BRANCHING PROCESSES AND THEIR APPLICATIONS: LECTURE 10: Branching processes with immigration; transient phenomena

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1 Galton-Watson processes with immigration

The Galton-Watson process with immigration: is specified by

$$f(s) = \mathbf{E}s^{\xi},\ g(s) = \mathbf{E}s^{\eta} = \sum_{k=1}^{\infty} \mathbf{P}(\eta = k)\, s^k,$$

and

$$Y(n+1) = \xi_1^{(n)} + ... + \xi_{Y(n)}^{(n)} + \eta^{(n)}, \qquad \eta^{(n)} \stackrel{d}{=} \eta,\ \text{and iid.}$$

We have

$$\begin{aligned} \Phi(n+1,s) &= \mathbf{E}\left[s^{Y(n+1)}|Y(0)=0\right] \\ &= \mathbf{E}\left[s^{\xi_1^{(n)}+...+\xi_{Y(n)}^{(n)}+\eta^{(n)}}|Y(0)=0\right] \\ &= g(s)\Phi(n,f(s)) = ... = \prod_{k=0}^{n+1} g(f_k(s)). \end{aligned}$$

Theorem 1 *If $g'(1) < \infty$ and $A = f'(1) < 1$ then there exists the limit*

$$\Phi(s) = \mathbf{E}s^Y = \lim_{n\to\infty} \Phi(n,s) = \prod_{k=0}^{\infty} g(f_k(s)) > 0.$$

Proof. Indeed,

$$1 - g(f_k(s)) \le g'(1)\,(1 - f_k(s)) \le g'(1)A^k\,(1-s)\,.$$

Hence

$$\sum_{k=0}^{\infty} (1 - g(f_k(s))) < \infty$$

which shows that $1 \geq \prod_{k=0}^{\infty} g(f_k(s)) > 0$ for all $s \in [0,1]$ finishing the proof.

Theorem 2 *If* $g^{'}(1) = b < \infty$ *and* $f'(1) = 1,\ B = f"(1)/2 \in (0, \infty)$ *then for* $\theta = b/B$

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{Y(n)}{Bn} \leq x\right) = F(x) = \frac{1}{\Gamma(\theta)} \int_0^x y^{\theta-1} e^{-y} dy. \tag{1}$$

Remark 3 *Since the denisity of* $F(x)$ *is*

$$p(x) = \frac{1}{\Gamma(\theta)} x^{\theta-1} e^{-x}$$

and

$$\int_0^{\infty} e^{-\lambda x} p(x) dx = \frac{1}{(1+\lambda)^{\theta}},\ \lambda > 0,$$

we will prove instead of (1) that

$$\lim_{n\to\infty} \mathbf{E}\left[e^{-\lambda Y(n)/Bn}\right] = \frac{1}{(1+\lambda)^{\theta}},\ \lambda > 0.$$

Proof. For fixed $\lambda > 0$ define $m = m(n, \lambda)$ by

$$1 - f_m(0) \geq 1 - e^{-\frac{\lambda}{Bn}} \geq 1 - f_{m+1}(0).$$

Since $1 - e^{-x} \sim x,\ x \to +0$, and

$$1 - f_k(0) \sim B^{-1} k^{-1},\ k \to \infty,$$

we have

$$\frac{1}{Bm} \sim \frac{\lambda}{Bn},\ n \to \infty,$$

implying $m \sim n/\lambda$. Clearly,

$$\mathbf{E}\exp\left\{-\lambda \frac{Y(n)}{Bn}\right\} = \Phi(n, e^{-\frac{\lambda}{Bn}}) = \prod_{k=0}^{n} g(f_k(e^{-\frac{\lambda}{Bn}})).$$

Hence

$$\begin{aligned}\prod_{k=m}^{n+m} g(f_k(0)) &= \prod_{k=0}^{n} g(f_k(f_m(0))) \\ &\leq \mathbf{E}\exp\left\{-\lambda \frac{Y(n)}{Bn}\right\} \leq \prod_{k=m+1}^{n+m+1} g(f_k(0)).\end{aligned} \tag{2}$$

