BRANCHING PROCESSES AND THEIR APPLICATIONS: LECTURE 5: Local time of a simple random walk; stationarity of supercritical populations which are known to die out

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1 Local time of the simple random walk

Consider again a simple random walk

$$S_0 = 1,\ S_k = X_1 + ... + X_k$$

with

$$\mathbf{P}(X_i = 1) = p, \qquad \mathbf{P}(X_i = -1) = 1 - p = q,\ p < q.$$

Let S_k^* be the random walk stopped at zero at moment $\tau = \min\{k : S_k = 0\}$. It is known that

$$\mathbf{P}(\tau < \infty) = \frac{p}{q}.$$

Set

$$Z(n) = \text{the number of } k \text{ such that } S_k^* = n + 1,\ S_{k+1}^* = n.$$

This is a branching process with geometric offspring distribution. Then for the local time $\ell(t)$ of the stopped random walk at level t :

$$\begin{aligned}
\ell(t) &= \text{the number of } k \text{ such that } S_k^* = t \\
&= (\text{the number of } k \text{ such that } S_{k-1}^* = t - 1 \text{ and } S_k^* = t) \\
&\quad + \big(\text{the number of } k \text{ such that } S_{k-1}^* = t + 1 \text{ and } S_k^* = t\big) \\
&= (\text{the number of } k \text{ such that } S_{k-1}^* = t \text{ and } S_k^* = t - 1) \\
&\quad + \big(\text{the number of } k \text{ such that } S_{k-1}^* = t + 1 \text{ and } S_k^* = t\big) \\
&= Z(t-1) + Z(t),\ t = 1, 2, ...,
\end{aligned}$$

where $Z(t)$ is the number of particles at moment t in a branching process with offspring generating function $f(s) = q(1 - ps)^{-1}$. Hence, to find the distribution of $\ell(t)$ it is necessary to study the joint distribution of $(Z(t-1), Z(t))$ for the processes with geometric probability generating functions. In fact, we establish the desired result in the general situation.

Theorem 1 *If $A < 1$ then for any fixed $m = 0, 1, ...$*

$$\lim_{n\to\infty} \mathbf{E}\left[s_1^{Z(n-m-1)} s_2^{Z(n-m)} | Z(n) > 0\right] = \frac{f^*(s_1 f(s_2)) - f^*(s_1 f(s_2 f_m(0)))}{A^{m+1}}.$$

Proof. We have

$$\begin{aligned} &\mathbf{E}\left[s_1^{Z(n-m-1)} s_2^{Z(n-m)}; Z(n) > 0\right] \\ = \ &\mathbf{E}\left[s_1^{Z(n-m-1)} s_2^{Z(n-m)}\right] - \mathbf{E}\left[s_1^{Z(n-m-1)} s_2^{Z(n-m)}; Z(n) = 0\right]. \end{aligned}$$

Now

$$\begin{aligned} \mathbf{E}\left[s_1^{Z(n-m-1)} s_2^{Z(n-m)}\right] &= \mathbf{E}\left[\mathbf{E}\left[s_1^{Z(n-m-1)} s_2^{Z(n-m)}\right] | Z(n-m-1)\right] \\ &= \mathbf{E}\left[s_1^{Z(n-m-1)} \mathbf{E}\left[s_2^{Z(n-m)} | Z(n-m-1)\right]\right] \\ &= \mathbf{E}\left[s_1^{Z(n-m-1)} f^{Z(n-m-1)}(s_2)\right] = f_{n-m-1}(s_1 f(s_2)) \end{aligned}$$

while

$$\begin{aligned} &\mathbf{E}\left[s_1^{Z(n-m-1)} s_2^{Z(n-m)}; Z(n) = 0\right] \\ = \ &\mathbf{E}\left[\mathbf{E}\left[s_1^{Z(n-m-1)} s_2^{Z(n-m)}; Z(n) = 0\right] | Z(n-m-1); Z(n-m)\right] \\ = \ &\mathbf{E}\left[s_1^{Z(n-m-1)} s_2^{Z(n-m)} \mathbf{E}\left[I\left\{Z(n) = 0\right\} | Z(n-m)\right]\right] \\ = \ &\mathbf{E}\left[s_1^{Z(n-m-1)} s_2^{Z(n-m)} \mathbf{P}(Z(n) = 0 | Z(n-m))\right] \\ = \ &\mathbf{E}\left[s_1^{Z(n-m-1)} s_2^{Z(n-m)} f_m^{Z(n-m)}(0)\right] = f_{n-m-1}(s_1 f(s_2 f_m(0))). \end{aligned}$$

As a result

$$\mathbf{E}\left[s_1^{Z(n-m-1)} s_2^{Z(n-m)}; Z(n) > 0\right] = f_{n-m-1}(s_1 f(s_2)) - f_{n-m-1}(s_1 f(s_2 f_m(0))).$$

