BRANCHING PROCESSES AND THEIR APPLICATIONS: LECTURE 14: Markov processes with immigration counted by random characteristics

June 9, 2005

1 Processes with immigration counted by random characteristics

Continuous time branching process with immigration and final product: Consider a BPI in which immigration occurs with rate ρ_0 and the reproduction function of the number of immigrants $g(s) = Es^{\eta}$. That is, at an immigration moment a random number η of children is produced and a final product $\chi^* \geq 0$ which is not changed later on (that is, a random variable which is INDEPENDENT on the moment of immigration),

$$\psi^{\chi}(s,\lambda) = \mathbf{E}s^{\eta}e^{-\lambda\chi^*},\ \psi^{\chi}(s,0) = g(s) = \mathbf{E}s^{\eta}.$$

The aboriginal individuals have the exponential life-time distribution with parameter ρ_1 and the reproduction function $f(s) = Es^{\xi}$. An aboriginal individual produces at the end of the life a random number ξ of children and a final product $\chi \geq 0$ with probability generating function

$$\varphi^{\chi}(s,\lambda) = \mathbf{E}s^{\xi}e^{-\lambda\chi}, \varphi^{\chi}(s,0) = f(s)$$

which is not changed later on. The random variables χ and χ^* are called random characteristics.

Let

$$Z_*^{\chi}(t) = \sum_D \chi_D + \sum_I \chi_I^*$$

where summation is for all particles D which died up to the moment t and all immigrants immigrated up to moment t.

The joint distribution of the components of the vector $(Z_*(t), Z_*^{\chi}(t))$, where $Z_*(t)$ is the number of particles in the process with immigration, is described by the following integral and differential equations. Setting

$$\Phi_*\left(t,s,\lambda\right) = \mathbf{E}\left[s^{Z(t)}e^{-\lambda Z_*^{\chi}(t)}|(Z(t), Z_*^{\chi}(t)) = (0,0)\right]$$

and $G_0(t) = 1 - e^{-\rho_0 t}$ we get

$$\begin{aligned}\Phi_*(t,s,\lambda) &= 1 - G_0(t) + \int_0^t \psi^\chi(\Phi(t-u,s,\lambda),\lambda)\Phi_*(t-u,s,\lambda)\,dG_0(u) \\ &= e^{-\rho_0 t} + \rho_0 \int_0^t \psi^\chi(\Phi(u,s,\lambda),\lambda)\Phi_*(u,s,\lambda)\,e^{-\rho_0(t-u)}du\end{aligned}$$

with

$$\Phi_*(t,s,\lambda) = 1.$$

This leads to

$$\begin{aligned}\frac{\partial \Phi_*(t,s,\lambda)}{\partial t} &= -\rho_0 e^{-\rho_0 t} - \rho_0^2 \int_0^t \psi^\chi(\Phi(u,s,\lambda),\lambda)\Phi_*(u,s,\lambda)\,e^{-\rho_0(t-u)}du \\ &\quad +\rho_0\psi^\chi(\Phi(u,s,\lambda),\lambda)\Phi_*(u,s,\lambda) \\ &= -\rho_0\Phi_*(t,s,\lambda) + \rho_0\psi^\chi(\Phi(t,s,\lambda),\lambda)\Phi_*(t,s,\lambda) \\ &= \rho_0(\psi^\chi(\Phi(t,s,\lambda),\lambda) - 1)\Phi_*(t,s,\lambda)\end{aligned}$$

implying

$$\Phi_*(t,s,\lambda) = \exp\left\{\int_0^t \rho_0(\psi^\chi(\Phi(u,s,\lambda),\lambda) - 1)du\right\}.$$

Here (RECALL) for

$$\Phi(t,s,\lambda) = \mathbf{E}\left[s^{Z(t)}e^{-\lambda Z^\chi(t)} | (Z(0), Z^\chi(0)) = (1,0)\right]$$

and $G(t) = 1 - e^{-\rho_1 t}$ we have

$$\Phi(t,s,\lambda) = s(1 - G(t)) + \int_0^t \varphi^\chi(\Phi(t-u,s,\lambda),\lambda)dG(u).$$

