Prof. A. WAKOLBINGER SoSe 2010

Exercises to the course „Stochastic Processes“

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Assignment 6

Martingale stopping, moment generating functions, uniform integrability

Please hand in your written solutions on Friday, May 28, 2010, at the beginning of the course

21. ***Betting on a pattern.*** Let $Z = (Z_1, Z_2, \ldots)$ be a fair 01-coin tossing.

a) Let $T := \min\{n \geq 4 : (Z_{n-3}, Z_{n-2}, Z_{n-1}, Z_n) = (0,0,0,1)\}$. By considering a martingale $M_n = \sum_{j=0}^{\infty} G_n^{(j)} = \sum_{j=0}^{n-1} G_n^{(j)}$, $n = 0, 1, \ldots$ similar as in the Friday May 14 course, and by using the equality $\mathbf{E}M_T = \mathbf{E}M_0$ (see part b) of this exercise for a justification of this equality) find $\mathbf{E}T$, the *expected time to the first occurrence of the pattern* 0001.

b) (optional) Consider the same setting as part in a).

(i) Show that $\mathbf{E}T < \infty$, e.g. by defining a Geom(1/16)-distributed random variable $Y = h(Z)$ which obeys $T \leq 4Y$ a.s.

(ii) Why is $|M_{T\wedge n}| \leq T + 30$ for all n?

(iii) Use the baby version of the stopping theorem to conclude that $\mathbf{E}M_{T\wedge n} = \mathbf{E}M_0$ and proceed by dominated convergence to obtain the equality $\mathbf{E}M_T = \mathbf{E}M_0$.

c) (optional) Can you find a pattern *abcd* for which the expected time to its first occurrence is 30?

22. ***How fast is an asymmetric random walk?***
Let X be the p-q-random walk from Exercise 18. For $a \in \mathbb{N}$, define $S_a := \min\{n \geq 0 : X_n = a\}$, $T := \min\{n \geq 0 : X_n = -10\}$.

a) Find a function g such $X_n - \sum_{i=1}^{n} g(X_{i-1})$, $n = 0, 1, \ldots$, is a martingale.

b) Use the stopping theorem applied to that martingale to establish an equality relating $\mathbf{E}[X_{S_a \wedge T}]$ and $\mathbf{E}[S_a \wedge T]$.

c) From that equality, compute $\mathbf{E}[T]$ by letting a tend to ∞. (Note that $\mathbf{E}[X_{S_a\wedge T}]$ converges to $\mathbf{E}[X_T]$, since $\mathbf{P}(S_a < T)$ decays exponentially as $a \to \infty$.)

d) Just to train the concepts: With $S := S_{10}$, is $\mathcal{F}_{S\wedge T} \subset \mathcal{F}_T$? Is $\mathcal{F}_S \subset \mathcal{F}_T$?

23. ***Exponential moments of the normal distribution.*** Let Z be a standard normal random variable.

(i) Compute and sketch the graph of $s \to g(s) := \mathbf{E}[e^{sZ}]$. *(Hint: Complete the exponent* $sx - \frac{x^2}{2} = \frac{1}{2}(2sx - x^2)$ *to a negative square.)*

(ii) Let (W_n) be a random walk with $\mathrm{N}(\mu, \sigma^2)$-distributed increments, starting in 0. Assume the drift μ is negative. For $a > 0$, find an upper bound for the probability $p(a) := \mathbf{P}(W_n > a \text{ for some } n)$, with the property that $p(a)$ decays exponentially as $a \to \infty$.
(*Hint: Express* $\mathbf{E}[e^{s(\mu+\sigma Z)}]$ *in terms of the function g found in part (i), and use a result of the Friday May 7/ Tuesday May 11 course.*)

24. ***Where do the big bits come from?*** A family $(X_i)_{i\in I}$ of random variables is called *uniformly integrable* if $\sup_{i\in I} \mathbf{E}[|X_i|\, I_{|X_i|\geq c}] \to 0$ as $c \to \infty$.

(i) Show that uniform integrability implies *boundedness in* L^1, i.e. $\sup_{i\in I} \mathbf{E}[|X_i|] < \infty$.

(ii) Let U be uniformly distributed on $[0, 1]$. Find a sequence of random variables $g_n(U)$, $n = 1, 2, \ldots$, that is bounded in L^1 but not uniformly integrable. (*Hint: Try with all the* g_n *nonnegative and with integral 1.)*