BRANCHING PROCESSES AND THEIR APPLICATIONS: LECTURE 12: Limit theorems for continuous-time Markov processes. Markov processes with final product

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1 Limit theorems

Theorem 1 *For subcritical case* $\left[f^{'}(1) < 1\right]$

$$Q(t) = \mathbf{P}(Z(t) > 0 | Z(0) = 1) \sim Ke^{at}(1 + o(1)),\ K > 0,$$

with $a = \rho(f^{'}(1) - 1) = f^{(\rho)\prime}(1)$ *if and only if*

$$\mathbf{E}\xi \log^{+} \xi = \sum_{k=1}^{\infty} p_k k \log k < \infty$$

and

$$\lim_{n\to\infty} \mathbf{P}(Z(t) = k | Z(t) > 0) = P_k^*,\ \sum_{k=1}^{\infty} P_k^* = 1,$$

and

$$\begin{aligned} f^*(s) &= \sum_{k=1}^{\infty} P_k^* s^k = 1 - \exp\left\{a \int_0^s \frac{du}{f^{(\rho)}(u)}\right\} \\ &= 1 - \exp\left\{(f^{'}(1) - 1) \int_0^s \frac{du}{f(u) - u}\right\}. \end{aligned}$$

Proof. We have

$$\frac{\partial F(t;s)}{\partial t} = \rho(f(F(t,s)) - F(t,s)) = f^{(\rho)}(F(t,s)),\ F(0,s) = s,$$

and

$$F^*(t,s) = \mathbf{E}\left[s^{Z(t)} | Z(t) > 0\right] = 1 - \frac{R(t,s)}{Q(t)}$$

where $R(t,s)=1-F(t;s)$. Hence

$$t=\int_{R(t,s)}^{1-s}\frac{du}{f^{(\rho)}(1-u)}=\int_{Q(t)}^{1}\frac{du}{f^{(\rho)}(1-u)}.$$

One can show that under the conditions of the theorem

$$\begin{aligned}\int_0^s\frac{du}{f^{(\rho)}(u)} &= \int_{R(t,s)}^{Q(t)}\frac{du}{f^{(\rho)}(1-u)}\\ &= \frac{1}{f^{(\rho)\prime}(1)(1+\varepsilon(t,s))}\ln\frac{R(t,s)}{Q(t)}\end{aligned}$$

with

$$\varepsilon(t,s)\to 0, t\to\infty,\text{ uniformly in } s\in[0,1].$$

Hence

$$\begin{aligned}\lim_{t\to\infty}\frac{R(t,s)}{Q(t)} &= \exp\left\{f^{(\rho)\prime}(1)\int_0^s\frac{du}{f^{(\rho)}(u)}\right\}\\ &= \exp\left\{(f^{'}(1)-1)\int_0^s\frac{du}{f(u)-u}\right\}.\end{aligned}$$

Theorem 2 *If $a>0,\ f''(1)=2B<\infty$ then there exists a random variable W such that, as $t\to\infty$*

$$W_t=\frac{Z(n)}{e^{at}}\to W\ \ a.s.$$

and 1)

$$\lim_{t\to\infty}\mathbf{E}\left(W-W_t\right)^2=0;$$

2)

$$\mathbf{E}W=1,\ VarW=\sigma^2/(A^2-A);$$

3)

$$\mathbf{P}(W=0)=q=\mathbf{P}(Z(t)=0 \textit{ for some } t).$$

Theorem 3 *If $f^{'}(1)=1,\ f"(1)=2B<\infty$ then*

$$Q(t)\sim\frac{1}{\rho Bt},\ t\to\infty,$$

and

$$\mathbf{E}\left[\exp\left\{-\lambda\frac{Z(t)}{\rho Bt}\right\}|Z(t)>0\right]\to\frac{1}{1+\lambda}.$$

Proof. We have

$$\begin{aligned}t &= \int_{Q(t)}^{1}\frac{du}{f^{(\rho)}(1-u)}=\frac{1}{\rho}\int_{Q(t)}^{1}\frac{du}{Bu^2(1+o(1))}\\ &\sim \frac{1}{\rho B}\frac{1}{Q(t)}, t\to\infty.\end{aligned}$$

Hence

$$Q(t) \sim \frac{1}{\rho B t},\ t \to \infty.$$

The rest of the proof follows the same line as in the case of Galton-Watson processes.