In particular, for

$$F^0(t,s) = \mathbf{E}\left[s^{Z_*(t)} | \text{immigration, } Z_*(0) = 0\right] = \Phi_*(t,s,0), F^0(0,s) = 1$$

and

$$F^1(t,s) = \mathbf{E}\left[s^{Z(t)} | \text{no immigration, } Z(0) = 1\right] = \Phi(t,s,0),\ F^1(0,s) = s,$$

we get

$$\frac{\partial F^0(t,s)}{\partial t} = \rho_0(g(F^1(t,s)) - 1)F^0(t,s),\ g(s) = \mathbf{E}s^\eta, F^0(t,s) = 1,$$

and

$$F^0(t,s) = \exp\left\{\int_0^t \rho_0(g(F^1(u,s)) - 1)du\right\}.$$

and, recall,

$$\frac{\partial F^1(t,s)}{\partial t} = \rho_1\left(f(F^1(t,s)) - F^1(t,s)\right) = f^{(\rho_1)}\left(F^1(t,s)\right), \quad F^1(0,s) = s,$$

and

$$\frac{\partial F^1(t;s)}{\partial t} = f^{(\rho_1)}(s)\,\frac{\partial F^1(t;s)}{\partial s},\ F^1(0,s) = s.$$

Theorem 1 *If $g'(1) < \infty$ and $f^{'}(1) < 1$ then*

$$\begin{aligned}\lim_{t\to\infty} F^0(t,s) &= \exp\left\{\int_0^\infty \rho_0(g(F^1(u,s)) - 1)du\right\} \\ &= \exp\left\{\int_s^1 \frac{\rho_0(g(y)-1)}{\rho_1\left(f(y)-y\right)}dy\right\}.\end{aligned}$$

Proof. Since

$$\begin{aligned}0 &\leq 1 - g(F^1(u,s)) \leq g'(1)\left(1 - F^1(u,s)\right) \\ &\leq g'(1)e^{\rho_1\left(f^{'}(1)-1\right)u}\,(1-s)\end{aligned}$$

the integral converges uniformly in $s \in [0,1]$. Hence

$$\lim_{t\to\infty} F^0(t,s) = \exp\left\{\int_0^\infty \rho_0(g(F^1(u,s)) - 1)du\right\}.$$

Now

$$\begin{aligned}&\frac{d}{ds}\int_0^\infty \rho_0(g(F^1(u,s)) - 1)du \\ =\ & \rho_0\int_0^\infty \frac{dg(F^1(u,s))}{dF^1}\frac{\partial F^1(u,s)}{\partial s}du \\ =\ & \rho_0\int_0^\infty \frac{dg(F^1(u,s))}{dF^1}\frac{\partial F^1(u,s)}{\partial s}du \\ =\ & \rho_0\int_0^\infty \frac{dg(F^1(u,s))}{dF^1}\frac{\partial F^1(u,s)}{\partial u}\frac{du}{f^{(\rho_1)}(s)} \\ =\ & \frac{\rho_0}{f^{(\rho_1)}(s)}\int_0^\infty \frac{\partial g(F^1(u,s))}{\partial u}du = \frac{\rho_0}{f^{(\rho_1)}(s)}g(F^1(u,s))|_0^\infty \\ =\ & \frac{\rho_0\left(1 - g(s)\right)}{f^{(\rho_1)}(s)}.\end{aligned}$$

Differentiation is also justified since convergence

$$\lim_{t\to\infty}\frac{\rho_0}{f^{(\rho_1)}(s)}g(F^1(t,s)) = \frac{\rho_0}{f^{(\rho_1)}(s)}$$

is uniform in $s \in [0,1)$.

Similarly one can prove the following statement for the continuous-time process:

Theorem 2 *If $g^{'}(1) = b < \infty$ and $f'(1) = 1$, $f"(1) = \sigma^2 \in (0, \infty)$ then for $\theta = 2b\rho_0/\rho_1\sigma^2$*

$$\lim_{t\to\infty} P\left(\frac{2Z(t)}{\sigma^2 t} \le x\right) = \frac{1}{\Gamma(\theta)}\int_0^x y^{\theta-1}e^{-y}dy. \qquad (1)$$

