BRANCHING PROCESSES AND THEIR APPLICATIONS: LECTURE 7: Limit theorems for critical processes; reduced supercritical and subcritical processes

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1 Conditional limit theorem for critical processes

Assume

$$A = f^{'}(1) = 1,\ f''(1) = 2B \in (0, \infty)\,. \tag{1}$$

Theorem 1 *Under (1)*

$$Q(n) = \mathbf{P}\left(Z(n) > 0\right) \sim \frac{1}{Bn},\ n \to \infty, \tag{2}$$

and

$$\lim_{n\to\infty} \mathbf{E}\left[\exp\left\{-\lambda \frac{Z(n)}{Bn}\right\}|Z(n) > 0\right] = \frac{1}{1+\lambda}. \tag{3}$$

Remark.

$$\frac{1}{1+\lambda} = \int_0^\infty e^{-\lambda x} e^{-x} dx$$

and, therefore,

$$\lim_{n\to\infty} P\left(\frac{Z(n)}{Bn} \leq y|Z(n) > 0\right) = \int_0^y e^{-x} dx = 1 - e^{-y}.$$

giving the exponential law.

Proof. Expanding $f(s)$ in a vicinity of point $s = 1$ we have

$$1 - f(s) = 1 - s - B(1-s)(1-s)^2$$

where

$$B(1-s) = \frac{f"(\theta)}{2}, \theta = \theta(s) \in [s, 1]\,,$$

and

$$B(y) \to B = \frac{f"(1)}{2},\ y \to 0.$$

Thus,

$$1 - f_{k+1}(0) = 1 - f(f_k(0)) = 1 - f_k(0) - B(1 - f_k(0))(1 - f_k(0))^2$$

or

$$Q(k+1) = Q(k) - B(Q(k))Q^2(k).$$

Observe that as $k \to \infty$

$$\begin{aligned} 1 &\le \frac{Q(k)}{Q(k+1)} = \frac{1 - f_k(0)}{1 - f_{k+1}(0)} = \frac{1 - f_k(0)}{1 - f(f_k(0))} \\ &\le \frac{1 - f_k(0)}{f'(f_k(0))\,(1 - f_k(0))} = \frac{1}{f'(f_k(0))} \to 1. \end{aligned}$$

Thus, we can write

$$Q(k+1) = Q(k) - B^*(k)Q(k)Q(k+1)$$

where

$$B^*(k) = B(Q(k))\frac{Q(k)}{Q(k+1)} = B + \varepsilon(k)$$

and $\varepsilon(k) \to 0, k \to \infty$, $|\varepsilon(k)| < C$. Hence it follows that

$$\frac{1}{Q(k+1)} - \frac{1}{Q(k)} = B + \varepsilon(k)$$

which, after summation from $k = 0$ to $n - 1$ gives

$$\frac{1}{Q(n)} - 1 = Bn + \sum_{k=0}^{n-1} \varepsilon(k).$$

Dividing this by n we get

$$\frac{1}{nQ(n)} = B + \frac{1}{n} + \frac{1}{n}\sum_{k=0}^{n-1} \varepsilon(k) \to B,\ n \to \infty,$$

since for any $\delta > 0$ one can find $K = K(\delta)$ such that $|\varepsilon(k)| < \delta$ for all $k > K$ and, therefore,

$$\begin{aligned} \frac{1}{n}\left|\sum_{k=0}^{n-1} \varepsilon(k)\right| &\le \frac{1}{n}\sum_{k=0}^{K} |\varepsilon(k)| + \frac{1}{n}\sum_{k=K+1}^{n} |\varepsilon(k)| \\ &\le \frac{CK}{n} + \frac{\delta(n-K)}{n} \le \frac{CK}{n} + \delta. \end{aligned}$$

This proves the first part of the theorem.

