BRANCHING PROCESSES AND THEIR APPLICATIONS: LECTURE 11: Branching processes with immigration at zero; transient phenomena; continuous time Markov branching processes

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1 The Galton-Watson process with immigration at zero:

$$f(s) = \mathbf{E}s^{\xi}, \; g(s) = \mathbf{E}s^{\eta} = \sum_{k=1}^{\infty} \mathbf{P}(\eta = k) s^k.$$

We have

$$Y(n+1) = \xi_1^{(n)} + ... + \xi_{Y(n)}^{(n)} + \eta^{(n)} I \{Y(n) = 0\}.$$

$$\xi_i^{(n)} \stackrel{d}{=} \xi, \; \eta^{(n)} \stackrel{d}{=} \eta \text{ and iid.}$$

If

$$\Pi(n,s) = \mathbf{E}s^{Y(n)}$$

then

$$\begin{aligned} \Pi(n+1,s) &= \Pi(n, f(s)) - \Pi(n,0) + \Pi(n,0)g(s) \\ &= \Pi(n, f(s)) - (1-g(s))\Pi(n,0) \\ &= \Pi(0, f_{n+1}(s)) - \sum_{k=0}^{n} (1 - g(f_k(s))\Pi(n-k,0). \end{aligned}$$

In particular, if $Y(0) = 0$ then

$$\Pi(n+1,0) = 1 - \sum_{k=0}^{n} (1 - g(f_k(0))\Pi(n-k,0).$$

If $A < 1$ and

$$g^{'}(1) = b, \; g(0) > 0,$$

then we have a stationary distribution for the process $Y(n)$ as $n \to \infty$.

Indeed, it is known that if a Markov chain is irreducible and nonperiodic then either

1) for any pair of states $p_{ij}^{(n)} \to 0, n \to \infty$, and, therefore, there exists no stationary distribution;

or

2) all the states are ergodic, that is,

$$\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j > 0$$

and in this case $\{\pi_j\}$ is a stationary distribution and no other stationary distributions exists.

In our case take $p_{00}^{(n)} = \Pi(n,0) = P\left(Y(n) = 0\right)$. Assuming that there is NO stationary distribution we get by dominated convergence theorem a contradiction:

$$\lim_{n\to\infty} \Pi(n+1,0) = 0 = 1 - \lim_{n\to\infty} \sum_{k=0}^{n} (1 - g(f_k(0))\Pi(n-k,0) = 1$$

since the series

$$\sum_{k=0}^{\infty} (1 - g(f_k(0)) \le b \sum_{k=0}^{\infty} (1 - f_k(0)) \le b \sum_{k=0}^{\infty} A^k < \infty.$$

Thus, we have a stationary distribution

$$\Pi\left(s\right) = \mathbf{E}s^{Y} = \lim_{n\to\infty} \mathbf{E}s^{Y(n)}$$

where

$$\Pi\left(s\right) = \Pi\left(f(s)\right) - \pi_0\left(1 - g(s)\right)$$

or

$$\Pi\left(s\right) = 1 - \pi_0 \sum_{k=0}^{\infty} \left(1 - g(f_k\left(s\right))\right).$$

From here

$$\pi_0 = 1 - \pi_0 \sum_{k=0}^{\infty} \left(1 - g(f_k\left(0\right))\right)$$

leading to

$$\pi_0 = \frac{1}{1 + \sum_{k=0}^{\infty} \left(1 - g(f_k\left(0\right))\right)}.$$

Hence

$$\Pi\left(s\right) = 1 - \frac{\sum_{k=0}^{\infty} \left(1 - g(f_k\left(s\right))\right)}{1 + \sum_{k=0}^{\infty} \left(1 - g(f_k\left(0\right))\right)} \equiv 1 - \frac{R(s)}{1 + R(0)}$$

with

$$R(s) = \sum_{k=0}^{\infty} (1 - g(f_k(s)))$$

Introduce the following classes of functions: $K_1 = K(b_1, b_2, \gamma_1(y)) = \{g\}$ of probability generating functions (PGF) specified by $b_1, b_2, \gamma_1(y)$:

$$g(1-y) = 1 - (b + \alpha_1(y))y$$

where

$$0 < b_1 \leq b \leq b_2, \ \sup_{g \in K_1} |\alpha_1(y)| \leq \gamma_1(y) = o(1), y \to 0,$$

and $K_2 = K_2(B_3, B_4, \gamma_2(y)) = \{f\}$ of PGF specified by $B_3, B_4, \gamma_2(y)$:

$$\begin{aligned} f(1-y) &= 1 - Ay + (B + \alpha_3(y))y^2 + \\ &= 1 - \frac{Ay}{1 + (BA^{-1} + \alpha_4(y))y} \end{aligned}$$

where

$$0 < B_3 \leq B \leq B_4, \ \sup_{f \in K_2} |\alpha_i(y)| \leq \gamma_2(y) = o(1), y \to 0, \ i = 3, 4.$$

Let H be the class of immigration processes such that $g \in K_1, \ f \in K_2$.

