Exercises to the course „Stochastic Processes“

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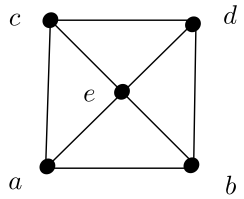
Assignment 7
Markov chains and martingales

Please hand in your written solutions on Friday, June 4, 2010, at the beginning of the course

25. ***A simple random walk on a finite graph.*** Consider the simple random walk on the graph with vertex set $S := \{a, b, c, d, e\}$ and edges as in the figure.



(i) For $\mu := \delta_a$ (i.e. $X_0 = a$ a.s.), compute the weights $\mu_1(i) := \mathbf{P}_\mu(X_1 = i)$ and $\mu_2(i) := \mathbf{P}_\mu(X_2 = i)$, $i \in S$.

(ii) Find a probability measure π on S such that $\mathbf{P}_\pi(X_1 \in (.)) = \pi(\cdot)$.

26. ***Martingales and harmonic functions*** Let S be a countable set, P be a transition matrix on S, and h be a real-valued function defined on S. Assume that for all $n \in \mathbb{N}_0$ and $i \in S$,

$$\sum_{k\in S}(P^n)_{ik}|h(k)| < \infty$$

or in other words that $h(X_n)$ is $\mathbf{P}_i$-integrable.
(a) Show that the following two assertions are equivalent:

(1) For all $i \in S$, $(h(X_n))_{n\geq 0}$ is an $\mathbb{F}^X$-martingale under $\mathbf{P}_i$,

(2) h is P-*harmonic*, that is $Ph = h$.

(b) In the situation of Exercise 25, use the martingale convergence theorem to show that every P-harmonic function h is constant.

27. ***Conditioning a Markov chain on its future.*** Let X be simple random walk on $\{0, 1, \dots, 10\}$ stopped when first hitting $\{0, 10\}$, and let T be the first hitting time of $\{0, 10\}$.

a) Let X start in state 2. What is the probability that X makes its first step to state 1, given $\{X_T = 10\}$?

b) Argue that, when conditioned on the event $\{X_T = 10\}$, X remains a Markov chain, and compute the transition matrix of this conditioned chain on its state space $\{1, 2, \dots, 10\}$.

28. ***Critical branching processes die out a.s.*** Let ξ_k, $k = 1, 2, \dots$, be i.i.d. copies of an $\mathbb{N}_0$-valued random variable ξ, with $\mathbf{E}[\xi] = 1$ and $\mathbf{P}(\xi = 0) > 0$. We define a transition matrix P on $\mathbb{N}_0$ by putting for $i, j \in \mathbb{N}_0$

$$P_{ij} := \mathbf{P}(\sum_{k=1}^{i} \xi_k = j)\,.$$

(a) Which are the transient, and which the recurrent states?

(b) Let (Z_n) be a Markov chain with transition matrix P. Why does Z_n converge a.s. to some random variable Z_∞ as $n \to \infty$?

(c) Why is $Z_\infty = 0$ a.s.?