Prof. A. WAKOLBINGER SoSe 2010

Exercises to the course „Stochastic Processes"

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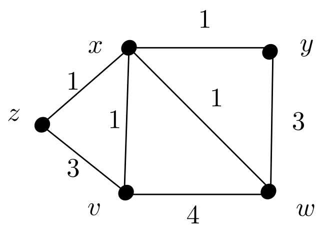
Assignment 9

Equilibria, excursions, Metropolis, renewals

Please hand in your written solutions on Friday, June 18, 2010, at the beginning of the course

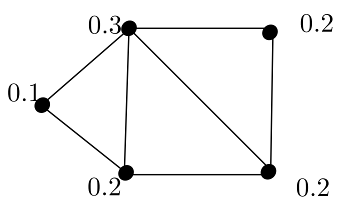
33. *A random walk on a little network*



a) Consider the network with vertex set $S = \{v, w, x, y, z\}$ and edge weights (*conductances*) c_{ij} as in the figure. Let P be the transition matrix on S for which, for each i, the transition probabilities P_{ij}, $j \neq i$, are proportional to c_{ij}, $j \neq i$. (E.g., $P_{wy} = 3/8$.) Calculate the equilibrium distribution. (Hint: Show that $\pi_i = \text{const} \cdot c_i$, with $c_i :=$ the sum of the conductances of the outgoing edges from vertex i, satisfiy the equations of detailed balance.)

b) For the random walk specified in a) (that is the Markov chain on S with transition matrix P), what is the expected number of visits to z during an excursion from w?

34. *Slow down if necessary!*



Let Q be the transition matrix of the simple random walk on the graph depicted in the figure, with vertices v, w, x, y, z as in the previous exercise. Following the recipe of Metropolis, we want to modify the transition rule given by Q in such a way that the probability distribution π with the weights given in the figure becomes an equilibrium distribution. If the current state is x and the Q-rule says "move on to w", what is the probability with which we reject it and stay in x?

35. *Expected payoff along an excursion.* Let (X_n) be a simple random walk on $\mathbb{Z}$, starting in 0.

a) For $i \in \mathbb{Z}$, what is the expected number of visits to i during an excursion from 0? (Hint: What are the invariant measures?)

b) Let $\alpha > 0$. Each time the random walker visits a state $i \neq 0$, she earns the payoff $|i|^{-\alpha}$. Consequently, her expected payoff during an excursion from 0 is $\mathbf{E}_0[\sum_{n=1}^{T_0-1} |X_n|^{-\alpha}]$, where T_0 is the first return time to 0. For which α is this expected payoff finite, and for which ones is it infinite?

36. *Coin tossing forever.* A coin with success probability p is tossed at each time $n \in \mathbb{Z}$. Let $S_0 < S_1 < \dots$ denote the time points of success after time 0, and $S_{-1} > S_{-2} > \dots$ those before (and including) time 0.

a) What is the probability of the event $\{S_0 - S_{-1} = g\}$, $g = 1, 2, \dots$?

b) Compute the size-biasing of a geometric distribution with parameter p.