# BRANCHING PROCESSES AND THEIR APPLICATIONS: LECTURE 9: Maximum of a critical process

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## 1 Maximum of the critical Galton-Watson process

Let $Z(n),\ n = 0, 1, 2, \dots$ be a critical Galton − Watson process (GWP) with $Z(0) = 1$, $\xi$ be the 'offspring variable' (the distribution of which coincides with that of $Z(n)$ conditioned that $Z(n) = 1$) with the generating function (g.f.) $f(s)$. Denote by

$$M_n = \max_{0 \le k \le n} Z_k \qquad \text{and} \qquad M = \max_n M_n$$

the partial and global maxima of the process $\{Z(n)\}_{n \ge 0}$ respectively. Recall that, in the critical case, $M < \infty$ a.s., for the process becomes extinct in a finite time $\tau = \min\{n : Z(n) = 0\}$ with probability one.

**Theorem 1** *If*

$$f(s) = \mathbf{E}s^{\xi}, \qquad \mathbf{E}\xi = f'(1-) = 1,\ f"(1) = 2B \in (0, \infty), \tag{1}$$

*then*

$$\lim_{x \to \infty} x\, \mathbf{P}(M > x) = 1. \tag{2}$$

We prove this theorem in several steps. Let $\zeta_n,\ n = 1, 2, \dots$ be i.i.d. random variables having the same distribution as $\zeta = \xi - 1$ and hence having zero means. Put

$$S_0 = 1, \qquad S_n = S_{n-1} + \zeta_n, \quad n \ge 1.$$

Recall that, without the loss of generality, we may assume that the process $\{Z(n)\}$ is embedded into the r.w. $\{S_n\}$: for $V_{-1} = 0$,

$$Z(n) = S_{V_{n-1}}, \qquad V_n = \sum_{k=0}^{n} Z_k, \qquad n \ge 0. \tag{3}$$

Clearly $V_k$ are stopping times for the r.w. $\{S_n\}$, and the latter can be replaced in (3) by the stopped r.w.

$$S_n^* = S_{n\wedge\tau_0}, \qquad \tau_0 = \min\{k \geq 1: \ S_k = 0\}.$$

Denote by

$$M^* = \max_{n\geq 0} S_n^*$$

the global maximum of the stopped r.w. $\{S_n^*\}$. It is obvious from (3) that $M \leq M^*$, and for the maximum $M^*$, it was proved in Pakes (Journal of Applied Probability, 15(1978), pp.292-299)) that, if (1) holds then

$$\lim_{x\to\infty} x\,\mathbf{P}(M^* > x) = 1. \tag{4}$$

The following lemma shows that, on the other hand, $M^*$ cannot be 'essentially greater' than $M$, and hence the relation (4) implies the same asymptotics for the distribution of $M$.

**Lemma 2** . *For any critical GWP $\{Z(n)\}$, there exists a function $\varepsilon = \varepsilon(x) \to 0$ as $x \to \infty$, such that*

$$(1-\varepsilon)\,\mathbf{P}\left(M^* > (1+\varepsilon)x\right) \leq \mathbf{P}(M > x) \leq \mathbf{P}(M^* > x) \leq \frac{1}{x}. \tag{5}$$

**Proof of Lemma 1**. The second inequality in (5) is obvious from (3), while the last inequality in (5) follows from the Doob inequality for the stopped r.w. $S_n^*$ which is clearly a martingale with $ES_n^* = ES_0^* = 1$. Thus it remains only to prove the first inequality in (5).

Letting $y = (1+\varepsilon)x, \ \varepsilon > 0$, we get

$$\begin{aligned}\mathbf{P}\left(M > x\right) &\geq \mathbf{P}\left(M > x; M^* > y\right) = \mathbf{P}\left(M > x | M^* > y\right)\mathbf{P}\left(M^* > y\right) \quad (6)\\ &= \left(1 - \mathbf{P}\left(M \leq x | M^* > y\right)\right)\mathbf{P}\left(M^* > y\right).\end{aligned}$$

Put

$$T = \min\{k \geq 1: \ S_k^* > y\}, \qquad m = \min\{j \geq 1: \ V_j > T\}.$$

Since $\{M^* > y\} = \{T < \infty\}$, we see that

$$\begin{aligned}\mathbf{P}\left(M \leq x | M^* > y\right) &= \mathbf{P}\left(M \leq x | T < \infty\right)\\ &\leq \mathbf{P}\left(Z_m \leq x, Z_{m+1} \leq x | T < \infty\right)\\ &\leq \mathbf{P}\left(Z_m \leq x, S_{V_m} - S_T \leq x - y | T < \infty\right). \quad (7)\end{aligned}$$