Therefore

$$\begin{aligned} \mathbf{E}\left[s_1^{Z(n-m-1)} s_2^{Z(n-m)} | Z(n) > 0\right] &= \frac{f_{n-m-1}(s_1 f(s_2)) - f_{n-m-1}(s_1 f(s_2 f_m(0)))}{1 - f_n(0)} \\ &= \frac{1 - f_{n-m-1}(0)}{1 - f_n(0)} \frac{f_{n-m-1}(s_1 f(s_2)) - f_{n-m-1}(s_1 f(s_2 f_m(0)))}{1 - f_{n-m-1}(0)}. \end{aligned}$$

Now

$$\lim_{n\to\infty} \frac{1 - f_{n-m-1}(0)}{1 - f_n(0)} = \lim_{n\to\infty} \frac{1 - f_{n-m-1}(0)}{1 - f_{m+1}(f_{n-m-1}(0))} = \frac{1}{f'_{m+1}(1)} = \frac{1}{A^{m+1}} \quad (1)$$

while as $n \to \infty$

$$\begin{aligned}
& \frac{f_{n-m-1}(s_1 f(s_2)) - f_{n-m-1}(s_1 f(s_2 f_m(0)))}{1 - f_{n-m-1}(0)} \\
= \quad & \frac{(f_{n-m-1}(s_1 f(s_2)) - f_{n-m-1}(0)) - (f_{n-m-1}(s_1 f(s_2 f_m(0))) - f_{n-m-1}(0))}{1 - f_{n-m-1}(0)} \\
\to \quad & f^{*}\left(s_1 f(s_2)\right) - f^{*}\left(s_1 f(s_2 f_m(0))\right). \qquad (2)
\end{aligned}$$

Combining (1) and (2) proves the theorem.

Corollary 2 *If $A < 1$ and $\mathbf{E}\xi \log^{+} \xi < \infty$ then*

$$\lim_{m\to\infty} \lim_{n\to\infty} \mathbf{E}\left[s_1^{Z(n-m-1)} s_2^{Z(n-m)} | Z(n) > 0\right] = \frac{s_1 s_2 f^{*\prime}\left(s_1 f(s_2)\right) f^{\prime}(s_2)}{A} K$$

where K is the same as in the theorem describing the asymptotic behavior of the survival probability of a subcritical process.

Proof. As $m \to \infty$

$$\begin{aligned}
& \frac{f^{*}\left(s_1 f(s_2)\right) - f^{*}\left(s_1 f(s_2 f_m(0))\right)}{A^{m+1}} \\
\approx \quad & s_1 s_2 f^{*\prime}\left(s_1 f(s_2)\right) f^{\prime}(s_2) \frac{1 - f_m(0)}{A^{m+1}} \to \frac{s_1 s_2 f^{*\prime}\left(s_1 f(s_2)\right) f^{\prime}(s_2)}{A} K.
\end{aligned}$$

We know that for the geometric case

$$f^{*}\left(s\right) = \frac{(1-A)s}{1-As} = \frac{(q-p)s}{q-ps}.$$

From here by direct calculations we get

Corollary 3 *If the offspring generating function is geometric then for $A = p/q < 1$*

$$\begin{aligned}
& \lim_{n\to\infty} \mathbf{E}\left[s^{\ell(n-m)} | \max_{k} S_k^{*} > n\right] \\
= \quad & \lim_{n\to\infty} \mathbf{E}\left[s^{Z(n-m-1)+Z(n-m)} | Z(n) > 0\right] = \frac{f^{*}\left(sf(s)\right) - f^{*}\left(sf(sf_m(0))\right)}{A^{m+1}} \\
= \quad & \frac{(1-A)^2 pqs^2}{A\left(1-2ps\right)\left(1 - A^{m+1} - p(2 - A^{m+1} - A^{m})s\right)}.
\end{aligned}$$

Proof. We have

$$
\begin{aligned}
& \frac{f^*\left(sf(s)\right) - f^*\left(sf(sf_m(0))\right)}{A^{m+1}} \\
= \quad & \frac{1}{A^{m+1}}\left(\frac{(1-A)qs}{1-2ps} - \frac{(1-A)qs}{1-p(1+f_m(0))s}\right) \\
= \quad & \frac{(1-A)qs}{A^{m+1}}\left(\frac{1}{1-2ps} - \frac{1}{1-p(1+f_m(0))s}\right) \\
= \quad & \frac{(1-A)pqs^2(1-f_m(0))}{A^{m+1}\left(1-2ps\right)\left(1-p(1+f_m(0))s\right)} \\
= \quad & \frac{(1-A)^2pqs^2}{A\left(1-2ps\right)\left(1-A^{m+1}\right)\left(1-p(1+f_m(0))s\right)}
\end{aligned}
$$

and since

$$
1+f_m(0) = 2 - \frac{A^m(1-A)}{1-A^{m+1}} = \frac{2-A^{m+1}-A^m}{1-A^{m+1}}
$$

this changes to

$$
\frac{(1-A)^2pqs^2}{A\left(1-2ps\right)\left(1-A^{m+1}-p(2-A^{m+1}-A^m)s\right)}.
$$

Letting $m \to \infty$ we get the following statement.

