BRANCHING PROCESSES AND THEIR APPLICATIONS: LECTURE6: Limit theorems for supercritical processes

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1 The rate of convergence to the extinction probability

Theorem 1 *If $A > 1$ and $P > 0$ then*

$$P - f_n(0) = K_1 \left[f^{'}(P)\right]^n + O\left(\left[f^{'}(P)\right]^{2n}\right), \; K_1 > 0.$$

Proof. We have

$$P - f_n(0) = P\left(1 - \frac{f_n(0)}{P}\right) = P\left(1 - g_n(0)\right)$$

and by our previous results for subcritical processes

$$1 - g_n(0) \sim K\left[g^{'}(1)\right]^n \sim K\left[f^{'}(P)\right]^n, \, 0 < K \leq 1.$$

Moreover, for

$$H(s) = \frac{1 - g(s)}{g'(1)(1-s)} = \frac{P - f(sP)}{f'(P)P\left(1-s\right)}$$

we have established that

$$1 - g_n(0) = \left[g^{'}(1)\right]^n \prod_{t=0}^{n-1} H(g_t(0))$$

and

$$K = \prod_{t=0}^{\infty} H(g_t(0)).$$

Clearly, $H(1)=1$ and $H(s)$ has finite derivatives of any order at point 1 since these derivatives corresponds to the derivatives of the function $f(s)$ at point $P<1$. In particular,

$$H'(1)=\lim_{s\uparrow 1}\left(1-\frac{1-g(s)}{g'(1)(1-s)}\right)(1-s)^{-1}=\frac{g''(1)}{2g'(1)}=\frac{Pf''(P)}{2f'(P)}.$$

Therefore,

$$\begin{aligned} H(g_t(0)) &= 1-H'(1)\,(1-g_t(0))+o(1-g_t(0)) \\ &= 1-H'(1)K\left[g^{'}(1)\right]^t(1+o(1)). \end{aligned}$$

Hence, as $n\to\infty$

$$\begin{aligned} 1-\prod_{t=n}^{\infty}H(g_t(0)) &\le H'(1)K(1+o(1))\sum_{t=n}^{\infty}\left[f^{'}(P)\right]^t \\ &= O(\left[f^{'}(P)\right]^n) \end{aligned}$$

in view of the inequality

$$\left|1-\prod_{j=n}^{\infty}(1-a_j)\right|\le\sum_{j=n}^{\infty}a_j, 0<a_j<1,$$

which one can prove by induction. It remains to obsrve that

$$\begin{aligned} \prod_{t=0}^{n-1}H(g_t(0)) &= K\left(\prod_{t=n}^{\infty}H(g_t(0))\right)^{-1} \\ &= \frac{K}{1-O([f'(P)]^n)}=K\left(1+O(\left[f^{'}(P)\right]^n)\right) \end{aligned}$$

and to write

$$\begin{aligned} P-f_n(0) &= P\,(1-g_n(0))=P\left[g^{'}(1)\right]^n\prod_{t=0}^{n-1}H(g_t(0)) \\ &= P\left[f^{'}(P)\right]^n K\left(1+O(\left[f^{'}(P)\right]^n)\right) \\ &= K_1\left[f^{'}(P)\right]^n+O(\left[f^{'}(P)\right]^{2n}) \end{aligned}$$

giving the desired result.

2 Slightly supercritical populations for single type processes

We derive a lower bound for the survival probability which is valid for any supercritical Galton-Watson process (with finite variance).