Further,

$$\mathbf{E}\left[\exp\left\{-\lambda\frac{Z(n)}{Bn}\right\}|Z(n)>0\right]=1-\frac{1-f_n\left(\exp\left\{-\frac{\lambda}{Bn}\right\}\right)}{Q(n)}.$$

Now let $m(n)$ be such that

$$f_m(0)\le\exp\left\{-\frac{\lambda}{Bn}\right\}\le f_{m+1}(0)$$

or

$$1-f_m(0)\ge 1-\exp\left\{-\frac{\lambda}{Bn}\right\}\ge 1-f_{m+1}(0)$$

or

$$Q(m)\ge\frac{\lambda}{Bn}(1+\varepsilon^*(n))\ge Q(m+1).$$

where $\varepsilon^*(n)\to 0,\ n\to\infty$. Hence,

$$\frac{1}{Bm}\sim Q(m)\sim\frac{\lambda}{Bn}=\frac{1}{B\left(n/\lambda\right)}.$$

Consequently, $m\sim[n/\lambda]$. Thus, in view of

$$1-f_n(f_{m+1}(0))\le 1-f_n\left(\exp\left\{-\frac{\lambda}{Bn}\right\}\right)\le 1-f_n(f_m(0))$$

we have

$$\begin{aligned}&1-f_n\left(\exp\left\{-\frac{\lambda}{Bn}\right\}\right)\\ \sim\ &1-f_{n+m}(0)\sim\frac{1}{B(n+m)}\\ \sim\ &\frac{1}{Bn(1+\lambda^{-1})}=\frac{\lambda}{Bn(1+\lambda)}.\end{aligned}$$

Hence

$$\frac{1-f_n\left(\exp\left\{-\frac{\lambda}{Bn}\right\}\right)}{Q(n)}\sim\frac{Bn\lambda}{Bn(1+\lambda)}=\frac{\lambda}{1+\lambda}$$

and, therefore,

$$\begin{aligned}\lim_{n\to\infty}\mathbf{E}\left[\exp\left\{-\lambda\frac{Z(n)}{Bn}\right\}|Z(n)>0\right]&=1-\lim_{n\to\infty}\frac{1-f_n\left(\exp\left\{-\frac{\lambda}{Bn}\right\}\right)}{Q(n)}\\ &=1-\frac{\lambda}{1+\lambda}=\frac{1}{1+\lambda}\end{aligned}$$

proving the theorem.

Example. In the pure geometric case

$$f(s)=\frac{1}{2-s}\ ,\quad f'(1)=1,\quad f"(1)=2$$

we have an easy explanation for this result. Indeed, we know from the previous lectures that

$$P(\max S_k^* > n) = \frac{1}{n+1}.$$

Further, if

$$Z(n) = \#\left(j : S_{j-1}^* = n+1, S_j^* = n\right)$$

then

$$\begin{aligned} & P(Z(n) = k; \max S_k^* > n) \\ = & \frac{1}{n+1}\left(1 - \frac{1}{n+1}\right)^{k-1}\frac{1}{n+1} \end{aligned}$$

and

$$P(Z(n) \geq k; \max S_k^* > n) = \frac{1}{n+1}\left(1 - \frac{1}{n+1}\right)^{k-1}.$$

Hence

$$P(Z(n) \geq k \mid \max S_k^* > n) = \left(1 - \frac{1}{n+1}\right)^{k-1}$$

and with $k = ny$

$$\lim_{n\to\infty} P(Z(n) \geq ny | \max S_k^* > n) = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right)^{ny-1} = e^{-y}.$$

2 Reduced processes

Let $Z(n), n = 0, 1, ...$ be a Galton-Watson process and let $Z(m, n)$ be the number of particles in the process at time $m \leq n$ having nonempty offspring at time n. The process $\{Z(m, n), m \leq n\}$ is called the reduced process.