Corollary 4 *If the offspring generating function is geometric then for* $A = p/q < 1$

$$
\begin{aligned}
& \lim_{m\to\infty}\lim_{n\to\infty}\mathbf{E}\left[s^{\ell(n-m)}|\max_k S_k^* > n\right] \\
= \quad & \lim_{m\to\infty}\lim_{n\to\infty}\mathbf{E}\left[s^{Z(n-m-1)+Z(n-m)}|Z(n)>0\right] = \frac{s^2 f^{*\prime}\left(sf(s)\right)f'(s)}{A}K \\
= \quad & \frac{(1-A)^2pqs^2}{A\left(1-2ps\right)^2} = \frac{(q-p)^2s^2}{\left(1-2ps\right)^2}.
\end{aligned}
$$

2 Supercritical populations which are know to die out

As we have mentioned, supercritical populations, which are known to die out later, behave as subcritical populations. Now we confirm this by means of the following theorem.

Theorem 5 *For a supercritical process there exists the limit*

$$
\lim_{n\to\infty}\mathbf{E}\left[s^{Z(n)}|n<\tau<\infty\right] = g^*(s) = \sum_{k=1}^{\infty} G_k^* s^k, \quad g^*(1) = 1,
$$

where $g^*(s)$ solves the equation

$$f'(P)(1-g^*(s)) = 1-g^*\left(\frac{f(sP)}{P}\right). \qquad (3)$$

Remark. Equation (3) is similar to that for subcritical processes and this is not a coincidence.

Proof. We have

$$\begin{aligned}\mathbf{E}\left[s^{Z(n)}; n<\tau<\infty\right] &= \sum_{k=1}^{\infty}\mathbf{P}\left(Z(n)=k; n<\tau<\infty\right)s^k \\ &= \sum_{k=1}^{\infty}\mathbf{P}\left(Z(n)=k\right)(sP)^k = f_n(sP)-f_n(0).\end{aligned}$$

Thus,

$$\mathbf{E}\left[s^{Z(n)}|n<\tau<\infty\right] = \frac{\mathbf{E}\left[s^{Z(n)}; n<\tau<\infty\right]}{\mathbf{P}\left(n<\tau<\infty\right)} = \frac{f_n(sP)-f_n(0)}{P-f_n(0)}. \qquad (4)$$

Let now

$$g(s) = \frac{f(Ps)}{P},\ g_0(s)=s,\ g_{n+1}(s)=g(g_n(s)).$$

We show that

$$g_n(s) = \frac{f_n(Ps)}{P}, n=0,1,\ldots.$$

Indeed, for $n=1$ this is true and if this is true for some n then we have by induction hypothesis that

$$\begin{aligned}g_{n+1}(s) &= g(g_n(s)) = \frac{f(Pg_n(s))}{P} \\ &= \frac{f(Pf_n(Ps)/P)}{P} = \frac{f(f_n(Ps))}{P} = \frac{f_{n+1}(Ps)}{P}.\end{aligned}$$

Now we can rewrite (4) as

$$\begin{aligned}\mathbf{E}\left[s^{Z(n)}|n<\tau<\infty\right] &= \frac{f_n(sP)-f_n(0)}{P-f_n(0)} \\ &= \frac{g_n(s)-g_n(0)}{1-g_n(0)} = \mathbf{E}\left[s^{Z^*(n)}\mid Z^*(n)>0\right],\end{aligned}$$

where $Z^*(n)$, $n=0,1,\ldots$ a branching process developing in accordance with probability generating function $g(s)$. Note that $g'(1)=f'(P)<1$ and therefore, we have a *subcritical* process. According to our previous theorems

$$\lim_{n\to\infty}\mathbf{E}\left[s^{Z^*(n)}\mid Z^*(n)>0\right] = g^*(s),$$

where $g^*(s)$ solves the equation

$$g'(1)(1-g^*(s)) = 1-g^*\left(g(s)\right)$$

or, in terms of $f(s)$

$$f'(P)(1-g^*(s)) = 1 - g^*\left(\frac{f(sP)}{P}\right).$$

Example. If

$$f(s) = \frac{q}{1-ps} = \frac{1}{1+A(1-s)}$$

with $A = p/q > 1$, then $P = q/p = 1/A$. Therefore,

$$g(s) = \frac{f(Ps)}{P} = \frac{A}{1+A(1-sA^{-1})} = \frac{A}{1+A-s} = \frac{1}{1+A^{-1}(1-s)}.$$

By direct calculations similar to those in the subcritical case

$$\begin{aligned} \lim_{n\to\infty} \mathbf{E}[s^{Z(n)} \,|\, n \quad &< \quad \tau < \infty) = \frac{s(A-1)}{A-s} = Es^{Z^*} \\ &= \quad \frac{s(1-A^{-1})}{1-A^{-1}s} = \lim_{n\to\infty} \mathbf{E}\left[s^{Z^*(n)} \,|\, Z^*(n) > 0\right]. \end{aligned}$$

Thus,

$$\mathbf{P}(Z^* = k) = (1-1/A)/(1/A)^{k-1}, k = 1, 2, \ldots, \text{ and } \mathbf{E}[Z^*] = 1/(1-A^{-1}).$$

Thus, both types of models, sub- or super-critical, yield the same type of distributions of the size of now extinct species.