PROF. A. WAKOLBINGER SoSe 2010

Exercises to the course „Stochastic Processes“

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Assignment 3

Conditional expectation: prediction, projection, compensation.

Please hand in your written solutions on Friday, May 7, 2010, at the beginning of the course

9. *Conditional independence.* (i) Let Z_1 be independent of (Z_2, Z_3). Show that this implies the *conditional independence of Z_3 and Z_1 given Z_2*, i.e., for all bounded, measurable h,

$$\mathbf{E}[h(Z_3)|(Z_1, Z_2)] = \mathbf{E}[h(Z_3)|Z_2] \quad \text{a.s.}$$

(ii) (Optional) Let (X_i) be a simple random walk on $\mathbb{Z}$ starting in the origin. Is X_1 independent of (X_2, X_3)? Is X_1 conditionally independent of X_3 given X_2?

10. *Pull out what is known.* Let Y be an integrable random variable, $\mathcal{F}$ a sub-σ-field, and H be a real-valued $\mathcal{F}$-measurable random variable. Assume that HY is integrable.
In the course we learned that for all $n \in \mathbb{N}$ and $F \in \mathcal{F}$

$$|(|H| \wedge n)\mathbf{E}[Y|\mathcal{F}]| = |\mathbf{E}[(|H| \wedge n)Y\big|\mathcal{F}]| \le \mathbf{E}[\,|H \wedge n|\,|Y|\,\big|\,\mathcal{F}] \le \mathbf{E}[\,|HY|\,\big|\,\mathcal{F}], \tag{1}$$

$$\mathbf{E}[HI_{\{|H|\le n\}}\mathbf{E}[Y|\mathcal{F}]I_F] = \mathbf{E}[HI_{\{|H|\le n\}}YI_F]. \tag{2}$$

(i) Use dominated convergence to show that $H\mathbf{E}[Y|\mathcal{F}]$ is integrable, and that for all $F \in \mathcal{F}$

$$\mathbf{E}[H\mathbf{E}[Y|\mathcal{F}]I_F] = \mathbf{E}[HYI_F].$$

(ii) Conclude that

$$\mathbf{E}[HY|\mathcal{F}] = H\mathbf{E}[Y|\mathcal{F}] \qquad \text{a.s.}$$

11. *A refined decomposition of the variance.* Let Y be a square-integrable random variable (i.e. $\mathbf{E}[Y^2] < \infty$). For a natural number n, consider sub-σ-fields

$$\{\Omega, \emptyset\} =: \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_n.$$

Put $V_i := \mathbf{E}[Y|\mathcal{F}_i]$, $i = 0, 1, \ldots, n$, and $V_{n+1} := Y$.

(i) What is V_0? Why is $\mathbf{E}[V_i|\mathcal{F}_{i-1}] = V_{i-1}$ a.s.?

(ii) Show that the random variables $V_i - V_{i-1}$, $i = 1, 2, \ldots, n+1$, are uncorrelated, and find a decomposition of the variance of Y along the filtration $(\mathcal{F}_i)$.

(iii) (Optional) As an example, consider a simple random walk (X_n) on $\mathbb{Z}$ starting in the origin, putting $\mathcal{F}_i := \mathcal{F}(X_1, \ldots, X_i)$ and $Y := X_{10}$. What are the V_i, and what is the decomposition of the variance of Y along the filtration $(\mathcal{F}_i)$?

12. *A branching process.* Let $\xi_{n,k}$, $n = 1, 2, \ldots$, $k = 1, 2, \ldots$, be independent, identically distributed random variables, taking the values $0, 1, 2, 3$ each with probability $1/4$. We define inductively

$$Z_1 := 1, \qquad Z_{n+1} := \sum_{k=1}^{Z_n} \xi_{n,k}, \qquad n = 1, 2, \ldots$$

(You are invited to think of Z_n as the size of the n-th generation of an asexual population, with all individuals reproducing independently, and $\xi_{n,k}$ being the number of children of the k-th individual in the n-th generation.)
What is $\mathbf{E}[Z_n|(Z_1, \ldots, Z_{n-1})]$? Can you find a sequence (m_n) of real numbers such that $(Z_n/m_n)_{n=1,2,\ldots}$ is a martingale? Compute $\mathbf{E}[Z_n]$.