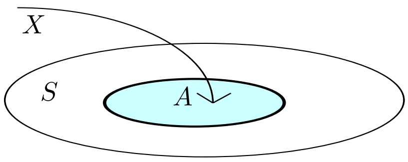
## $\sigma$-algebras, measurable mappings, and $\sigma$-fields of events

Recall our logo of Elementare Stochastik, see [KW] p. 3:




For certain subsets $A \subset S$, we considered the *events* $\{X \in A\}$, and assigned to them *probabilities* in a consistent (countably additive) way. If $S$ is countable, it does not cause trouble to do this for *all* subsets $A$ of $S$. If, however, $S$ is uncountable, it turns out that admitting *all* subsets $A$ of $S$ would exclude any non-trivial theory. One thus restricts to a (large enough) class of "interesting" subsets of $S$. Here and below, $S$ is always a non-empty set.

Definition: A collection $\mathfrak{S}$ of subsets of $S$ is called a *$\sigma$-algebra on $S$* $:\Longleftrightarrow$
(i) $S \in \mathfrak{S}$,
(ii) together with $A$, also $S \setminus A =: A^c \in \mathfrak{S}$,
(iii) together with $A_1, A_2, \ldots$, also $\bigcup_{i\in\mathbb{N}} A_i \in \mathfrak{S}$.

The pair $(S, \mathfrak{S})$ is then called a *measurable space*. A subset $A$ of $S$ is called $\mathfrak{S}$*-measurable* if it belongs to $\mathfrak{S}$.[1]

Usually, a collection $\mathfrak{C}$ of subsets of $S$ is specified which one wants to have in $\mathfrak{S}$ by all means. Then $\mathfrak{S}$ can be defined as the smallest of all $\sigma$-algebras on $S$ that contain $\mathfrak{C}$ as a sub-collection. This is called the $\sigma$-algebra *generated by* $\mathfrak{C}$ *(on $S$)*, and denoted by the symbol $\sigma(\mathfrak{C})$.

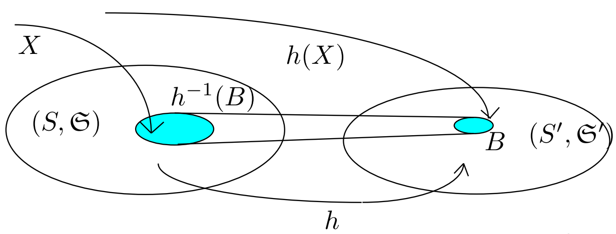
The most prominent example is the *$\sigma$-algebra of Borel sets*[2] on $\mathbb{R}$, usually denoted by $\mathcal{B}$. It is generated by the collection of all intervals on the real line.

The only upgrade of our basic logo (see top of this page) was to endow $S$ with a $\sigma$-algebra $\mathfrak{S}$. Also, the figure in [KW] page 19 needs an upgrade, as shown on next page.

We consider two measurable spaces $(S, \mathfrak{S})$, $(S', \mathfrak{S}')$, and a mapping $h : S \to S'$. We want $h(X)$ to be an $S'$-valued random variable, with the equality $\{h(X) \in B\} = \{X \in h^{-1}(B)\}$ for all $B \in \mathfrak{S}'$. So there is a requirement on $h$:

For all $B \in \mathfrak{S}'$, the preimage $h^{-1}(B)$ must belong to $\mathfrak{S}$.

A mapping $h$ with this property is called ($\mathfrak{S}$-$\mathfrak{S}'$-*)measurable*.




[1] Note that we are here speaking of measurability before speaking of any concrete measure. The idea is that $\sigma$-algabras are good candidates for collections of subsets on which measures can be defined.

[2] named after Émile Borel (1873-1956)

The collection of events

$$\mathcal{F}(X) := \{\{X \in A\} : A \in \mathfrak{S}\}$$

is called the *σ-field of events generated by X*. It has the following properties:

(i) it contains the *certain event* and the *impossible event*[3],
(ii) together with an event $E$, it contains also the *complementary event* $E^c$,
(iii) together with a countable family of events $E_1, E_2, \ldots$, it contains also their "union" $\bigcup_i E_i$.

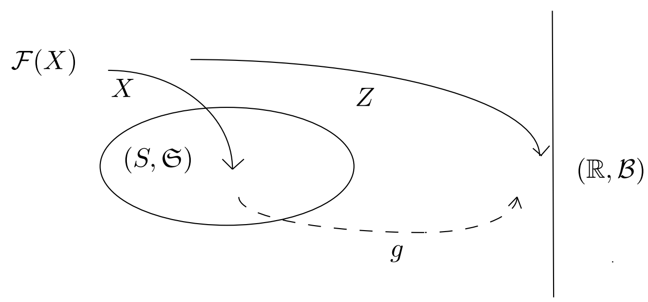
Any collection of events with the properties (i)-(iii) is called a *σ-field of events.* Henceforth we work with a basic $\sigma$-field $\mathcal{A}$ of events, assuming that $\mathcal{A}$ contains the $\sigma$-fields generated by all the random variables that come into play.

Any $\sigma$-field $\mathcal{F}$ of events with the property $\mathcal{F} \subset \mathcal{A}$ is briefly called a *sub-σ-field* ("Teilfeld").

Let $\mathcal{F}$ be a sub-$\sigma$-field, and $Z$ be a random variable. $Z$ is called *$\mathcal{F}$-measurable* if $\mathcal{F}(Z) \subset \mathcal{F}$.

Intuitively, this means that all possible information about $Z$ is contained in $\mathcal{F}$. For example, with an arbitrary $S$-valued random variable $X$, a random variable $Z$ of the form $Z = h(X)$ is $\mathcal{F}(X)$-measurable. (Indeed, $\{h(X) \in B\} = \{X \in h^{-1}(B)\} \in \mathcal{F}(X)$ for all $B \in \mathfrak{S}$.)
At least for real-valued $Z$, also the converse is true:

If a real-valued random variable $Z$ is $\mathcal{F}(X)$-measurable, then it is of the form $Z = g(X)$ for some measurable mapping $g : S \to \mathbb{R}$.




[3] denoted by $\Omega$ and $\emptyset$, respectively