_2$ defined as in Theorem 8 is a dead end of strong depth at least k.*

Proof. In the proof of Theorem 8 we have shown, that all elements $(\omega)_t$, $\omega \in S_{2k}, |t| \leq k$ have distance $d(1,(\omega)_t) \leq d(1,g_k)$ to the identity. Hence a geodesic from g_k to a point outside the ball of radius $d(1,g_k)$ has to contain an element $h=(\omega')_{\pm k}$, $\omega \in S_{2k}$ and we have seen in proof of Theorem 8 that $d(1,h) \leq d(1,g_k)-k$. □

The situation in H_n for $n>2$ is more complicated, but one still has a good combinatorial description of elements and generators. By a careful study of this discription it might be possible to show that H_n (or at least H_3) also has arbitrary deep dead ends.

To prove that the lamplighter group has unbounded strong depth only a few changes in the proof of [4] are needed. We do not know about any dead ends of large strong depth in the group of Riley and Warshall and therefore conclude by posing the following question: Does there exist a group, which has unbounded strong depth of dead ends with respect to one set of generators but strong depth bounded above with respect to a different set of generators?

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