Note that the ratio of the estimates from above and below is $g(f_{n+m+1}(0))/g(f_m(0))$ and tends to 1 as $m \to \infty$. Thus, it is enough to analyse the left-nad side only. On account of

$$g(s) = 1 - b(1-s) + o(1-s) = e^{-b(1-s)+o(1-s)}$$

as $s \uparrow 1$, we have as $n \to \infty$

$$\begin{aligned}
\prod_{k=m}^{n+m} g(f_k(0)) &\approx \exp\left\{-b\sum_{k=m}^{n+m}(1-f_k(0))\right\} \\
&\approx \exp\left\{-\frac{b}{B}\sum_{k=m}^{n+m}\frac{1}{k}\right\} \\
&\approx \exp\left\{-\frac{b}{B}\ln\frac{n+m}{m}+o(1)\right\} \\
&= \exp\left\{-\frac{b}{B}\ln\frac{n+n/\lambda}{n/\lambda}+o(1)\right\} \\
&\approx \exp\left\{-\frac{b}{B}\ln(\lambda+1)+o(1)\right\} \\
&\to \frac{1}{(\lambda+1)^{\theta}}.
\end{aligned}$$

The right-nand side of (2) approaches the same value as $n \to \infty$. Thus,

$$\lim_{n\to\infty}\mathbf{E}\exp\left\{-\lambda\frac{Y(n)}{Bn}\right\} = \frac{1}{(\lambda+1)^{\theta}}.$$

The theorem is proved.

1.1 Transient phenomena

Introduce the following classes of functions: $K_1 = K(b_1, b_2, \gamma_1(y)) = \{g\}$ of probability generating functions (PGF) specified by $b_1, b_2, \gamma_1(y)$:

$$g(1-y) = 1-(b+\alpha_1(y))y = e^{-(b+\alpha_2(y))y}$$

where

$$0 < b_1 \le b \le b_2, |\alpha_i(y)| \le \gamma_1(y) = o(1), i = 1,2$$

and $K_2 = K_2(B_3, B_4, \gamma_2(y)) = \{f\}$ of PGF specified by $B_3, B_4, \gamma_2(y)$:

$$\begin{aligned}
f(1-y) &= 1 - Ay + (B+\alpha_3(y))y^2 + \\
&= 1 - \frac{Ay}{1+(BA^{-1}+\alpha_4(y))y}
\end{aligned}$$

where

$$0 < B_3 \le B \le B_4, |\alpha_i(y)| \le \gamma_2(y) = o(1), i = 3,4.$$

Let I be the class of immigration processes such that $g \in K_1,\ f \in K_2$.

Theorem 4 *If $b = g'(1) < \infty$ and $A = f'(1) < 1$ and $B = f''(1)/2 < \infty$ then there exists the limit*

$$\lim_{A\nearrow 1}\mathbf{P}\left(\frac{Y(1-A)}{B} \le x\right) = \frac{1}{\Gamma(\theta)}\int_0^x y^{\theta-1}e^{-y}dy,\ \theta = b/B.$$

Proof. We know that

$$\Phi(s) = \mathbf{E}s^Y = \prod_{k=0}^{\infty} g(f_k(s))$$

It follows from the conditions of the theorem that for any $\varepsilon \in (0, B)$ there exists $\delta = \delta(\varepsilon) > 0$ such that for all $0 < 1 - s < \delta$ a

$$\frac{A\,(1-s)}{1 + (B+\varepsilon)\,(1-s)} \le 1 - f(s) \le \frac{A\,(1-s)}{1 + (B-\varepsilon)\,(1-s)}$$

and

$$e^{-b(1+\varepsilon)y} \le g(1-y) \le e^{-b(1-\varepsilon)y}, 0 < y < \delta.$$

Let

$$f^{\pm}(s) = 1 - \frac{A\,(1-s)}{1 + B(1 \pm \varepsilon)\,(1-s)}$$

and

$$f_n^{+}(s) = f^{+}(f_{n-1}^{+}(s)),\ f_n^{-}(s) = f^{-}(f_{n-1}^{-}(s)).$$

Since the functions are fractional-linear and the derivative is less than 1 it is not difficult to show that

$$f_n^{\pm}(s) = 1 - \frac{A^n(1-s)}{1 + B(1 \pm \varepsilon)(1-s)\frac{1-A^n}{1-A}} \qquad (3)$$

and that the inequalities

$$1 - f_n^{+}(s) \le 1 - f_n(s) \le 1 - f_n^{-}(s)$$

are preserved in the mentioned vicinity of point $s = 1$.