Theorem 1 *If* $\{Y(n)\} \in H$ *then*

$$\lim_{A \nearrow 1} \mathbf{P}\left(\frac{\ln Y}{\ln \frac{1}{1-A}} \leq x \right) = x, \ x \in (0, 1].$$

Proof. It follows from the conditions of the theorem that for any $\varepsilon \in (0, B_2)$ there exists $\delta = \delta(\varepsilon) > 0$ such that for all $0 < y < \delta$ and all $\{Y(n)\} \in H$

$$\frac{A(1-s)}{1 + B(1+\varepsilon)(1-s)} \leq 1 - f(s) \leq \frac{A(1-s)}{1 + B(1-\varepsilon)(1-s)}$$

and

$$b(1-\varepsilon)y \leq 1 - g(1-y) \leq b(1+\varepsilon)y.$$

Let for $\varepsilon \in (0, B_2)$

$$f^{\pm}(s) = 1 - \frac{A(1-s)}{1 + B(1 \pm \varepsilon)(1-s)}$$

and

$$f_n^+(s) = f^+(f_{n-1}^+(s)), f_n^-(s) = f^-(f_{n-1}^-(s)).$$

Since the functions are fractional-linear and the derivative of f^+ and f^- at point $s = 1$ are less than 1 it is not difficult to show that

$$f_n^{\pm}(s) = 1 - \frac{A^n(1-s)}{1 + B(1 \pm \varepsilon)(1-s)\frac{1-A^n}{1-A}}$$

and that the inequalities are preserved. Thus, for s sufficiently close to 1 we have

$$\sum_{k=0}^{\infty}\left(1-g(f_k^+(s))\right) \le R(s) \le \sum_{k=0}^{\infty}\left(1-g(f_k^-(s))\right).$$

In particular, if M is such that $f_M(0) > 1-\varepsilon$ then

$$\begin{aligned}\sum_{k=M}^{\infty}\left(1-g(f_k^+(f_M(0)))\right) &\le R(0)=\sum_{k=0}^{\infty}(1-g(f_k(0))) \\ &= \sum_{k=0}^{M-1}(1-g(f_k(0)))+\sum_{k=0}^{\infty}(1-g(f_k(f_M(0)))) \\ &\le \sum_{k=0}^{M-1}(1-g(f_k(0)))+\sum_{k=0}^{\infty}\left(1-g(f_k^-(f_M(0)))\right) \\ &\le \sum_{k=0}^{M-1}(1-g(f_k(0)))+\sum_{k=0}^{\infty}\left(1-g(f_k^-(0))\right)\end{aligned}$$

and using the arguments to calculate integral in the previous section one can show that

$$R(0) \sim -\frac{b}{B}\ln(1-A)\,,\ \ A\uparrow 1. \tag{1}$$

Now for $x\in(0,1)$ let $s=\exp\{-\lambda(1-A)^x\}$. Clearly,

$$1-s\sim\lambda(1-A)^x\,,\ \ A\uparrow 1.$$

Select $m=m(\lambda,A,B)$:

$$f_m^-(0)\le\exp\{-\lambda(1-A)^x\}\le f_{m+1}^-(0)$$

that is

$$\lambda(1-A)^x\sim\frac{A^m}{1+B(1-\varepsilon)\frac{1-A^m}{1-A}}$$

or

$$\lambda(1-A)^xA^{-m}\sim\frac{1}{1+B(1-\varepsilon)\frac{1-A^m}{1-A}}.$$

Observe, that under our choice of m we have $A^m\sim 1$ since assuming $A^m<c<1$ we would have as $A\uparrow 1$

$$\lambda(1-A)^x\sim\frac{A^m(1-A)}{B(1-\varepsilon)\,(1-A^m)}\le\frac{c(1-A)}{B(1-\varepsilon)\,(1-c)}$$