2.1 Reduced supercritical processes

Theorem 2 *If $A > 1$ then for any $m = 0, 1, ...$*

$$\lim_{n\to\infty} \mathbf{E}\left[s^{Z(m,n)}|Z(n) > 0\right] = \frac{f_m(P + (1-P)s) - P}{1-P}.$$

Proof. We have

$$\begin{aligned} \mathbf{E}\left[s^{Z(m,n)}|Z(n > 0\right] &= \frac{\mathbf{E}\left[s^{Z(m,n)}; Z(n) > 0\right]}{\mathbf{P}(Z(n) > 0)} = \frac{\mathbf{E}\left[s^{Z(m,n)}\right] - \mathbf{E}\left[s^{Z(m,n)}; Z(n) = 0\right]}{\mathbf{P}(Z(n) > 0)} \\ &= \frac{f_m\left(f_{n-m}(0) + (1 - f_{n-m}(0)s\right) - f_n(0)}{1 - f_n(0)} \end{aligned} \quad (4)$$

since

$$\mathbf{E}\left[s^{Z(m,n)}; Z(n) = 0\right] = \mathbf{P}(Z(n) = 0) = f_n(0)$$

and

$$
\begin{aligned}
\mathbf{E}\left[s^{Z(m,n)}\right] &= \mathbf{E}\left[\mathbf{E}\left[s^{Z(m,n)}|Z(m)\right]\right] = \mathbf{E}\left[(f_{n-m}(0)+(1-f_{n-m}(0))s)^{Z(m)}\right] \\
&= f_m\left(f_{n-m}(0)+(1-f_{n-m}(0))s\right).
\end{aligned}
$$

Passing in (4) to the limit as $n \to \infty$, using the continuity of $f_m(y)$ for $y \in [0,1)$ and recalling that $\lim_{n\to\infty} f_{n-m}(0) = P$ we prove the theorem.

The moment

$$D(n) = n - \max\{j : Z(j,n) = 1\}$$

is called the distance to the most recent mutual ancestor of the population at time n. Clearly,

$$\{n - D(n) \geq m\} = \{Z(m,n) = 1\}.$$

Corollary 3 *If $A > 1$ then for any $m = 0, 1, ...$*

$$\lim_{n\to\infty} P\left(n - D(n) = m \,|\, Z(n) > 0\right) = \left(f'(P)\right)^m - \left(f'(P)\right)^{m+1}.$$

Proof. We have

$$\mathbf{P}\left(n - D(n) \geq m | Z(n) > 0\right) = \mathbf{P}\left(Z(m,n) = 1 | Z(n) > 0\right)$$

and by the previous theorem

$$\lim_{n\to\infty} \mathbf{P}\left(Z(m,n) = 1 | Z(n) > 0\right) = coef_s\left[\frac{f_m(P+(1-P)s)-P}{1-P}\right].$$

We have by Taylor's formula

$$\frac{f_m(P+(1-P)s)-P}{1-P} = \frac{1}{1-P}\sum_{k=1}^{\infty}\frac{f_m^{(k)}(P)}{k!}(1-P)^k s^k.$$

Thus,

$$\lim_{n\to\infty} \mathbf{P}\left(Z(m,n) = 1 | Z(n) > 0\right) = \frac{1}{1-P}\frac{f_m^{'}(P)}{1!}(1-P) = \left(f'(P)\right)^m.$$

Hence

$$
\begin{aligned}
\lim_{n\to\infty} \mathbf{P}\left(n - D(n) = m | Z(n) > 0\right) &= \lim_{n\to\infty} \mathbf{P}\left(Z(m,n) = 1 | Z(n) > 0\right) \\
&\quad - \lim_{n\to\infty} \mathbf{P}\left(Z(m+1,n) = 1 | Z(n) > 0\right) \\
&= \left(f'(P)\right)^m - \left(f'(P)\right)^{m+1}.
\end{aligned}
$$

In partucular, we see that the most recent common ancestor in supercritical processes is located at the begining of the evolution of the process.

