Exercises to the course „Stochastic Processes“

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Assignment 10

Stationary renewals, Gamma distribution, Compound Poisson processes

Please hand in your written solutions on Friday, June 25, 2010, at the beginning of the course

37. ***Waiting for the next renewal.*** a) Let X be binomial$(10, 1/4)$-distributed. Find its *second moment* $\mathbf{E}[X^2]$.
b) Let G be a random variable whose distribution is the size-biasing of the binomial$(10, 1/4)$-distribution. Compute $\mathbf{E}[G]$.
c) In a stationary renewal process on $\mathbb{Z}$ whose lifetime distribution is binomial$(10, 1/4)$-distributed, what is the expected time to the next renewal from the time point 100? And how many renewals do you expect between times 101 and 110 (boundaries included)?

38. ***The Gamma Distribution***
For $k \in (0, \infty)$, the Gamma(k)-distribution has density $g_k(y)\,dy$ with

$$g_k(y) := \frac{1}{\Gamma(k)} y^{k-1} e^{-y}, \quad y > 0$$

(Recall that $\Gamma(k) = \int_0^\infty x^{k-1} e^{-x} dx$)
a) Let Y be Gamma(k)-distributed, $k > 0$. Show that

$$\mathbf{E}\, e^{-\beta Y} = (1 + \beta)^{-k}, \quad \beta > 0.$$

b) Let $Y_i \quad (i = 1, 2)$ be Gamma(k_i)-distributed and independent. What is the distribution of $Y_1 + Y_2$? (Hint: You may use that the distribution of an $\mathbb{R}_+$-valued random variable Z is determined by its *Laplace transform*, i.e. the collection of all expectations $\mathbf{E}[e^{-\beta Z}]$, $\beta > 0$.)

c) (*optional, for gourmets*) Apparently, the size-biasing of the Gamma(5)-distribution is the Gamma(6)-distribution. Can you interpret this in the light of a stationary renewal process on the real time axis? (Hint: Think of a stationary renewal process on $\mathbb{R}$ whose lifetimes arise as sums of 5 independent standard exponential random variables. One of these covers the origin of the time axis ....)

39. *Three Poisson points.* Let V_1, V_2, V_3 be independent standard exponential random variables, and put $T_1 := V_1$, $T_2 := V_1 + V_2$, $T_3 := V_1 + V_2 + V_3$.
a) What is the conditional density of (T_1, T_2) given T_3?
b) What is the conditional density of (T_1, T_2) given the event $E := \{T_3 > 3\} \cap \{T_2 < 3\}$?
c) What is the conditional probability of the event $\{T_1 \in [0, 1]\} \cap \{T_2 \in [2, 3]\}$, given the event E from part b)?

40. *Compound Poisson processes.* Let $Z = \sum \delta_{T_i}$ be a stationary Poisson point process on $\mathbb{R}_+$ with intensity $\lambda \in \mathbb{R}_+$, and $H_1, H_2, \ldots$ be i.i.d. real-valued random variables with mean μ and variance σ^2. We define the *Poisson counting process* $X_t := Z([0, t])$, $t \geq 0$, and the *compound Poisson process* $Y_t := \sum_{i: T_i \leq t} H_i$, $t \geq 0$.

a) Find $\mathbf{E}[Y_t | X_t]$ and $\mathbf{E}[Y_t]$.

b) Using the variance decomposition as stated in Handout 3, compute $\mathbf{Var}[Y_t]$.