which is impossible for $x<1$. This implies

$$\lambda(1-A)^x\sim\frac{1}{1+B(1-\varepsilon)\frac{1-A^m}{1-A}}. \tag{2}$$

Now

$$\begin{aligned} R(e^{-\lambda(1-A)^x}) &\le \sum_{k=0}^{\infty}\left(1-g(f_k^-\left(e^{-\lambda(1-A)^x}\right))\right) \\ &\le \sum_{k=0}^{\infty}\left(1-g(f_{k+m}^-(0))\right) = \sum_{k=m}^{\infty}\left(1-g(f_k^-(0))\right) \end{aligned}$$

and we have calculated that

$$\sum_{k=m}^{\infty}\left(1-g(f_k^-(0))\right) \le \frac{b\,(1+\varepsilon)\,(1-A)}{B(1-\varepsilon)\ln A}\ln\frac{1+B(1-\varepsilon)\frac{1}{1-A}}{1+B(1-\varepsilon)\frac{1-A^m}{1-A}} \tag{3}$$

or, in view of (2)

$$\begin{aligned} \sum_{k=m}^{\infty}\left(1-g(f_k^-(0))\right) &\le \frac{b\,(1+\varepsilon)\,(1-A)}{B(1-\varepsilon)\ln A}\ln\left(\lambda(1-A)^x\left(1+B(1-\varepsilon)\frac{1}{1-A}\right)\right) \\ &\sim \frac{b\,(1+\varepsilon)}{B(1-\varepsilon)}\ln\left((1-A)^x\frac{1}{1-A}\right) = \frac{(x-1)b\,(1+\varepsilon)}{B(1-\varepsilon)}\ln(1-A) \end{aligned} \tag{4}$$

Similarly, specifying $m = m(\lambda, A, B)$:

$$f_m^+(0) \le \exp\{-\lambda(1-A)^x\} \le f_{m+1}^+(0)$$

one can show that

$$\sum_{k=m}^{\infty}\left(1-g(f_k^+(0))\right) \ge -\frac{(x-1)b\,(1-\varepsilon)}{B(1+\varepsilon)}\ln(1-A)\,. \tag{5}$$

Since $\varepsilon > 0$ can be taken arbitrary small, it follows from (1), (4) and (5) that

$$\begin{aligned} \lim_{A\nearrow 1}\Pi\left(e^{-\lambda(1-A)^x}\right) &= 1-\lim_{A\nearrow 1}\frac{R(s)}{1+R(0)} \\ &= 1-\frac{\frac{(x-1)b}{B}\ln(1-A)}{1-\frac{b}{B}\ln(1-A)} = 1+(x-1) = x. \end{aligned}$$

Hence

$$\lim_{A\nearrow 1}\mathbf{E}e^{-\lambda Y(1-A)^x} = x.$$

Therefore,

$$\lim_{A\nearrow 1}\mathbf{P}\left(Y(1-A)^x < 1\right) = \lim_{A\nearrow 1}\mathbf{P}\left(\ln Y + x\ln(1-A) < 0\right) = x$$

or

$$\lim_{A\nearrow 1}\mathbf{P}\left(\frac{\ln Y}{\ln\frac{1}{1-A}} < x\right) = x.$$

1.1 Queueing systems with batch service

$M^{[X]}|G|1$

Λ- the intensity of the input Poisson flow. The customers arrive in batches of random size. The size of the $i-$th group is $\eta^{(i)}$

$$g(s) = \mathbf{E}s^{\eta} = \sum_{k=1}^{\infty} \mathbf{P}\left(\eta = k\right) s^k.$$

The first customer $\rightarrow$ to the server

$\nu(1)$- the number of customers coming during the service time of the $first$ customer.

$\nu(2)$- the number of customers coming during the service time of all first $\nu(1)$ customers.

$\nu(j)$- the number of customers coming during the service time of all $\nu(j-1)$ customers.

If NO customers arrive during the service time of a group of customers then we wait for the new batch and take all of them. We have

$$\nu(n+1) = \xi_1^{(n)} + ... + \xi_{\nu(n)}^{(n)} + \eta^{(n)} I\left\{\nu(n) = 0\right\}.$$

$$\xi_i^{(n)} \overset{d}{=} \xi, \text{ and iid.}$$

This is a BRANCHING PROCESS WITH IMMIGRATION AT ZERO. Clearly,

$$\begin{aligned} \mathbf{E}s^{\xi} &= \sum_{j=0}^{\infty} \mathbf{P}\left(\xi = j\right) s^j = \sum_{k=0}^{\infty} \int_0^{\infty} e^{-\Lambda u} \frac{(\Lambda u)^k}{k!} g^k(s) dG(u) \\ &= \int_0^{\infty} e^{-\Lambda u(1-g(s))} dG(u) = f(s). \end{aligned}$$

Direct calculations show that

$$A = \mathbf{E}\xi = f^{'}(1) = \Lambda g^{'}(1) \int_0^{\infty} u dG(u) = \Lambda g^{'}(1) m$$

where m is the expected service time of a customer. Hence we can apply the previous theorem to study the queueing system under heavy traffic when $A = \Lambda g^{'}(1) m \nearrow 1$.

