BRANCHING PROCESSES AND THEIR APPLICATIONS:
Lecture 15: Crump-Mode-Jagers processes and queueing systems with processor sharing

June 17, 2005

1 Crump-Mode-Jagers process counted by random characteristics

We give here only an informal description of the Crump-Mode-Jagers process counted by random characteristics or, what is the same, of the general branching process counted by random characteristics. A particle, say, x, of this process is characterised by three random processes

$$(\lambda_x, \xi_x(\cdot), \chi_x(\cdot))$$

which are iid copies of a triple $(\lambda, \xi(\cdot), \chi(\cdot))$ and whose components have the following sense:

if a particle was born at moment σ_x then

$\lambda_x -$ is the life-length of the particle;

$\xi_x(t-\sigma_x)$ - is the number of children produced by the particle within the time-interval $[\sigma_x, t)$; $\xi_x(t-\sigma_x) = 0$ if $t - \sigma_x < 0$;

$\chi_x(t-\sigma_x) \geq 0-$ is a stochastic process subject to changes ONLY within the time-interval $[\sigma_x, \sigma_x + \lambda_x)$ while outside the interval it has the form

$$\chi_x(t-\sigma_x) = \begin{cases} 0 & \text{if} \quad t-\sigma_x < 0 \\ \chi_x(\lambda_x) & \text{if} \quad t-\sigma_x \geq \lambda_x \end{cases}$$

(it is NOT assumed that $\chi_x(t)$ is a nondecreasing function in $t \geq 0$).

The stochastic process

$$Z^{\chi}(t) = \sum_x \chi_x(t-\sigma_x)$$

where summation is taken over all particles x born in the process up to moment t is called the general branching process counted by random characteristics.

Examples:

1) $\chi(t) = I\{t \in [0, \lambda)\} -$ in this case $Z^{\chi}(t) = Z(t)$ is the number of particles existing in the process up to moment t;

2)

$$\chi(t) = tI\{t \in [0, \lambda)\} + \lambda I\{\lambda < t\}$$

then

$$Z^{\chi}(t) = \int_0^t Z(u)du;$$

3) $\chi(t) = I\{t \geq 0\}$ then $Z^{\chi}(t)$ is the total number of particles born up to moment t.

Classification. $E\xi(\infty) <, =, > 1$ - subcritical, critical and supercritical, respectively.

Let

$$0 \leq v(1) \leq v(2) \leq ... \leq v(n) \leq ...$$

be the birth moments of the children of the initial particle. Then

$$\xi_0(t) = \#\{n : v(n) \leq t\}$$

is the number of children born by the initial particle up to moment t. We have

$$Z^{\chi}(t) = \chi_0(t) + \sum_{x \neq 0} \chi_x(t - \sigma_x) = \chi_0(t) + \sum_{v(n) \leq t} Z_n^{\chi}(t - v(n))$$

where $Z_n^{\chi}(\cdot)$, $n = 1, 2, ...$ are iid copies of $Z^{\chi}(\cdot)$. Hence it follows that

$$\begin{aligned}
\mathbf{E}Z^{\chi}(t) &= \mathbf{E}\chi(t) + \mathbf{E}\left[\sum_{v(n) \leq t} Z_n^{\chi}(t - v(n))\right] \\
&= \mathbf{E}\chi(t) + \mathbf{E}\left[\sum_{v(n) \leq t} \mathbf{E}\left[Z_n^{\chi}(t - v(n)) | v(1), v(2), ..., v(n), ...\right]\right] \\
&= \mathbf{E}\chi(t) + \mathbf{E}\left[\sum_{v(n) \leq t} \mathbf{E}\left[Z_n^{\chi}(t - v(n)) | v(n)\right]\right] \\
&= \mathbf{E}\chi(t) + \mathbf{E}\left[\sum_{u \leq t} \mathbf{E}\left[Z^{\chi}(t - u)\right](\xi_0(u) - \xi_0(u-))\right] \\
&= \mathbf{E}\chi(t) + \int_0^t \mathbf{E}Z^{\chi}(t - u)\,\mathbf{E}\xi(du).
\end{aligned}$$