Note that $f(s) = \mathbf{E}s^\xi$ and write $Q = 1 - P$ for the survival probability of a Galton-Watson process so that for some $\theta \in [s, 1]$

$$\begin{aligned} Q &= 1 - f(1-Q) \\ &= f'(1)Q - \frac{f"(\theta)}{2!}Q^2 \geq f'(1)Q - \frac{f"(1)}{2!}Q^2. \end{aligned}$$

Since $\mathbf{E}[\xi(\xi-1)] = f"(1)$ this inequality leads to

$$Q \geq AQ - \mathbf{E}[\xi(\xi-1)]Q^2/2.$$

Division by Q (known to be strictly positive in the supercritical case) and rearranging gives

$$Q \geq \frac{2(A-1)}{\mathbf{E}[\xi(\xi-1)]}.$$

Furthermore

$$\mathbf{E}[\xi(\xi-1)] = \sigma^2 + A(A-1).$$

Thus, for *any* Galton-Watson process with reproduction mean $A = 1 + \varepsilon > 1$ and variance σ^2 we can conclude that

$$Q \geq \frac{2\varepsilon}{\sigma^2 + A\varepsilon}. \tag{1}$$

Of course, this is also true for $\varepsilon = 0$.

To get a feeling for the accuracy of the bound, consider the case of binary splitting, $f(s) = q + ps^2$, $q + p = 1$, with $\mathbf{E}[\xi] = A = 2p > 1$ and $\mathbf{E}[\xi(\xi-1)] = f''(1) = 2p$. The probability P of extinction of this process is the minimal solution of the equation

$$f(x) = q + px^2 = x$$

and for $p > 1/2$ equals q/p. Therefore, the survival probability Q is

$$Q = 1 - P = \frac{p-q}{p} = \frac{2p-1}{p} = \frac{2(2p-1)}{2p} = \frac{2(A-1)}{\mathbf{E}[\xi(\xi-1)]}.$$

Thus, despite its simplicity, the estimate (1) is *sharp* in the sense that there is no smaller bound valid for all Galton-Watson processes, and it is natural to suspect that for little ε indeed

$$Q \approx \frac{2\varepsilon}{\sigma^2 + A\varepsilon} \approx \frac{2(A-1)}{\sigma^2}. \tag{2}$$

To obtain it in a strict form, consider Galton-Watson processes with reproduction generating functions

$$f^{(\varepsilon)}(s) = \mathbf{E}[s^{\xi^{(\varepsilon)}}] = \sum_{k=0}^{\infty} \mathbf{P}(\xi^{(\varepsilon)} = k)s^k, \quad \mathbf{E}[\xi^{(\varepsilon)}] = 1 + \varepsilon \geq 1.$$

Thus, we assume that $A = 1 + \varepsilon$ is only "slightly" larger than one.

It would be nice to express Q, at least approximately in terms of a few natural and more easily determined characteristics of $\xi^{(\varepsilon)}$ rather than as a solution of the complicated equation $f^{(\varepsilon)}(P) = P$. This is done by the following theorem.

Theorem 2 *Assume that the reproduction generating functions*

$$f^{(\varepsilon)}(s) = \mathbf{E}[s^{\xi^{(\varepsilon)}}],\ \varepsilon \geq 0,$$

are such that $\mathbf{E}[\xi^{(\varepsilon)}] = 1 + \varepsilon$ *and for some* $\varepsilon_0 > 0$

$$\sup_{0 \leq \varepsilon \leq \varepsilon_0} \mathbf{E}[(\xi^{(\varepsilon)})^3] = c_3 < \infty, \quad \inf_{0 \leq \varepsilon \leq \varepsilon_0} \mathbf{E}[\xi^{(\varepsilon)}(\xi^{(\varepsilon)} - 1)] = c_2 > 0, \inf_{0 \leq \varepsilon \leq \varepsilon_0} f^{(\varepsilon)}(0) = c_0 > 0. \tag{3}$$

Then

$$Q = \frac{2\varepsilon}{\mathbf{E}[\xi^{(\varepsilon)}(\xi^{(\varepsilon)} - 1)]} + o(\varepsilon) \quad as \quad \varepsilon \to 0. \tag{4}$$

If further

$$\sigma_\varepsilon^2 = Var[\xi^{(\varepsilon)}] \to \sigma_0^2 > 0,$$

as $\varepsilon \to 0$, *then*

$$Q = 2\varepsilon/\sigma_0^2 + o(\varepsilon).$$

We do not prove this theorem.