2 Continuous time Markov processes

A stochastic process $\{Z(t,\omega), t \geq 0\}$ on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is called a continuous time Markov branching process if

1) the state-space - nonnegative integers;

2) stationary Markov Chain with respect to the σ-algebra $\mathcal{F}_t = \sigma\left\{Z(s,\omega), s \leq t\right\}$;

3) for all $t \geq 0, i = 0, 1, 2, ...$ and $|s| \leq 1$ the following branching property is valid:

$$\sum_{j=0}^{\infty} P_{ij}(t)s^j = \left(\sum_{j=0}^{\infty} P_{1j}(t)s^j\right)^i = (F(t,s))^i .$$

2.1 Construction

$$P_{ij}(\tau, \tau + t) = P\{Z(\tau + t) = j | Z(\tau) = i\} = P_{ij}(t) .$$

Now probabilistic interpretation: if there are i particles at some moment then each of them has exponential remaining life-length with parameter, say, ρ, and then dies producing children in accordance with the pgf

$$f(s) = \sum_{k=0}^{\infty} \mathbb{P}(\xi = k)s^k = \sum_{k=0}^{\infty} p_k s^k, 0 \leq s \leq 1,$$

independently of other individuals.

Thus, for $j \geq i - 1$, $i \neq j$

$$\begin{aligned} P_{ij}(\Delta t) &= \rho i p_{j-i+1}\Delta t + o(\Delta t), \\ P_{ii}(\Delta t) &= 1 - \rho i \Delta t + o(\Delta t), \\ P_{ij}(\Delta t) &= o(\Delta t),\ j < i - 1. \end{aligned}$$

From here one can deduce the forward

$$\frac{d}{dt}P_{ij}(t) = -j\rho P_{ij}(t) + \rho \sum_{k=1}^{j+1} kP_{ik}(t)p_{j-k+1}$$

and backward Kolmogorov equations

$$\frac{d}{dt}P_{ij}(t) = -i\rho P_{ij}(t) + i\rho \sum_{k=i-1}^{\infty} p_{k-i+1}P_{kj}(t).$$

with boundary

$$P_{ij}(+0) = \delta_{ij}.$$

From here for $f^{(\rho)}(s) = \rho(f(s) - s)$ and $i = 1$ we have for

$$F(t,s) = E\left[s^{Z(t)} | Z(0) = 1\right]$$

the following equations

$$\frac{\partial F(t;s)}{\partial t} = f^{(\rho)}(s)\frac{\partial F(t;s)}{\partial s},\ F(0,s) = s,$$

and

$$\begin{aligned} \frac{\partial F(t;s)}{\partial t} &= -\rho \sum_{j=0}^{\infty} P_{1j}(t)s^j + \rho \sum_{j=0}^{\infty} s^j \sum_{k=0}^{\infty} p_k P_{kj}(t) \\ &= \rho(f(F(t,s)) - F(t,s)) = f^{(\rho)}(F(t,s)), \\ F(0,s) &= s. \end{aligned} \tag{6}$$

2.2 Classification

Let

$$A(t) = EZ(t).$$

Then

$$\frac{\partial}{\partial s}\frac{\partial F(t;s)}{\partial t} = f^{(\rho)\prime}(F(t,s))\frac{\partial F(t,s)}{\partial s}$$

or, setting $s = 1$

$$\frac{dA(t)}{dt} = \rho(f^{'}(1) - 1)A(t),\ A(0) = 1.$$

Solving this equation we get

$$A(t) = e^{at}, a = \rho(f^{'}(1) - 1).$$

A continuous time Markov branching process is called supercritical, critical, subcritical if, respectively $f^{'}(1) > 1, = 1, < 1$.