2 Branching processes counted by random characteristics (branching processes with final product)

We consider continuous time Markov branching process with exponential life-time distribution with parameter ρ and the reproduction function $f(s)$.

Now we suppose that at the end of life any particle produces along with random number ξ of children a final product $\chi \geq 0$ which is not changed later on and denote by $\varphi^{\chi}(s,\lambda)$ the joint probability generating function of the vector (ξ,χ) specified by

$$\varphi^{\chi}(s,\lambda) = \mathbf{E}s^{\xi}e^{-\lambda\chi}.$$

χ is called a random characteristics or the final product.

Examples. $\chi = I\{\xi = k\}, \chi = I\{\xi \geq k\}, \chi = I\{l_{\xi} < x\}$ and so on.

Let

$$Z^{\chi}(t) = \sum_{A} \chi_D$$

where the summation is taken over all particles D which died up to the moment t.

We deduce integral and diferential equations for the probability generating fucntion of the pair $(Z(t), Z^{\chi}(t))$ assuming that the final product of a particle IS INDEPENDENT of her life-length. We have by the totla probability formula for

$$\Phi(t,s,\lambda) = \mathbf{E}\left[s^{Z(t)}e^{-\lambda Z^{\chi}(t)} | (Z(0), Z^{\chi}(0)) = (1,0)\right]$$

and $G(t) = 1 - e^{-\rho t}$:

$$\Phi(t,s,\lambda) = s(1-G(t)) + \int_0^t \varphi^{\chi}(\Phi(t-u,s,\lambda),\lambda)dG(u).$$

Hence

$$\frac{\partial \Phi(t,s,\lambda)}{\partial t} = \rho\left(\varphi^{\chi}(\Phi(t,s,\lambda),\lambda) - \Phi(t,s,\lambda)\right),\ \Phi(0,s,\lambda) = s.$$

In particular, for

$$\Phi(t,\lambda) := \mathbf{E}\left[e^{-\lambda Z^{\chi}(t)} | (Z(0), Z^{\chi}(0)) = (1,0)\right] = \Phi(t,1,\lambda)$$

we get

$$\Phi\left(t,\lambda\right)=\left(1-G(t)\right)+\int_{0}^{t}\varphi^{\chi}(\Phi\left(t-u,\lambda\right),\lambda)dG(u) \tag{1}$$

and

$$\frac{\partial\Phi(t,\lambda)}{\partial t}=\rho\left(\varphi^{\chi}(\Phi\left(t,\lambda\right),\lambda)-\Phi\left(t,\lambda\right)\right)$$

with

$$\Phi\left(0,\lambda\right)=1.$$

Thus, if

$$A^{\chi}(t)=\mathbf{E}Z^{\chi}(t),$$

then denoting by l the lifelength of the initial particle we get from (1) by differentiating with respect to λ and setting $\lambda=0$:

$$\begin{aligned}
A^{\chi}(t) &= \mathbf{E}\xi\int_{0}^{t}A^{\chi}(t-u)dG(u)+\int_{0}^{t}\mathbf{E}[\chi|l=u]dG(u) \\
&= \text{(by independence of } \chi \text{ of the lifelength)} \\
&= \mathbf{E}\xi\int_{0}^{t}A^{\chi}(t-u)dG(u)+\mathbf{E}\chi G(t)
\end{aligned}$$

or

$$\frac{d}{dt}A^{\chi}(t)=(\mathbf{E}\xi-1)A^{\chi}(t)+\mathbf{E}\chi,\ A^{\chi}(0)=0,$$

giving

$$A^{\chi}(t)=\frac{\mathbf{E}\chi}{\mathbf{E}\xi-1}e^{(\mathbf{E}\xi-1)t}-\frac{\mathbf{E}\chi}{\mathbf{E}\xi-1}$$

if $\mathbf{E}\xi\neq1$ and

$$A^{\chi}(t)=t\mathbf{E}\chi$$

if $\mathbf{E}\xi=1$.

