Exercises to the course „Stochastic Processes“

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Assignment 4
Martingale convergence

Please hand in your written solutions on Friday, May 14, 2010, at the beginning of the course

13. *Constant expectations - and yet no martingale.* We know that every martingale has constant expectation. What about the converse? *(Hint: Let Z be uniform on $\{-1, 1\}$, and put $X_n := nZ$.)*

14. *A geometric random walk.* Let $X = (X_0, X_1, \dots)$ be a simple random walk on $\mathbb{Z}$, and $\mathbb{F}$ be its natural filtration.
a) For which sequence (c_n) is $\exp(X_n - c_n)$ a martingale?
b) For that (c_n), what is the a.s. limit of $\exp(X_n - c_n)$ as $n \to \infty$?

15. *Predicting backwards.*
Let X_1, $X_2, \dots$ be i.i.d., integrable random variables, and

$$S_n := X_1 + \ldots + X_n.$$

a) Compute
i) $\mathbf{E}[X_1 | X_1 + X_2]$ ii) $\mathbf{E}[X_1 | (S_n, S_{n+1}, \ldots)]$.

b) Show that $(X_1 + \ldots + X_n)/n$ converges a.s. as $n \to \infty$

Hints: a) i) Use a symmetry argument.
a) ii) Use the equality $\mathcal{F}(S_n, S_{n+1}, S_{n+2}, \ldots) = \mathcal{F}(S_n, X_{n+1}, X_{n+2}, \ldots)$, and recall Exercise 9.
b) Use Exercise 16 c). By the way, Exercise 16 is quite easy if you followed the Friday April 30 course.

16. *Backward martingales.*
Let $(\mathcal{F}_{-n})_{n \geq 0}$ be a family of sub-σ-fields which is *decreasing* in the following sense:

$$\mathcal{F}_0 \supseteq \mathcal{F}_{-1} \supseteq \mathcal{F}_{-2} \supseteq \ldots.$$

For an integrable $\mathcal{F}_0$-measurable random variable Z, put

$$M_{-n} := \mathbf{E}[Z \,|\, \mathcal{F}_{-n}].$$

Show that

a)$(M_{-n}, M_{-n+1}, \ldots, M_0)$ is a martingale with respect to the filtration $(\mathcal{F}_{-n}, \mathcal{F}_{-n+1}, \ldots, \mathcal{F}_0)$,

b)
$$\mathbf{E}U_n \leq \frac{|a| + \mathbf{E}|Z|}{b - a} \quad ,$$
where $a < b$, and U_n is the number of $[a, b]$-upcrossings of the path $(M_{-n}, M_{-n+1}, \ldots, M_0)$,

c) M_{-n} converges a.s. as $n \to \infty$.