Thus, we get the following renewal-type equation for $A^{\chi}(t) = \mathbf{E}Z^{\chi}(t)$ and $\mu(t) = \mathbf{E}\xi(t)$:

$$A^{\chi}(t) = \mathbf{E}\chi(t) + \int_0^t A^{\chi}(t - u)\mu(du). \qquad (1)$$

Malthusian parameter: a number α is called the Malthusian parameter of the process if

$$\int_0^\infty e^{-\alpha t}\mu(dt) = 1 \tag{2}$$

(such a solution not always exists). For the critical processes $\alpha = 0$, for the supercritical processes $\alpha > 0$, for the subcritical processes $\alpha < 0$ (if exists).

If the Malthusian parameter exists we can rewrite (1) as

$$C^\chi(t) = e^{-\alpha t}\mathbf{E}\chi(t) + \int_0^t C^\chi(t-u)d\left(\int_0^u e^{-\alpha y}\mu(dy)\right)$$

where $C^\chi(t) = e^{-\alpha t}A^\chi(t)$. In view of (2) and given that, say, $e^{-\alpha t}\mathbf{E}\chi(t)$ is directly Riemann integrable and

$$\int_0^\infty e^{-\alpha t}\mathbf{E}\chi(t)dt < \infty, \ \int_0^\infty te^{-\alpha t}\mu(dt) < \infty$$

we can apply the key renewal theorem to conclude that if the measure

$$M(t) = \int_0^t e^{-\alpha y}\mu(dy)$$

is non-lattice then

$$\lim_{t\to\infty} C^\chi(t) = \lim_{t\to\infty} e^{-\alpha t}A^\chi(t) = \int_0^\infty e^{-\alpha t}\mathbf{E}\chi(t)dt\left(\int_0^\infty te^{-\alpha t}\mu(dt)\right)^{-1}.$$

In particular, if $G(t)$ is the life-length distribution of particles and $\chi(t) = I\{t \in [0,\lambda)\}$ we get

$$\mathbf{E}\chi(t) = \mathbf{P}(\lambda > t) = 1 - G(t)$$

and

$$\lim_{t\to\infty} e^{-\alpha t}\mathbf{E}Z(t) = \frac{\int_0^\infty e^{-\alpha t}(1-G(t))\,dt}{\int_0^\infty te^{-\alpha t}\mu(dt)}$$

if the respective integrals converge.

2 M|G|1 system with processor sharing discipline

The model: a Poisson flow of customers with intensity Λ comes to a system with one server which has unit service intensity. The service time distribution of a particular customer is (if there are no other customers in the queue) $B(u)$. If there are M customers in the system at some moment T they are served simultaneously with intensity M^{-1} each.

Let

$$W_{l_1,\dots,l_{N-1}}(l_N)$$

be the waiting time for the end of service of a customer which arrived to the queue at the moment when the queue had $N-1$ customers with remaining service times $l_1, ..., l_{N-1}$.

The question is to study the properties of the random variable $W_{l_1,...,l_{N-1}}(l_N)$ when $l_N \to \infty$.

To solve this problem we construct an auxiliary general branching process.

Construction of the branching process.

Consider a general branching process in which initially at time $t=0$ there are N particles with remaining life-lengths $l_1, ..., l_{N-1}, l_N$ and which constitute the zero generation of this process. The life-length distribution of any newborn particle λ_x is $P(\lambda_x \le u) = B(u)$, the reproduction process $\xi_x(t)$ of the number of children produced by a particle up to moment t has the probability generating function

$$\mathbf{E}s^{\xi_x(t)} = \int_0^t e^{\Lambda(s-1)u} dB(u) + e^{\Lambda(s-1)t}(1-B(t))$$

that is, this is an ordinary Poisson flow with intensity Λ stopped when the particle dies:

$$\mathbf{E}s^{\xi_x(t)} = \mathbf{E}s^{Poi_\Lambda(t\wedge\lambda_x)}.$$

Let $Z(t; l_1, ..., l_{N-1}, l_N)$ denote the number of particles in the process at moment t with the mentioned initial conditions. We use a simplified notation $Z(t)$ if at moment $t=0$ there is only one particle of zero age in the process.

