Exercises to the course „Stochastic Processes“

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Assignment 8

Equilibria and time reversal

Please hand in your written solutions on Friday, June 11, 2010, at the beginning of the course

29. ***Memory or not?*** [1] A die is rolled repeatedly. Which of the following are Markov chains? For those that are, supply the transition matrix.
(a) The largest number X_n shown up to the n-th roll.
(b) The number N_n of sixes in n rolls.
(c) At time n, the time C_n since the most recent six.[2]
(d) At time n, the time B_n until the next six.

30. ***All transient?*** (1) For a Markov chain on S, argue that $\mathbf{E}_i[V_j] \leq \mathbf{E}_j[V_j]$, $i, j \in S$. (Recall that we defined V_j as the expected number of visits in state j after time 0.)
(2) Assume that S is finite. Can it be that all states in S are transient?
(3) *A Markov chain on a finite state space has an equilibrium distribution.* True or false?

31. ***Time reversal.*** Let P be a transition matrix on S, having an equilibrium distribution π.
a) Show that

$$Q_{ji} := \frac{1}{\pi_j} P_{ij}\, \pi_i$$

is a transition matrix on $S_\pi := \{i \in S : \pi_i > 0\}$.
b) Compute $\mathbf{P}_\pi(X_0 = i | X_1 = j)$ and $\mathbf{P}_\pi(X_0 = i | X_1 = j,\, X_2 = k)$.
c) Show that

$$\mathbf{P}_\pi((X_n, X_{n-1}, \ldots, X_0) = (i_n, \ldots, i_0)) = \pi_{i_n} Q_{i_n, i_{n-1}} \ldots Q_{i_1, i_0}.$$

Congratulations! You have just shown that the time reversal of a stationary Markov chain is again a stationary Markov chain.

32. ***Time reversal of the residual lifetime process.*** Consider a renewal chain with life time distribution ρ on $\mathbb{N}$ and finite expected life time $\mathbf{E}R$. Let ν be the equilibrium distribution of the residual lifetime process (Y_n), and P be its transition matrix. (Recall that $\nu_k = \frac{1}{\mathbf{E}R}\mathbf{P}[R \geq k]$.)

a) Express the *dual transition matrix* Q (in the sense of Exercise 31) in terms of ρ.

b) The *hazard function* is the conditional probability that the lifetime R equals k, given that it exceeds $k-1$. What is the relation to Q?

Congratulations! You have just seen that the time reversal of the stationary residual lifetime process is the stationary age process.

[1] taken from G. Grimmett, D. Stirzaker, Probability and Random Processes, 3rd. ed., Oxford University Press, 2001

[2] Here, C_0 is literally defined if (and only if) the rolling of the die has started suitably before time 0. However, if you insist on starting to roll the die only at time 0, you may define C_0 in an arbitrary way, and each time add a one as long as no six appears