Proof. We have

$$F^0(t,s) = \exp\left\{\int_0^t \rho_0(g(F^1(u,s)) - 1)du\right\}.$$

Let

$$s = \exp\left\{-\frac{2\lambda}{\rho_1\sigma^2 t}\right\}$$

and let $T = T(t,\lambda)$ be defined through

$$F^1(T,0) = \exp\left\{-\frac{2\lambda}{\rho_1\sigma^2 t}\right\}.$$

Clearly,

$$\frac{2\lambda}{\rho_1\sigma^2 t} \sim \frac{2}{\rho_1\sigma^2 T}$$

implying $T = T(t,\lambda) \sim t/\lambda$.Then

$$\begin{aligned}
\mathbf{E}e^{-\lambda\frac{2Z(t)}{\rho_1\sigma^2 t}} &= \exp\left\{\int_0^t \rho_0(g(F^1(u,s)) - 1)du\right\} \\
&= \exp\left\{\int_0^t \rho_0(g(F^1(u+T,s)) - 1)du\right\} \\
&= \exp\left\{-(1+\varepsilon(t,s))\int_0^t \rho_0 g^{'}(1)\left(1 - F^1(u+T,s)\right))du\right\} \\
&= \exp\left\{-(1+\varepsilon_1(t,s))\frac{2\rho_0 b}{\rho_1\sigma^2}\int_0^t \frac{1}{u+t/\lambda}du\right\} \\
&\sim \exp\left\{-\theta\int_0^t \frac{1}{u+t/\lambda}du\right\} = \exp\left\{-\theta\ln\left(\frac{t(1+1/\lambda)}{t/\lambda}\right)\right\} \\
&= \frac{1}{(1+\lambda)^\theta}
\end{aligned}$$

proving the theorem.

2 System $M|G|1$ with retrials (repeated calls)

Consider an $M|G|1$ system with Poisson flow of customers having intensity ρ_0 and the following service discipline: a just arriving customer is immediately served if the server is idle else the customer joins the queue and repeats its attempts with exponentially distributed time-intervals with parameter ρ_1 until success.

Assume that the vectors $(\xi_i, \pi_i), i = 1, 2, ...$ are iid and have components which are equal, respectively, to the number of new customers arriving during the service time of the respective customer and its service.

Suppose that initially there were $n+1$ customers, one is marked and the server is idle. The problem is to evaluate the waiting time of the marked customer.

Under our assumptions for the time-interval σ_1,

$$\mathbf{P}\left(\sigma_1 > t\right) = e^{-t(\rho_1(n+1)+\rho_0)}$$

the server is idle then some of the customers comes to the service and is served within the time-interval π_1 and the server remains idle for the time-interval σ_2 whose distribution depends on the number of customers staying in the queue just after the moment $\sigma_1 + \pi_1$ so on. Let N be the number of customers served BEFORE the marked one was taken to the service. In this case the waiting time of the marked customer is

$$V_n = \sigma_1 + \pi_1 + \sigma_2 + \pi_2 + ... + \sigma_N + \pi_N + \sigma_{N+1}$$

while the total number R_n of unsuccessful calls of the marked customer until success equals

$$R_n = r_1 + r_2 + ... + r_N + 1$$

where r_i is the number of attempts to call by the marked customer between the end of the services of the $(i-1)-$ th and i-th customers. Hence, as before, we may assume that at the each, say, $i-$th customer, produces at the end of the service a final product χ_i and the number of new customers arriving to the system during the service time of the i-th customer is just ξ_i. Suppose that $(\xi_i, \chi_i), i = 1, 2, ...$ are iid and set

$$T_n^{*\chi} = \chi_1 + \chi_2 + ... + \chi_N.$$

Clearly,

$$R_n = T_n^{*r} + 1, V_n = T_n^{*\pi} + \sigma_1 + \sigma_2 + ... + \sigma_N + \sigma_{N+1}.$$

2.0.1 The associated BP with immigration.

Now we construct an associated branching process with immigration. We have two types of particles 0 and 1. The life-lengths of the particles of the respective types are exponential with parameters

$$\rho_0, \rho_1.$$

Each particle, say D, produces at the end of her life (ξ_D, χ_D) and, additionally, a particle of the type 0 produces exactly *one* particle of type 0.

Thus, in our previous setting this is a Markov process with immigration rate ρ_0 and the reproduction function

$$\psi^{\chi}(s,\lambda) = \varphi^{\chi}(s,\lambda),\ \psi^{\chi}(s,0) = f(s)$$

(since in the case under consideration there is no difference in the service times of immigrants and aboriginal individuals).

Denote by $\sigma_1^{'}, \sigma_2^{'}, ...$ the splitting moments of the BPI and let the process start by $n+1$ particles of type 1 (with one of them marked) and 1 particle of type 0.

Thus, we have interpretation - a particle of type zero - a customer from outside, a particle of type 1 - a customer from the queue.

