BRANCHING PROCESSES AND THEIR APPLICATIONS
LECTURE 2: Elementary properties of generating functions; branching processes and simple random walk

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1 Elementary properties of generating functions.

Let

$$f(s) = \mathbf{E}s^{\xi} = \sum_{k=0}^{\infty} P(\xi = k)\, s^k = \sum_{k=0}^{\infty} p_k s^k$$

be the offspring probability generating function. Assume $p_0 + p_1 < 1$. Then

1) $f(s)$ is strictly convex and increasing in $[0,1]$; $f'(s) > 0, f"(s) > 0$;
2) $f(0) = p_0 = P(Z(1) = 0|Z(0) = 1)$;
2) if $A \leq 1$ then $f(s) > s$, $s \in [0,1)$, since $f'(s) - 1 < 0$, $s \in [0,1)$;
3) if $A > 1$ then $f(s) = s$ has a unique root r in $[0,1)$ and $f(s) > s$ if $s < r$ and $f(s) < s$ if $s > r$ since $f'(0) - 1 < 0$, $f'(1-) - 1 = A - 1 > 0$ and $f"(s) > 0$.

Extinction probability

$$f_n(s) = \mathbf{E}s^{Z(n)} = \sum_{k=0}^{\infty} P(Z(n) = k)\, s^k,$$

$$f_n(0) = \mathbf{P}(Z(n) = 0) \leq \mathbf{P}(Z(n+1) = 0) = f_{n+1}(0).$$

It follows that the sequence

$$P(n) = \mathbf{P}(\text{ extinction by generation } n) = \mathbf{P}(Z(n) = 0) = f_n(0), n = 1, 2 \dots$$

must increase to the extinction probability, which we denote by P,

$$\lim_{n\to\infty} P(n) = P.$$

Since $f(0) < r = f(r)$

$$P(n) = f_n(0) = f(f_{n-1}(0)) = f(P(n-1)) < f(r) = r$$

and the function f is continuous, it follows that $P = f(P)$. Hence $P = r$.

Thus, the subcritical and critical processes die with probability 1 while supercritical with probability $P < 1$ being the smallest root of $f(s) = s$, $s \in [0, 1)$.

Example. For binary splitting, we must solve $q + px^2 = x$ with the result that $P = q/p$ provided $p > 1/2$ and one otherwise. For geometrically distributed offspring, the equation

$$q/(1 - xp) = x$$

yields the same result, $P = q/p$ for $p > 1/2$. In this case $A = p/q$ so that also $P = 1/A$. A process with a mean reproduction of, say 1.2, therefore has a population doubling time of four generations (if $Z(0) = N$, $\mathbb{E}[Z(4)] = N(1.2)^4 = N \times 2.07$), and an extinction probability for each particular family higher than 80 % ($P = 1/A = 1/1.2 = 0.83$ if $Z(0) = 1$).

2 Branching processes and simple random walk

Branching process: Consider a branching process with geometric probability generating function for the offspring number:

$$f(s) = \frac{q}{1 - ps} = Es^{\xi},\ p + q = 1,\ pq > 0. \tag{1}$$

It follows from the consideration above that the probability of extinction of this process, being a solution of $f(P) = P$, is

$$P = \min\left\{\frac{q}{p}, 1\right\}$$

and, besides the standard recurrence relation

$$Z(n+1) = \xi_1^{(n)} + ... + \xi_{Z(n)}^{(n)} \tag{2}$$

is valid, where $\xi_i^{(n)}$ are iid, $\xi_i^{(n)} \overset{d}{=} \xi$ with $\mathbf{P}(\xi = j) = qp^j,\ j = 0, 1, ...$

Random walk: Consider a random walk

$$S_0 = 1, S_k = X_1 + ... + X_k$$

with

$$\mathbf{P}(X_i = 1) = p, \qquad \mathbf{P}(X_i = -1) = 1 - p = q.$$

Let S_k^* be the random walk stopped at zero at moment $\tau = \min\{k : S_k = 0\}$. It is known that

$$\mathbf{P}(\tau < \infty) = \min\left\{\frac{q}{p}, 1\right\}.$$

Set

$$Y(n) = \text{the number of } k \text{ such that } S_k^* = n+1, S_{k+1}^* = n.$$

Then the random variable

$$Y(1) = \text{the number of } k \text{ such that } S_k^* = 2, S_{k+1}^* = 1$$

has the following probability law:

$$\mathbf{P}(Y(1) = 0) = q, \ \mathbf{P}(Y(1) = 1) = pq$$

and, in general, the Geometric distribution with

$$\mathbf{P}(Y(1) = j) = \mathbf{P}(\eta = j) = qp^j.$$

Besides,

$$Y(n+1) = \eta_1^{(n)} + ... + \eta_{Y(n)}^{(n)}$$

where $\eta_i^{(n)} \overset{d}{=} \eta$.