We will consider also the process with immigration $X(t; l_1, ..., l_{N-1}, l_N)$ which has the same initial conditions and development as $Z(t; l_1, ..., l_{N-1}, l_N)$ but, in addition, given $X(t; l_1, ..., l_{N-1}, l_N) = 0$ it starts again by *one* individual of zero age after a random time r_i having distribution $P(r_i \le u) = 1 - e^{-\Lambda u}$ (if the process dies out for the $i-$th time). $X(t)$ is used if we initially start by the process $Z(t)$.

Now let $\sigma_{x_1} \le \sigma_{x_2} \le ...$be the sequential moments of jumps of the process $X(t; l_1, ..., l_{N-1}, l_N)$. We construct by the general branching process the following queueing system with $S(T)$ being the number of customers in the queue at moment T:

1) the queue has N customers at $T=0$ with remaining service times $l_1, ..., l_{N-1}, l_N$;

2) the moment T_i of the $i-$th jump of the queue size $S(\cdot)$ is specified as

$$T_i = \int_0^{\sigma_{x_i}} X(y; l_1, ..., l_{N-1}, l_N) dy + \int_0^{\sigma_{x_i}} I\left\{X(y; l_1, ..., l_{N-1}, l_N) = 0\right\} dy.$$

3) the service discipline is such that at each moment T the number of customers in the queue and their remaining service times coincide with the number of individuals and the remaining life-lengths of individuals in the branching process at moment $t(T)$ where

$$T = \int_0^{t(T)} X(y; l_1, ..., l_N) dy + \int_0^{t(T)} I\left\{X(y; l_1, ..., l_N) = 0\right\} dy.$$

Thus, $t \leftrightarrow T$ is a random change of time.

Theorem. The described queueing system is a processor-sharing system with service time of customers $B(u)$ and a Poisson flow of customers with intensity of arrivals Λ.

Proof. Let $S(T)$ be the number of customers in the queue at time T and let $\Theta_1, \Theta_2, ...$ be the moments of changes the size of the queue. Let us show that the evolution of the constructed queue coincides with the evolution of a queueing system with processor sharing discipline. It is enough to show that this is true for $T \in [0, \Theta_1]$ and then, using the memoryless property of the Poisson flow to show in a similar way that this is true for $T \in [\Theta_1, \Theta_2]$ and so on.

To demonstrate this it is enough to check that:

1) $\Theta_1 = Nl_1 \wedge ... \wedge Nl_N \wedge d$ where $P(d \le u) = 1 - e^{-\Lambda u}$;

2) If $\Theta_1 = Nl_i$ then at this moment the i-th customer comes out of the queue; if $\Theta_1 = d$ then *one* new customer arrives;

3) at any moment $T \in [0, \Theta_1]$ the remaining service times of the initial N customers are $l_1 - N^{-1}T, ..., l_N - N^{-1}T$.

Let θ_1 be the first moment of change of $X(t; l_1, ..., l_N)$. Clearly,

$$\theta_1 = l_1 \wedge ... \wedge l_N \wedge d_1 \wedge ... \wedge d_N$$

where $P(d_i \le u) = 1 - e^{-\Lambda u}$ and where the sense of d_i is the birth of an individual by the initial particle labelled i. On the interval $u \in [0, \theta_1]$ the processing time of the queueing system T and the time t passed from the start of the evolution of the general branching process are related by $T = Nt$. Hence 3) is valid.

Further, $\Theta_1 = N(l_1 \wedge ... \wedge l_N \wedge d_1 \wedge ... \wedge d_N) = Nl_1 \wedge ... \wedge Nl_N \wedge (N(d_1 \wedge ... \wedge d_N))$ and

$$P(N(d_1 \wedge ... \wedge d_N) \ge y) = \left(e^{-y/N}\right)^N = e^{-y}.$$

This proves 1). Point 2) is evident.

