BRANCHING PROCESSES AND THEIR APPLICATIONS:
LECTURE 3: Survival probability for subcritical processes

V.A.Vatutin
Department of Discrete Mathematics
Steklov Mathematical Institute
Gubkin street, 8
119991 Moscow
RUSSIA
e-mail: vatutin@mi.ras.ru

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1 Asymptotic behavior of the survival probability for subcritical processes

Theorem 1 *If* $A<1$ *then*

$$\mathbf{P}\left(Z(n)>0\right)=Q(n)\sim KA^{n}(1+o(1)),\ \ K>0,$$

if and only if

$$\begin{aligned}\mathbf{E}\xi\log^{+}\xi &= \mathbf{E}Z(1)\log^{+}Z(1)\\ &= \sum_{k=1}^{\infty}p_{k}k\log k<\infty.\end{aligned}$$

Note that this theorem implies

$$\frac{A^{n}}{Q(n)}=\frac{\mathbf{E}Z(n)}{\mathbf{P}\left(Z(n)>0\right)}=\mathbf{E}\left[Z(n)|Z(n)>0\right]\approx K^{-1},\,n\rightarrow\infty.$$

Lemma 2 *Let*

$$H(s)=\sum_{k=0}^{\infty}h_{k}s^{k}$$

be a probability generating function and let $\delta \in (0,1)$*. The series*

$$\sum_{n=0}^{\infty} [1 - H(1-\delta^n)] < \infty$$

if and only if

$$\sum_{k=1}^{\infty} h_k \log k < \infty.$$

Proof. Since

$$1 - H(1-\delta^{n+1}) \leq 1 - H(1-\delta^x) \leq 1 - H(1-\delta^n), x \in [n, n+1]$$

it follows that

$$1 - H(1-\delta^{n+1}) \leq \int_n^{n+1} (1 - H(1-\delta^x))\, dx \leq 1 - H(1-\delta^n)$$

and after summation over n from 1 to infinity we have

$$0 \leq \sum_{n=1}^{\infty} [1 - H(1-\delta^n)] - \int_1^{\infty} [1 - H(1-\delta^x)]\, dx \leq 1 - H(1-\delta).$$

Thus, we need to check with $\delta = e^{-\alpha}$ and the change of variables $y = 1 - \delta^x$ when

$$\int_1^{\infty} \left[1 - H(1 - e^{-\alpha x})\right] dx = \frac{1}{\alpha} \int_{1-\delta}^{1} \frac{1 - H(y)}{1-y} dy < \infty.$$

Clearly,

$$\int_{1-\delta}^{1} \frac{1 - H(y)}{1-y} dy = \int_0^1 \frac{1 - H(y)}{1-y} dy - \int_0^{1-\delta} \frac{1 - H(y)}{1-y} dy$$

and

$$\int_0^{1-\delta} \frac{1 - H(y)}{1-y} dy \leq \int_0^{1-\delta} \frac{1}{1-y} dy \leq \frac{1}{\delta} \int_0^{1-\delta} dy = \frac{1-\delta}{\delta} < \infty.$$

Thus, we need to establish when

$$\int_0^1 \frac{1 - H(y)}{1-y} dy < \infty.$$

Clearly,

$$\frac{1 - H(y)}{1-y} = \sum_{k=1}^{\infty} h_k \frac{1-y^k}{1-y} = \sum_{k=1}^{\infty} h_k \sum_{j=0}^{k-1} y^j$$

Hence (and equivalent with respect to the convergence of the integrals)

$$\int_0^1 \frac{1 - H(y)}{1-y} dy = \sum_{k=1}^{\infty} h_k \sum_{j=0}^{k-1} \frac{1}{j+1} \approx \sum_{k=1}^{\infty} h_k (\log k + \gamma)$$

where the last follows from

$$\frac{1}{j+2} \leq \frac{1}{x} \leq \frac{1}{j+1}, x \in [j+1, j+2]$$

and

$$\sum_{j=0}^{k-1} \frac{1}{j+2} \leq \int_1^k \frac{dx}{x} = \ln k \leq \sum_{j=0}^{k-1} \frac{1}{j+1}$$

proving the lemma.

