Exercises to the course „Stochastic Processes"

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Assignment 1

Random pairs, two-stage experiments, and conditional expectations

Please hand in your written solutions on Friday, April 23, 2010, at the beginning of the course

1. ***Predicting ratios in Pólya's urn.*** In Pólya's urn with white and blue balls, each day one ball is drawn completely at random and put back into the urn together with a new ball of the same colour. Let W_n and B_n denote the number of white and blue balls which are in the urn at the end of the n-th day. Compute for $n \in \mathbb{N}$
a) the conditional distribution of tomorrow's ratio $R_{n+1} := \frac{W_{n+1}}{W_{n+1}+B_{n+1}}$ given the "past history" $((W_1, R_1), \dots, (W_n, R_n))$,
b) the conditional expectation of R_{n+1} given $((W_1, R_1), \dots, (W_n, R_n))$.

2. ***Predicting a function of*** (X_1, X_2) ***based on*** X_1***.*** Let (X_1, X_2) be a random pair taking its values in $S_1 \times S_2$. We know from Elementary Probability (cf. [KW][1] p. 85 and p. 104) that the distribution of (X_1, X_2) can be obtained in a "two-stage" manner:

$$\mathbf{P}(X_1 \in da_1, X_2 \in da_2) = \mathbf{P}(X_1 \in da_1)\mathbf{P}_{a_1}(X_2 \in da_2).$$

$\mathbf{P}_{a_1}(X_2 \in .)$ is then the *conditional distribution of* X_2 *given* $\{X_1 = a_1\}$. Consequently, for a real-valued mapping h defined on $S_1 \times S_2$, the conditional expectation of $h(X_1, X_2)$ given $\{X_1 = a_1\}$ is

$$\mathbf{E}_{a_1}[h(X_1, X_2)] = \int h(a_1, a_2)\, \mathbf{P}_{a_1}(X_2 \in da_2) = \mathbf{E}_{a_1}[h(a_1, X_2)]$$

(cf. [KW] formulae (14.6) and (14.7)).
a) Give an argument why for *independent* X_1, X_2 the conditional expectation of $h(X_1, X_2)$ given $\{X_1 = a_1\}$ is $\mathbf{E}[h(a_1, X_2)]$.
b) For $p \in (0, 1)$, let X and Y be independent, Geom(p)-distributed random variables. Compute $\mathbf{E}[\min(X, Y)\,|X]$, or in words, the conditional expectation of $\min(X, Y)$ given X.

3. ***Poisson with a random parameter.*** We consider a two-stage experiment: first generate a positive random variable X, and given $\{X = \alpha\}$, let Y be Poisson(α)-distributed.
a) $\mathbf{E}[Y|X] =?$ b) $\mathbf{Var}[Y|X] =?$
c) For Exp(λ)-distributed X, compute $\mathbf{Var}[Y]$.

4. ***Betting on a random walk.*** Let $(X_1, X_2, \dots)$ be a simple random walk on $\mathbb{Z}$, i.e. $X_n = a + \sum_{i=1}^n \xi_i$, where $\xi_1, \xi_2, \dots$ are i.i.d. ± 1 with probability $1/2$, and $a \in \mathbb{Z}$. For $n \in \mathbb{N}$ let f be a real-valued function on $\mathbb{Z}^n$. Compute

$$\mathbf{E}[f((X_1, \dots, X_n))(X_{n+1} - X_n)\,|\,(X_1, \dots, X_n)].$$

[1] G. Kersting and A. Wakolbinger, Elementare Stochastik, Birkhäuser 2008

Interpretation: In a fairy-tale roulette (with rouge *and* noir *each having probability* 1/2*) you bet on* "rouge next" *with a strategy that depends on what you have seen so far. What is your expected gain?*