Corollary 1.

$$S(T) = X(t(T); l_1, ..., l_N).$$

Corollary 2.

$$W_{l_1,...,l_{N-1}}(l_N) = \int_0^{l_N} Z(y; l_1, ..., l_N)dy.$$

More detailed construction:

Let L be the life-length of a particle and let $0 \le \delta(1) \le \delta(2) \le ...$ be the birth moments of her children. Denote

$$\xi(t, L) = \#\{n : \delta(n) \le t\}.$$

Then the process generated by this particle can be treated as a process with immigration stopped at moment L where

$$Es^{\xi(t,L)} = e^{\Lambda(s-1)\min(t,L)},$$

and, since each newborn particle generates an *ordinary* process without immigration, we see that the offspring size of new particles at moment t in the process is

$$\int_0^t Z_{\xi(u,L)}(t-u)\xi(du,L)$$

where $Z_i(y)$ are independent branching processes initiated by one individual of zero age. Thus,

$$\begin{aligned} Z(y;l_1,...,l_N) &= I\left\{l_1 \geq y\right\} + \int_0^y Z_{\xi(u,l_1)}(y-u)\xi(du,l_1) \\ &\quad +... + I\left\{l_N \geq y\right\} + \int_0^y Z_{\xi(u,l_N)}(y-u)\xi(du,l_N) \end{aligned}$$

and, in particular, we have

$$\begin{aligned} W_{l_1,...,l_{N-1}}(l_N) &= \int_0^{l_N} Z(y;l_1,...,l_N)dy \\ &= \sum_{k=1}^N \min(l_N,l_k) + \sum_{k=1}^N \int_0^{l_N} dy \int_0^y Z_{\xi(u,l_k)}(y-u)\xi(du,l_k). \end{aligned}$$

Since the birth moments of new particles constitute a Poisson flow with intensity Λ we have $\mathbf{E}\left[\xi(u,l)|l\right] = \min\left(u,l\right)$. Hence

$$\begin{aligned} & \mathbf{E}\left[\int_0^{l_k} dy \int_0^y Z_{\xi(u,l_k)}(y-u)\xi(du,l_k)\right] \\ = & \mathbf{E}\left[\int_0^{l_k} dy \mathbf{E}\left[\int_0^y Z_{\xi(u,l_k)}(y-u)\xi(du,l_k)\left|\xi(u,l_k), 0 \leq u \leq l_k\right.\right]\right] \\ = & \mathbf{E}\left[\int_0^{l_k} dy \int_0^y \mathbf{E}\left[Z_{\xi(u,l_k)}(y-u)\left|\xi(u,l_k), 0 \leq u \leq l\right.\right]\xi(du,l_k)\right] \\ = & \mathbf{E}\left[\int_0^{l_k} dy \int_0^y \mathbf{E}\left[Z(y-u)\right]\xi(du,l_k)\right] \\ = & \mathbf{E}\left[\int_0^{l_k} dy \int_0^y \mathbf{E}\left[Z(y-u)\right]\mathbf{E}\left[\xi(du,l_k)|l_k\right]\right] \\ = & \mathbf{E}\left[\int_0^{l_k} dy \int_0^y \mathbf{E}\left[Z(y-u)\right]\Lambda du\right] = \Lambda \mathbf{E}\left[\int_0^{l_k} dy \int_0^y \mathbf{E}\left[Z(u)\right]du\right]. \end{aligned}$$

Hence

$$\mathbf{E}W_{l_1,...,l_{N-1}}(l_N) = \mathbf{E}\left[\sum_{k=1}^N \min(l_N,l_k) + \Lambda\sum_{k=1}^N \int_0^{l_k} dy \int_0^y \mathbf{E}\left[Z(u)\right]du\right].$$

One can prove also that if

$$\beta_1 = \mathbf{E} l_N = \int_0^\infty u dB(u) < \infty$$

and $\Lambda\beta_1 < 1$ then for *fixed* $l_1, ..., l_{N-1}$

$$\lim_{l_N \to \infty} W_{l_1,...,l_{N-1}}(l_N) = \frac{1}{1 - \Lambda\beta_1}$$

almost surely (in particular, if it comes to an empty system).