3 Unconditional limit theorem for the supercritical case

Theorem 3 *If* $A > 1, \sigma^2 < \infty$ *then there exists a random variable* W *such that, as* $n \to \infty$

$$W_n = \frac{Z(n)}{A^n} \to W \quad a.s.$$

and

1)

$$\lim_{n \to \infty} \mathbf{E}\left(W - W_n\right)^2 = 0,$$

2)

$$\mathbf{E}W = 1, \qquad VarW = \sigma^2/(A^2 - A)$$

3)

$$\mathbf{P}(W=0) = P = \mathbf{P}(Z(n)=0 \text{ for some } n).$$

Proof. Clearly, $\mathbf{E}W_n = 1$ and

$$\begin{aligned}
\mathbf{E}\left[W_n|W_{n-1}\right] &= \mathbf{E}\left[\frac{Z(n)}{A^n}|\frac{Z(n-1)}{A^{n-1}}\right] = \frac{1}{A^n}\mathbf{E}\left[Z(n)|Z(n-1)\right] \\
&= \frac{1}{A^n}\mathbf{E}\left[\sum_{k=1}^{Z(n-1)}\xi_k^{(n-1)}|Z(n-1)\right] = \frac{Z(n-1)}{A^n}E\xi = W_{n-1}
\end{aligned}$$

and, therefore, $\{W_n\}_{n\geq 1}$ form a non-negative martingale. Hence, there exists a random variable W such that as $n\to\infty$

$$W_n = \frac{Z(n)}{A^n} \to W \quad \text{a.s.}$$

From the previous results

$$\mathbf{E}W_n^2 = \frac{\mathbf{E}Z^2(n)}{A^{2n}} = \frac{\sigma^2(1-A^{-n})}{A^2-A} + 1$$

and, therefore,

$$\sup_n \mathbf{E}W_n^2 = \lim_{n\to\infty}\mathbf{E}W_n^2 = \frac{\sigma^2}{A^2-A} + 1 < \infty.$$

Now by properties of martingales we have according to the Doob theorem (Doob, 1953, p.319) that 1) and 2) are valid.

If $r = \mathbf{P}(W=0)$ then $\mathbf{E}W = 1$ implies $r < 1$ and

$$\begin{aligned}
r &= \sum_{k=0}^{\infty}\mathbf{P}\left(W=0|Z(1)=k\right)\mathbf{P}(Z(1)=k) = \\
&= \sum_{k=0}^{\infty}\mathbf{P}^k\left(W=0|Z(1)=1\right)\mathbf{P}(Z(1)=k) = f(r).
\end{aligned}$$

Hence, $r = P$.

We see also that

$$\mathbf{E}e^{-\lambda W_n} = f\left(\mathbf{E}\left[e^{-\lambda A^{-1}W_{n-1}}\right]\right)$$

or, passing to the limit as $n\to\infty$, we see that $\varphi(\lambda) = \mathbf{E}e^{-\lambda W}$ satisfies

$$\varphi(\lambda) = f\left(\varphi\left(\frac{\lambda}{A}\right)\right). \tag{5}$$

If

$$f(s) = \frac{q}{1-ps} = \frac{1}{1+A(1-s)}$$

with $A = p/q > 1$, then

$$f_n(s) = 1 - \frac{A^n(A-1)(1-s)}{A(A^n-1)(1-s)+A-1} \tag{6}$$

and

$$\begin{aligned}\mathbf{E}e^{-\lambda W_n} &= 1 - \frac{A^n(A-1)(1-e^{-\lambda A^{-n}})}{A(A^n-1)(1-e^{-\lambda A^{-n}})+A-1} \\ &\to 1 - \frac{\lambda(A-1)}{\lambda A + A - 1} = \frac{1}{A} + \left(1 - \frac{1}{A}\right)\frac{(1-\frac{1}{A})}{\lambda + 1 - \frac{1}{A}}\end{aligned}$$

and the limiting distribution is

$$\mathbf{P}\left(W \le x\right) = \frac{1}{A} + (1 - \frac{1}{A})(1 - e^{-(1-\frac{1}{A})x}).$$