Thus, we get the same stochastic process as in (2).

If $p \leq 1/2$ then the branching process dies out and if T is the moment of extinction then

$$\sigma = Z(0) + Z(1) + ... + Z(T-1)$$

is the total number of particles in the process and

$$\sigma = 2\tau - 1.$$

We know that for the probability generating function $f(s)$ of the form (1) with $p = 1/2$

$$1 - f_n(0) = \frac{1}{n+1}$$

while for $p < 1/2$

$$1 - f_n(0) = \frac{\left(\frac{p}{q}\right)^n \left(1 - \frac{p}{q}\right)}{1 - \left(\frac{p}{q}\right)^{n+1}}$$

In terms of our random walk interpretation this fact is easy to explain.

Indeed, assume that our random walk S_n starts from a point $S_0 = m \in [0, n+1]$ and stops when it hits for the first time either 0 or $n+1$. What is the probability to hit $n+1$ earlier than 0 (this corresponds to $P(Z(n) > 0)$)?

Let

$$\mathbf{P}_m(N) = \mathbf{P}(\text{ random walk starts at } m \text{ and hits } n+1 \text{ within the time interval } [0,N] \text{ and earlier than } 0).$$

Clearly, for $1 \le m \le n$ we have

$$\mathbf{P}_m(N) = p\mathbf{P}_{m+1}(N-1) + q\mathbf{P}_{m-1}(N-1)$$

while

$$\mathbf{P}_0(N) = 0, \mathbf{P}_{n+1}(N) = 1.$$

Letting $N \to \infty$ we get

$$\mathbf{P}_m = p\mathbf{P}_{m+1} + q\mathbf{P}_{m-1}, 1 \le m \le n,$$

with

$$\mathbf{P}_0 = 0,\ \mathbf{P}_{n+1} = 1.$$

It is necessary to search for a solution of this difference system of equations by solving the characteristic equation

$$\lambda^m = p\lambda^{m+1} + q\lambda^{m-1}$$

or

$$\lambda = p\lambda^2 + q$$

giving

$$\lambda_1 = 1, \lambda_2 = \frac{q}{p}.$$

Thus, for $p \neq q$ the general solution is

$$\mathbf{P}_m = a\lambda_1^m + b\lambda_2^m = a + b\left(\frac{q}{p}\right)^m.$$

Hence, on account of our boundary conditions

$$a + b = 0 \text{ and } a + b\left(\frac{q}{p}\right)^{n+1} = 1$$

and, therefore,

$$a = \frac{1}{1 - \left(\frac{q}{p}\right)^{n+1}} = -b.$$

Consequently,

$$\mathbf{P}_m = \frac{1 - \left(\frac{q}{p}\right)^m}{1 - \left(\frac{q}{p}\right)^{n+1}} = \frac{\left(\frac{p}{q}\right)^n \left(\left(\frac{p}{q}\right)^{1-m} - \frac{p}{q}\right)}{1 - \left(\frac{p}{q}\right)^{n+1}}$$

with

$$\mathbf{P}_1 = \frac{\left(\frac{p}{q}\right)^n \left(1 - \frac{p}{q}\right)}{1 - \left(\frac{p}{q}\right)^{n+1}} = \mathbf{P}\left(Z(n) > 0\right).$$

If $p = q$ then the general solution is

$$\mathbf{P}_m = a + bm$$

and, on account of our boundary conditions,

$$a = 0,\ b = \frac{1}{n+1}$$

leading to

$$\mathbf{P}_m = \frac{m}{n+1}.$$

Hence, in particular,

$$\mathbf{P}_1 = \frac{1}{n+1} = \mathbf{P}\left(Z(n) > 0\right).$$