Example 1. Let

$$f^{(\rho)}(s) = a(s-1) + \lambda(1-s)^{1+\alpha}, 0 < \alpha < 1, \lambda > \max\{a, 0\}.$$

Then the respective probability generating function $F(t;s)$ solves the equation

$$\frac{\partial F(t;s)}{\partial t} = a(F(t;s) - 1) + \lambda(1 - F(t;s))^{1+\alpha},\ F(0,s) = s.$$

Set

$$v = \frac{1}{1-F}.$$

Then

$$\frac{dv}{dt} = -\alpha a v + \lambda\alpha,\ v = \frac{1}{1-s}.$$

Hence

$$F(t,s) = 1 - \left[\frac{\lambda}{a}\left(1 - e^{-\alpha a t}\right) + e^{-\alpha a t}(1-s)^{-\alpha}\right]^{-1/\alpha},\ a \neq 0,$$

and

$$F(t,s) = 1 - \left[\alpha\lambda t + (1-s)^{-\alpha}\right]^{-1/\alpha},\ a = 0.$$

Example 2. Let

$$f^{(\rho)}(s) = a\left(s - 1 - (1-s)^{\alpha}\right), 0 < \alpha < 1, a > 0.$$

Then

$$\frac{\partial F(t;s)}{\partial t} = a(F(t;s) - 1 - (1 - F(t;s))^{\alpha}),\ F(0,s) = s.$$

Set

$$v = (1-F)^{1-\alpha}.$$

Then we get a linear equation whose solution is

$$F(t,s) = 1 - \left[1 - e^{-a(1-\alpha)t} + e^{-a(1-\alpha)t}(1-s)^{1-\alpha}\right]^{\frac{1}{1-\alpha}}.$$

Observe that

$$F(t,1) = \lim_{s\uparrow 1} F(t,s) = 1 - \left[1 - e^{-a(1-\alpha)t}\right]^{\frac{1}{1-\alpha}} < 1.$$

Thus, in this case we have the so-called explosion phenomena:

$$F(t,1) = \sum_{k=0}^{\infty} \mathbf{P}(Z(t)=k) = \mathbf{P}(Z(t)<\infty)$$

and $1 - F(t;1) = 1 - \mathbf{P}(Z(t)<\infty) = \mathbf{P}(Z(t)=\infty) > 0$ showing that within *any* finite time interval the number of individuals in the population becomes infinite with a positive probability!

2.2.1 Criterion

A Markov process does not explode if and only if for any $\varepsilon \in (0,1)$

$$\int_{1-\varepsilon}^{1} \frac{du}{1-f(u)} = \infty.$$

We prove the criterion in a more general situation later on.
Theorem. If $f^{'}(1) < \infty$ then the equation

$$\frac{\partial F(t;s)}{\partial t} = \rho(f(F(t,s)) - F(t,s)) = f^{(\rho)}(F(t,s)), F(0,s) = s$$

has a unique solution in the class of functions $F(t,s)$ with $F(t,1) = 1$.
Proof. Let $G(t) = 1 - e^{-\rho t}$. Then

$$F(t,s) = s\,(1 - G(t)) + \int_0^t f(F(t-u,s))dG(u).$$

If there are two solutions $F_1(t,s)$ and $F_2(t,s)$ then

$$\begin{aligned} |F_1(t,s) - F_2(t,s)| &\leq \int_0^t |f(F_1(t-u,s)) - f(F_2(t-u,s))| dG(u) \\ &\leq f^{'}(1)\int_0^t |F_1(t-u,s) - F_2(t-u,s)| dG(u) \\ &\leq f^{'}(1)G(t) \sup_{0\leq v\leq t} |F_1(v,s) - F_2(v,s)|. \end{aligned}$$

If $t_0 > 0$ is such that $f^{'}(1)G(t_0) < 1$ then

$$\sup_{0 \le v \le t_0} |F_1(v,s) - F_2(v,s)| \le f^{'}(1)G(t_0) \sup_{0 \le v \le t_0} |F_1(v,s) - F_2(v,s)|.$$

Hence

$$F_1(t,s) = F_2(t,s),\ \ 0 \le t \le t_0.$$

Again

$$\begin{aligned} |F_1(t,s) - F_2(t,s)| &\le \int_0^t |f(F_1(t-u,s)) - f(F_2(t-u,s))| dG(u) \\ &= \int_0^{t-t_0} |f(F_1(t-u,s)) - f(F_2(t-u,s))| dG(u) \\ &\le f^{'}(1) \int_0^{t-t_0} |F_1(t-u,s) - F_2(t-u,s)| dG(u) \\ &\le f^{'}(1)G(t-t_0) \sup_{0 \le v \le t-t_0} |F_1(t_0+v,s) - F_2(t_0+v,s)| \end{aligned}$$

if $t - t_0 \le t_0$ and this is for all $|s| \le 1$.

One can check that

$$\lim_{t\to\infty} F(t,s) = f(\lim_{t\to\infty} F(t,s))$$

Hence all the properties related with the extinction of the Markov continuous time processes are similar to those for the Galton-Watson processes.