2.2 Reduced subcritical processes

Theorem 4 *If $A < 1$ then for any $m = 0, 1, ...$*

$$\lim_{n\to\infty} \mathbf{E}\left[s^{Z(n-m,n)}|Z(n>0\right] = h(s) = \frac{f^*(f_m(0)+(1-f_m(0)s)-f^*(f_m(0))}{A^m}.$$

where

$$f^*(s) = \sum_{k=1}^{\infty} P_k^* s^k$$

is the limiting function for the conditional distribution of our subcritical process:

$$1 - f^*(f(s)) = A(1 - f^*(s)).$$

Proof. We have

$$\begin{aligned}
\mathbf{E}\left[s^{Z(n-m,n)}|Z(n>0\right] &= \frac{\mathbf{E}\left[s^{Z(n-m,n)}\right] - \mathbf{E}\left[s^{Z(m,n)}; Z(n)=0\right]}{\mathbf{P}\left(Z(n)>0\right)} \\
&= \frac{f_{n-m}\left(f_m(0)+(1-f_m(0)s\right) - f_{n-m}(f_m(0))}{1-f_n(0)} \\
&= \frac{1-f_{n-m}(0)}{1-f_n(0)} \frac{f_{n-m}\left(f_m(0)+(1-f_m(0)s\right) - f_{n-m}(f_m(0))}{1-f_{n-m}(0)} \\
&\to A^{-m}\left(f^*(f_m(0)+(1-f_m(0)s) - f^*(f_m(0))\right)
\end{aligned}$$

as $n \to \infty$ proving the theorem.

Corollary 5 *If $A < 1$ then for any $m = 0, 1, ...$*

$$\lim_{n\to\infty} \mathbf{P}\left(D(n) \le m | Z(n) > 0\right) = \frac{P_1^*}{p_1(m)},$$

where

$$p_1(m) = \mathbf{P}\left(Z(m) = 1 | Z(m) > 0\right) = \frac{\mathbf{P}\left(Z(m)=1\right)}{1-f_m(0)}$$

and

$$P_1^* = \lim_{m\to\infty} p_1(m).$$

Proof. Expanding $f^*(y)$ in Taylor's series at point $y = f_m(0)$ we have

$$\begin{aligned}
h(s) &= \frac{f^*(f_m(0)+(1-f_m(0)s) - f^*(f_m(0))}{A^m} \\
&= A^{-m} \sum_{k=1}^{\infty} \frac{f^{*(k)}(f_m(0))}{k!} (1-f_m(0))^k s^k \\
&= A^{-m} f^{*\prime}(f_m(0))(1-f_m(0))s + A^{-m} \sum_{k=2}^{\infty} \frac{f^{*(k)}(f_m(0))}{k!}(1-f_m(0))^k s^k.
\end{aligned}$$

In particular

$$\lim_{n\to\infty} \mathbf{P}\left(D(n) \le m | Z(n) > 0\right) = \frac{f^{*\prime}(f_m(0))(1 - f_m(0))}{A^m}. \tag{5}$$

Using

$$1 - f^*(f_m(s)) = A^m\left(1 - f^*(s)\right), \tag{6}$$

differentiating (6) in s and setting $s = 0$ we get

$$\frac{\partial f^*(f_m(s))}{\partial s}|_{s=0} = \frac{df^*(s)}{ds}|_{s=f_m(0)}\frac{\partial f_m(s)}{\partial s}|_{s=0} = A^m\frac{df^*(s)}{ds}|_{s=0}$$

implying

$$f^{*\prime}(f_m(0)) = \frac{df^*(s)}{ds}|_{s=f_m(0)} = \frac{A^m\frac{df^*(s)}{ds}|_{s=0}}{\frac{\partial f_m(s)}{\partial s}|_{s=0}} = \frac{A^m P_1^*}{\mathbf{P}\left(Z(m) = 1\right)}.$$

Substituting this into (5) proves the corollary.

Note that

$$\lim_{m\to\infty} \frac{P_1^*}{p_1(m)} = 1$$

and, therefore, the distribution of the distance to the most recent common ancestor in subcritical processes is pure discrete.