In particular convergence is either for all $\delta \in (0,1)$ or for none of them.

Proof of the theorem. Set

$$\begin{aligned} H(s) &= \frac{1-f(s)}{A(1-s)} = \frac{1}{A} \sum_{j=1}^{\infty} p_j \frac{1-s^j}{1-s} = \frac{1}{A} \sum_{j=1}^{\infty} p_j \sum_{j=0}^{j-1} s^k \\ &= \frac{1}{A} \sum_{k=0}^{\infty} s^k \sum_{j=k+1}^{\infty} p_j = \sum_{k=0}^{\infty} h_k s^k \leq 1. \end{aligned}$$

Clearly,

$$1 - f_{n+1}(s) = A(1-f_n(s)) \frac{1-f_{n+1}(s)}{A(1-f_n(s))} = A(1-f_n(s)) H(f_n(s)).$$

Consequently,

$$Q(n+1) = 1 - f(f_n(0)) = AQ(n)H(f_n(0))$$

implying for $K(n) = Q(n)A^{-n}$ that

$$K(n+1) = K(n)H(f_n(0)).$$

Thus,

$$K(n) = \prod_{t=0}^{n-1} H(f_t(0))) \downarrow K$$

and $K = \lim_{n\to\infty} K(n)$ is positive if and only if

$$\sum_t (1 - H(f_t(0))) < \infty.$$

Clearly,

$$1 - f_t(0) = \mathbf{P}(Z(t) > 0) \leq \mathbf{E}Z(t) = A^t = \delta_1^t$$

and

$$\begin{aligned} \mathbf{P}(Z(t) &> 0) = 1 - f_t(0) \geq P\left(Z(1) = 1, Z(2) = 1, ..., Z(t) = 1\right) \\ &= (1 - f(0))^t = \delta_2^t. \end{aligned}$$

Thus, by monotonicity of $H(s)$

$$\sum_t \left(1 - H(1 - \delta_2^t)\right) \le \sum_t (1 - H(f_t(0))) \le \sum_t \left(1 - H(1 - \delta_1^t)\right).$$

Now acccording to the previous lemma

$$\sum_t (1 - H(f_t(0))) < \infty$$

if and only if

$$\sum_{k=1}^{\infty} \sum_{j=k+1}^{\infty} p_j \log k = \sum_{j=1}^{\infty} p_j \sum_{k=1}^{j-1} \log k < \infty.$$

The last is valid if and only if

$$\sum_{j=1}^{\infty} p_j j \log j < \infty$$

since

$$\begin{aligned} (j-1)\ln(j-1) - j + 2 &= \int_1^{j-1} (\ln x)\, dx \le \sum_{k=1}^{j-1} \log k \\ &\le \int_1^{j} (\ln x)\, dx = j \ln j - j + 1. \end{aligned}$$

This proves the theorem.

NOTE THAT

$$K \le \frac{\mathbf{P}(Z(n) > 0)}{A^n}$$

for ALL n.

2 Practical estimates for the survival probability

Lemma 3 *If $\xi \ge 0$ with probability 1 and is not identical to zero then*

$$\mathbf{P}(\xi > 0) \ge \frac{(\mathbf{E}\xi)^2}{\mathbf{E}\xi^2}.$$

Proof. By Hölder inequality

$$\begin{aligned} \mathbf{E}\xi &= \mathbf{E}\left[\xi I\{\xi > 0\}\right] \le \sqrt{\mathbf{E}\xi^2 \mathbf{E} I^2 \{\xi > 0\}} \\ &= \sqrt{\mathbf{E}\xi^2 \mathbf{P}(\xi > 0)} \end{aligned}$$

as desired.

