Exercises to the course „Stochastic Processes"

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Assignment 5

Hitting probabilities via stopped (super-)martingales

Please hand in your written solutions on Friday, May 21, 2010, at the beginning of the course

17. ***How improbable is a large detour on the way to*** $-\infty$***?*** Let Y be a standard exponential random variable, and $V := Y - 2$.

a) Compute and sketch the graphs of $s \mapsto \mathbf{E}e^{sY}$ and $s \mapsto \mathbf{E}e^{sV}$.

b) Let $V_1, V_2, \dots$ be independent copies of V. For $a > 0$, let $p(a)$ denote the probability of the event $\{V_1 + \dots + V_n > a \text{ for some } n\}$. Using a result of the Friday May 7 course, show that $p(a) \le e^{-a/2}$. Is the exponent 1/2 the best possible constant in this estimate?

18. ***Hitting probabilities for the*** p-q-***random walk.*** Let p and q be nonnegative with $p + q = 1$. We consider the *p-q-random walk* $X = (X_n)$, with $X_n = Z_1 + \dots + Z_n$, $n = 0, 1, \dots$, and $Z_1, Z_2, \dots$ independent, identically distributed random variables with $\mathbf{P}(Z_1 = 1) = p$, $\mathbf{P}(Z_1 = -1) = q$.

a) Is $M = (M_n)$ with $M_n := \left(\frac{q}{p}\right)^{X_n}$ a martingale?

b) For $p = 1/4$ and $q = 3/4$, compute the probability that X hits $+10$ before it hits -10.

19. ***Predicting along a path: the Doob decomposition.*** Let $\mathbb{F} = (\mathcal{F}_n)_{n\ge0}$ be a filtration.

a) Let $(D_n)_{n\ge1}$ be an $\mathbb{F}$-adapted sequence of integrable random variables. Assume $\mathbf{E}[D_n|\mathcal{F}_{n-1}] = 0$ a.s. for all $n = 1, 2, \dots$. Show that $D_1 + \dots + D_n$, $n = 0, 1, \dots$ is a martingale.

b) Let (Y_n) be an $\mathbb{F}$-adapted sequence of integrable random variables. Show that $D_n := Y_n - Y_{n-1} - \mathbf{E}[Y_n - Y_{n-1}|\mathcal{F}_{n-1}]$, $n = 1, 2, \dots$ fulfills the requirements of a) and conclude that $M_n := Y_n - Y_0 - \sum_{i=1}^n \mathbf{E}[Y_i - Y_{i-1}|\mathcal{F}_{i-1}]$, $n = 0, 1, \dots$ is a martingale.

c) For a p-q-random walk as in Exercise 18, and a given function $f : \mathbb{Z} \to \mathbb{R}$, find a function $g : \mathbb{Z} \to \mathbb{R}$ such that $f(X_n) - \sum_{i=1}^n g(X_{i-1})$, $n = 0, 1, \dots$ is a martingale.

20. ***Making friends with*** $\exp$***,*** $\mathbf{E}[\exp(sV)]$ ***and dominated convergence***

a) Let v^+, v^- be nonnegative numbers. By sketching the graphs of the functions $s \mapsto e^{sv^+}$ and $s \mapsto e^{-sv^-}$, argue that for $0 < s \le \alpha$

$$v^+ \le \frac{e^{sv^+} - 1}{s} \le v^+ e^{sv^+} \le \frac{1}{\alpha}\alpha v^+ e^{\alpha v^+} \le \frac{1}{\alpha} e^{2\alpha v^+} \qquad \text{and} \qquad \left|\frac{e^{-sv^-} - 1}{s}\right| \le v^-$$

b) Conclude from a) that for $v \in \mathbb{R}$ and $0 < s \le \alpha \in \mathbb{R}$,

$$\left|\frac{e^{sv} - 1}{s}\right| \le \frac{1}{\alpha} e^{2\alpha v} + |v|.$$

c) Let V be an integrable random variable and assume $\mathbf{E}[e^{\eta V}] < \infty$ for some $\eta > 0$. Use b) and dominated convergence to show that

$$\lim_{s\downarrow0} \frac{1}{s}\left(\mathbf{E}[e^{sV}] - 1\right) = \mathbf{E}[V].$$