Passing to the limit as $t\to\infty$ we get for

$$\begin{aligned}
\Phi\left(\lambda\right) &= :Ee^{-\lambda Z^{\chi}(\infty)}=\lim_{t\to\infty}\Phi\left(t,\lambda\right) \\
&= \lim_{t\to\infty}\mathbf{E}\left[e^{-\lambda Z^{\chi}(t)}|(Z(0),Z^{\chi}(0))=(1,0)\right]
\end{aligned}$$

(since $Z^{\chi}(t)$ is nondecreasing this limit always exists) that

$$\Phi\left(\lambda\right)=\varphi^{\chi}(\Phi\left(\lambda\right),\lambda).$$

This is a reflection of the relation

$$Z^{\chi}(t)\overset{d}{=}\left[\chi_{0}+Z_{1}^{\chi}(t-l_{0})+...+Z_{\xi}^{\chi}(t-l_{0})\right]I\left\{l_{0}\leq t\right\}$$

and, therefore,

$$Z^{\chi}(\infty)\overset{d}{=}\chi_{0}+Z_{1}^{\chi}(\infty)+...+Z_{\xi}^{\chi}(\infty).$$

In particular, for the total number of particles born in the process ($\chi = 1$) we get

$$\varphi^{\chi}(s,\lambda) = \mathbf{E}s^{\xi}e^{-\lambda\chi} = e^{-\lambda}\mathbf{E}s^{\xi} = e^{-\lambda}f(s)$$

and

$$\Phi(\lambda) = e^{-\lambda}f(\Phi(\lambda)).$$

For instance, for the case

$$f(s) = \frac{1}{2-s} \tag{2}$$

we get

$$\Phi(\lambda) = 1 - \sqrt{1-e^{-\lambda}} \text{ or } (= 1 - \sqrt{1-s}).$$

2.1 Branching processes and Queueng system with SIRO (service in random order) discipline

System with one server and the infinite capacity queue.

1) **Standard application**.

The service-time distribution is $B(x)$. Initially there is 1 customer in the queue and the server is idle (free). New customers arrive in accordance with the Poisson flow with intensity Λ. The problemis to find the distribution of the length of the busy period.

We associate with the queueing system the following branching process.

The individuals in this process have exponential life-lenght with parameter 1 and each individal produces at the end of his life a random number of children having the same distribution as the number of customers arriving during the service time of the respective customer, and a random product (characteristics)

$$\chi = \text{ the service time.}$$

Then denoting by l_0 the lifelength of the initial individual (customer) in the associated continuous time branching process we see that

$$Z^{\chi}(t) = \left[\chi_0 + Z_1^{\chi}(t-l_0) + ... + Z_{\xi}^{\chi}(t-l_0)\right] I\{l_0 \leq t\}$$

and, therefore, pasing to the limit as $t \to \infty$ we get

$$Z^{\chi}(\infty) \stackrel{d}{=} \chi + Z_1^{\chi}(\infty) + ... + Z_{\xi}^{\chi}(\infty).$$

In our case

$$\begin{aligned}
\varphi^{\chi}(s,\lambda) &= \mathbf{E}s^{\xi}e^{-\lambda\chi} = \int_0^{\infty} e^{-\lambda x}\mathbf{E}\left[s^{\xi}|\chi = x\right]dB(x) \\
&= \int_0^{\infty} e^{-\lambda x}e^{-\Lambda(1-s)x}dB(x) = \beta(\lambda + \Lambda(1-s))
\end{aligned}$$

and

$$f(s) = \mathbf{E}s^{\xi} = \varphi^{\chi}(s,1) = \beta(\Lambda(1-s))$$

leading to

$$f'(1) = \frac{d\beta\left(\Lambda\left(1-s\right)\right)}{ds}|_{s=1} = \Lambda\int_0^\infty x dB(x) = \Lambda\mathbf{E}l = \mathbf{E}\xi.$$

In particular, for $E\xi < 1$

$$\mathbf{E}Z^\chi(\infty) = \frac{\mathbf{E}l}{1-\Lambda\mathbf{E}l} = \frac{\mathbf{E}l}{1-\mathbf{E}\xi}.$$

This shows that

$$\Phi\left(\lambda\right) = \mathbf{E}e^{-\lambda Z^\chi(\infty)}$$

meets the equation

$$\Phi\left(\lambda\right) = \beta\left(\lambda + \Lambda\left(1 - \Phi\left(\lambda\right)\right)\right).$$