Hence we have

$$\begin{aligned} A^n &= \mathbf{E}Z(n) \geq \mathbf{P}\left(Z(n) > 0\right) \geq \frac{\left(\mathbf{E}Z(n)\right)^2}{\mathbf{E}Z^2(n)} \\ &= \frac{A^{2n}}{\sigma^2 \frac{A^{n-1}(A^n-1)}{A-1} + A^{2n}} = \frac{A^{n+1}(1-A)}{\sigma^2(1-A^n) + A^{n+1}\left(1-A\right)}. \end{aligned} \tag{1}$$

In the preceding section we claimed that the extinction probability P is one, if $A \leq 1$. In the subcritical case this is easily proved, and we can even get a sharp bound. Set

$$\mathbf{P}_N(Z(n) > 0) = \mathbf{P}(Z(n) > 0 | Z(0) = N), \quad \mathbf{E}_N[Z(n)] = \mathbf{E}[Z(n)|Z(0) = N].$$

By Chebyshev's inequality

$$\mathbf{P}_N(Z(n) > 0) = \mathbf{P}(Z(n) \geq 1 | Z(0) = N) \leq \mathbf{E}_N[Z(n)] = NA^n, \tag{2}$$

where N is the number of founders of the population and, clearly, $\lim_{n\to\infty} NA^n = 0$.

Theorem 4 *Consider a subcritical Galton-Watson process, starting from $Z(0) = N$ individuals. Then*

$$N\mathbf{P}_1(Z(n) > 0)\left(1 - \mathbf{P}_1(Z(n) > 0)\right)^{N-1} \leq \mathbf{P}_N(Z(n) > 0) \leq N\mathbf{P}_1(Z(n) > 0).$$

If the reproduction variance $\sigma^2 < \infty$, then

$$N(1-A)A^{n+1}/\sigma^2 \approx \frac{NA^{n+1}\left(1-A^n\right)^{N-1}(1-A)}{\sigma^2(1-A^n) + A^{n+1}(1-A)} \tag{3}$$

$$\leq \mathbf{P}_N(Z(n) > 0\,) \leq NA^n. \tag{4}$$

For $N = 1$ the theorem is known. To treat the case $Z(0) = N > 1$ denote for breivity by $R = \mathbf{P}_1(Z(n) = 0)$ the probability of extinction in the first n generations for a process with one single ancestor. Then

$$\mathbf{P}_N(Z(n) > 0) = 1 - R^N \leq N(1-R) = N\mathbf{P}_1(Z(n) > 0) \leq NA^n.$$

On the other hand

$$1 - x^N \geq N(1-x)x^{N-1}$$

and this gives

$$1 - R^N \geq N(1-R)R^{N-1} \geq N\mathbf{P}_1(Z(n) > 0)(1-A^n)^{N-1}.$$

Example

We consider an example borrowed from

Caswell,H., Fujiwara, M., and Brault S., *Declining survivial probability threatens the North Atlanitic right whale*, Proc. Nath. Acad. USA, 96 (1999), 3308-3313.

They study the threats posed by a decline of survival probabilities for the North Atlantic right whale, within the framework of the following model.

A female right whale may produce 0, 1, or 2 females the following year. It is assumed that the death of a parent results in the death of a calf in the first year. Thus, a female at time n produces no offspring if she dies before $n+1$, one offspring (herself) if she survives without reproducing female offspring and two offspring (herself and her calf) if she survives and gives birth to a female calf. Generation length is then one year. Let p be the survival probability and μ be the probability of begetting a female calf. The reproduction generating function of the process becomes

$$f(s) = 1 - p + p(1-\mu)s + p\mu s^2$$

with mean $A = p(1-\mu) + 2p\mu = p(1+\mu)$. Caswell *et al.* (1999), using different sources, give the following estimates for p, μ, and, as a result, for A:

	$\mu = 0.051$	$\mu = 0.038$
$p = 0.94$	$A = 0.988$	$A = 0.976$

Applying formulas (3) and (4) to the data, we obtain the following estimates from below of the number n of generations (years) which the population of whales (now having around 150 female members) can survive with probability higher than 0.99 and from above for the number of generations within which the population will die out with a probability greater than 0.99:

	A	0.988	0.976
survival with probability ≥ 0.99 for at least n years	$n \geq$	357	177
extinction with probability ≥ 0.99 within at most n years	$n \leq$	796	395.

In particular this shows that provided reproduction conditions remain the same in thefuture, then under the worst scenario the whale population will die out within 400 years with a probability of more than 99 percent.