But $V_m - T \leq Z_m$ by the definition of $m$, and, on the event $\{Z_m \leq x\}$, one has

$$S_{V_m} - S_T \geq \min_{j\leq x}\left(S_{T+j} - S_T\right).$$

By the strong Markov property, the last expression does not depend on $T$ and has, conditioned that $T < \infty$, the same distribution as $\min_{j\leq x} S_j$. Therefore the right hand side of (7) does not exceed

$$\mathbf{P}\left(\min_{j\le x} S_j \le -x\varepsilon\right) = \mathbf{P}\left(x^{-1}\min_{j\le x} S_j \le -\varepsilon\right). \tag{8}$$

Further, by the strong law of large numbers $x^{-1}S_x \to 0$ a.s. as $x \to \infty$, and hence

$$x^{-1}\max_{j\le x}|S_j| \to 0 \quad as \quad x \to \infty,$$

so that, for any fixed $\varepsilon > 0$, the probability (8) tends to 0 as $x \to \infty$. This means that, for some positive function $\varepsilon(x) \to 0$ as $x \to \infty$,

$$\mathbf{P}\left(\min_{j\le x} S_j \le -x\varepsilon(x)\right) \le \varepsilon(x). \tag{9}$$

In view of (6) and (7) for $y = (1+\varepsilon(x))\,x$, relation (9) gives

$$\mathbf{P}(M > x) \ge (1-\varepsilon(x))\mathbf{P}\left(M^* > (1+\varepsilon(x))x\right).$$

Lemma 1 is proved.

**Proof of Theorem 1** follows immediately from Lemma 1 and relation (4).

The next lemma gives both upper and lower bounds for the expectations $EM_n$ in terms of the tail $P(M > x)$.

**Lemma 3** *For any* $t > 0$,

$$-t\,\mathbf{P}(Z(n) > 0) \le \mathbf{E}M_n - \int_0^t \mathbf{P}(M > x)\,dx \le \frac{2nB}{t}. \tag{10}$$

**Proof of Lemma** 3. For any $t > 0$,

$$\mathbf{E}M_n = \int_0^\infty \mathbf{P}(M_n > x)\,dx \le \int_0^t \mathbf{P}(M > x)\,dx + \int_t^\infty \mathbf{P}(M_n > x)\,dx. \tag{11}$$

To estimate the last integral, observe that, since $\{Z(n)\}$ is a martingale, the Doob's inequality yields

$$\mathbf{P}(M_n > x) = \mathbf{P}(\max_{0\le k\le n} Z(n) > x) \le x^{-2}\mathbf{E}Z^2(n) = 2x^{-2}Bn.$$

Therefore,

$$\int_t^\infty \mathbf{P}(M_n > x)\,dx \le 2\int_t^\infty x^{-2}Bn\,dx = \frac{2Bn}{t}.$$

The right inequality in (10) is proved.

On the other hand,

$$\begin{aligned}
\mathbf{E}M_n &\geq \mathbf{E}\left(M_n; Z(n)=0\right) = \mathbf{E}\left(M; Z(n)=0\right) \\
&\geq \int_0^t \mathbf{P}\left(M > x; Z(n)=0\right) dx = \int_0^t \mathbf{P}\left(M > x\right) dx \\
&\quad - \int_0^t \mathbf{P}\left(M > x; Z(n)>0\right) dx \\
&\geq \int_0^t \mathbf{P}\left(M > x\right) dx - t\mathbf{P}\left(Z(n)>0\right).
\end{aligned}$$

Lemma 3 is proved.

**Theorem 4** *If conditions (1) are valid then*

$$\lim_{n\to\infty} \frac{\mathbf{E}Z(n)}{\log n} = 1. \tag{12}$$

**Proof.** First we note that Theorem 1 yields

$$\int_0^n \mathbf{P}\ (M > x)\ dx\ = (1+\theta)\ \log n \tag{13}$$

where $\theta = \theta(n) \to 0,\ n \to \infty$. We know that under conditions (1) the non-extinction probability of the process has the asymptotic representation

$$\mathbf{P}\left(Z(n) > 0\right) \sim \frac{1}{Bn}.$$

Thus, for any $\delta > 0$ there exists $n_0 = n_0(\delta)$ such that for all $n \geq n_0$

$$\begin{aligned}
\mathbf{E}M_n &\geq \int_0^n \mathbf{P}\left(M > x\right) dx - n\mathbf{P}\left(\tau > n\right) \\
&\geq (1-\delta)\log n - \frac{3n}{nB} \\
&= (1-\delta)\log n - \frac{3}{B}
\end{aligned} \tag{14}$$

To make use of the right inequality in (10) we let $t = n$. We have that for any $\delta > 0$ there exists $n_0 = n_0(\delta)$ such that for all $n \geq n_0$

$$\mathbf{E}M_n \leq (1+\delta)\log n + 2B. \tag{15}$$

Now relations (14) and (15) mean that, for any positive $\delta > 0$,

$$1-\delta \leq \liminf_{n\to\infty} \frac{\mathbf{E}M_n}{\log n} \leq \limsup_{n\to\infty} \frac{\mathbf{E}M_n}{\log n} \leq 1+\delta.$$

Since $\delta > 0$ is arbitrary, the theorem follows.