We follow the evolution of the queue at the moments $\sigma_1, \sigma_1 + \pi_1 + \sigma_2, \sigma_1 + \pi_1 + \sigma_2 + \pi_2 + \sigma_3, ...$

We record how many new customers come to the system, which customer is served (from outside or from the queue) and what is the amount of the final product it produces. Thus, the final product produced up to the service moment of the marked customer coincides with the amount of the final product in the BPI up to the splitting moment of the marked particle,

$$\mathbf{P}\left(\tau \leq x\right) = 1 - e^{-\rho_1 x}.$$

One can see that

$$\sigma_1 \stackrel{d}{=} \sigma_1', \sigma_2 \stackrel{d}{=} \sigma_2', ...$$

and using this relation check that the following statement is valid.

Theorem 3 *the BPI and the queueing system can be specified on a common probability space in such a way that*

$$T_n^{*\chi} = Z_1^{\chi}(\tau) + ... + Z_n^{\chi}(\tau) + Z_*^{\chi}(\tau)\ a.s.$$

and

$$\sigma_1 = \sigma_1', \sigma_2 = \sigma_2', ...a.s.$$

where τ and the rv $Z_i^{\chi}(t), i = 1, 2, ..., n$ are independent and $P(\tau \leq x) = 1 - e^{-\rho_1 x}$ and

$$Z_i^{\chi}(\tau) \stackrel{d}{=} Z^{\chi}(\tau)$$

and $Z_*^{\chi}(t)$ is the final product produced in a BPI up to moment t which starts by one individual of type zero at time 0.

Since

$$\sigma_1 + \sigma_2 + ... + \sigma_N + \sigma_{N+1} \stackrel{a.s.}{=} \sigma_1' + \sigma_2' + ... + \sigma_N' + \sigma_{N+1}' = \tau$$

we get the following

Corollary 4

$$\begin{aligned} R_n &= r_1 + r_2 + ... + r_N + 1 \\ &= Z_1^r(\tau) + ... + Z_n^r(\tau) + Z_*^r(\tau) + 1 \\ &= T_n^{*r} + 1 \end{aligned}$$

and

$$\begin{aligned} V_n &= \sigma_1 + \pi_1 + \sigma_2 + \pi_2 + ... + \sigma_N + \pi_N + \sigma_{N+1} \\ &= T_n^{*\pi} + \sigma_1 + \sigma_2 + ... + \sigma_N + \sigma_{N+1} \\ &= Z_1^\pi(\tau) + ... + Z_n^\pi(\tau) + Z_*^\pi(\tau) + \tau. \end{aligned}$$

Now we recall that

$$\Phi\left(t, s, \lambda\right) = \mathbf{E}\left[s^{Z(t)} e^{-\lambda Z^\chi(t)} | (Z(0), Z^\chi(0)) = (1, 0)\right],$$

$$\Phi\left(t, \lambda\right) = \mathbf{E}\left[e^{-\lambda Z^\chi(t)} | (Z(0), Z^\chi(0)) = (1, 0)\right],$$

and

$$\Phi_*\left(t, \lambda\right) = \mathbf{E}\left[e^{-\lambda Z^\chi(t)} | \text{ immigration; } Z(0) = 0\right].$$

Then

$$\mathbf{E} e^{-\lambda T_n^{*\chi}} = \int_0^\infty e^{-t} \Phi^n(t, \lambda) \Phi_*\left(t, \lambda\right) dt$$

where

$$\frac{\partial \Phi(t, \lambda)}{\partial t} = \rho_1 \left(\varphi^\chi(\Phi\left(t, \lambda\right), \lambda) - \Phi\left(t, \lambda\right)\right)$$
$$\Phi\left(0, \lambda\right) = 1.$$

and

$$\Phi_*\left(t, \lambda\right) = \exp\left\{\int_0^t \rho_0(\varphi^\chi(\Phi\left(u, \lambda\right), \lambda) - 1) du\right\}.$$

Let $m = \mathbf{E}\xi - 1$. In the above situation we have as before (the proof is omitted)

Theorem 5 *As* $n \to \infty$

$$\frac{T_{*n}^\chi}{\rho_1 n \mathbf{E}\chi} \xrightarrow{d} \zeta$$

where the distribution function of ζ *is*

$$F_m(x) = 1 - \left(1 + mx\right)^{-1/m}, \ 0 \leq x \leq x_m,$$

where

$$x_m = -\frac{1}{m}, \ m < 0, \ x_m = \infty, \ m \geq 0,$$

and

$$F_0(x) = 1 - e^{-x}.$$