Exercises to the course „Stochastic Processes“

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Assignment 2

Random pairs, two-stage experiments, and conditional expectations

Please hand in your written solutions on Friday, April 30, 2010, at the beginning of the course

5. ***X-measurable random variables.*** Let S be a countable space and $\mathfrak{S}$ be the collection of all subsets of S. Let X be an S-valued random variable. Show that a real-valued random variable Z is $\mathcal{F}(X)$-measurable if and only if Z is of the form $g(X)$ for some function $g : S \to \mathbb{R}$.

(In the course we formulated - without proof - a version of this statement also for general measurable state spaces $(S, \mathcal{S})$. The proof in the discrete case, which you are invited to cook up here, is easy and still instructive.)

6. ***Predicting the square of a simple random walk***. For fixed $a \in \mathbb{Z}$, let $(X_0, X_1, X_2, \ldots)$ be simple random walk starting in a, as in Exercise 4. For $n \in \mathbb{N}$, let $\mathcal{F}_n$ be the σ-field generated by $(X_0, \ldots, X_n)$. Find

(i) $\mathbf{E}[X_{n+1}|\mathcal{F}_n]$ (ii) $\mathbf{E}[X_{n+1} - X_n|\mathcal{F}_n]$
(iii) $\mathbf{E}[X_{n+1}^2|\mathcal{F}_n]$ (iv) $\mathbf{E}[(X_{n+1} - X_n)^2|\mathcal{F}_n]$

7. ***σ-algebras.*** a) Let $S := \{1, 2\}$. Which of the following two collections $\mathcal{C}_1$, $\mathcal{C}_2$ are σ-algebras on S?
(i) $\mathcal{C}_1 = \{\{1, 2\}, \emptyset\}$ $\mathcal{C}_2 = \{\{1, 2\}, \{1\}, \emptyset\}$.

b) Let S be a non-empty set. Show that
(i) the intersection $\mathfrak{S}_1 \cap \mathfrak{S}_2$ of two σ-algebras $\mathfrak{S}_1$, $\mathfrak{S}_2$ on S is a σ-algebra,
(ii) the intersection of an arbitrary family of σ-algebras on S is a σ-algebra.

c) Let $\mathcal{C}$ be a collection of subsets of the non-empty set S. How would you formally define $\sigma(\mathcal{C})$:= “the smallest σ-algebra on S that contains $\mathcal{C}$”?

8. ***Slicing the square.*** Let X and Y be independent and uniform on the interval $[0, 6]$. Find

a) the conditional distribution
b) the conditional expectation

of Y given $X - Y = 1$.

Hint: What is the joint distribution of X and Y? Where are the points (a, b) with $a - b = 1$? Draw a picture.