Let now $m = m(\lambda, A, B)$:

$$f_m^{+}(0) \le \exp\{-\lambda(1-A)/B\} \le f_{m+1}^{+}(0).$$

In view of

$$1 - s \approx \lambda(1-A)/B \text{ as } A \uparrow 1$$

and (3) we have

$$\lambda(1-A)/B \approx \frac{A^m}{1 + B(1+\varepsilon)\frac{1-A^m}{1-A}},\ m \to \infty,$$

or

$$\begin{aligned}
\lambda(1+\varepsilon) &\approx \frac{\frac{B(1+\varepsilon)}{1-A}A^m}{1 + B(1+\varepsilon)\frac{1-A^m}{1-A}} \\
&= \frac{\frac{B(1+\varepsilon)}{1-A}A^m - 1 - \frac{B(1+\varepsilon)}{1-A} + 1 + \frac{B(1+\varepsilon)}{1-A}}{1 + B(1+\varepsilon)\frac{1-A^m}{1-A}} \\
&= -1 + \frac{1 + \frac{B(1+\varepsilon)}{1-A}}{1 + B(1+\varepsilon)\frac{1-A^m}{1-A}}
\end{aligned}$$

and, therefore,

$$\frac{1+\frac{B(1+\varepsilon)}{1-A}}{1+B(1+\varepsilon)\frac{1-A^m}{1-A}} \approx 1+\lambda(1+\varepsilon).$$

Thus, in the vicinity in question

$$\Phi(e^{-\lambda(1-A)/b}) \leq \prod_{k=0}^{\infty} g(f_{k+m+1}^{+}(0))$$

or

$$\begin{aligned}
\ln\Phi(e^{-\lambda(1-A)/b}) &\leq \sum_{k=0}^{\infty} \ln g(f_{k+m+1}^{+}(0)) \\
&\leq -b\,(1-\varepsilon)\sum_{k=m+1}^{\infty}(1-f_k^{+}(0)) \\
&= -b\,(1-\varepsilon)\sum_{k=m+1}^{\infty}\frac{A^k}{1+B(1+\varepsilon)\frac{1-A^k}{1-A}}.
\end{aligned}$$

Since under our choice of m the first term in the last sum tends to zero as $m\to\infty$ and summands are monotone in k it follows that as $m\to\infty$

$$\begin{aligned}
& -b\,(1-\varepsilon)\sum_{k=m+1}^{\infty}\frac{A^k}{1+B(1+\varepsilon)\frac{1-A^k}{1-A}} \\
\approx\ & -b\,(1-\varepsilon)\int_m^{\infty}\frac{A^t dt}{1+B(1+\varepsilon)\frac{1-A^t}{1-A}} \\
=\ & (\text{if } y=1-A^t) = \frac{b\,(1-\varepsilon)}{\ln A}\int_{1-A^m}^{1}\frac{dy}{1+B(1+\varepsilon)\frac{y}{1-A}} \\
=\ & \frac{b\,(1-\varepsilon)\,(1-A)}{B(1+\varepsilon)\ln A}\ln\frac{1+B(1+\varepsilon)\frac{1}{1-A}}{1+B(1+\varepsilon)\frac{1-A^m}{1-A}} \\
\approx\ & -\frac{b\,(1-\varepsilon)}{B(1+\varepsilon)}\ln\left(1+\lambda(1+\varepsilon)\right).
\end{aligned}$$

Thus,

$$\limsup_{A\nearrow 1}\Phi\left(e^{-\lambda(1-A)/B}\right) \leq (1+\lambda(1+\varepsilon))^{-\frac{b(1-\varepsilon)}{B(1+\varepsilon)}}.$$

and similarly, selecting $m=m(\lambda,A,B)$:

$$f_m^{-}(0) \leq \exp\{-\lambda(1-A)/B\} \leq f_{m+1}^{-}(0)$$

one can prove that

$$\liminf_{A\nearrow 1}\Phi\left(e^{-\lambda(1-A)/B}\right) \geq (1+\lambda(1+\varepsilon))^{-\frac{b(1-\varepsilon)}{B(1+\varepsilon)}}.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\lim_{A \nearrow 1} \Phi \left(e^{-\lambda(1-A)/B} \right) = (1+\lambda)^{-\frac{b}{B}}$$

as desired.

1.2 $\mathbf{M}^{[X]}|\mathbf{G}|\mathbf{1}$ systems with permanent customers and $FIFO-$ discipline.

Consider a queueing system with Poisson flow of customers with intensity Λ which arrive in batches whose sizes are spesified by a probability generating function $h(s)$. Assume that there is 1 permanent customer in the queue. The service time of the permanent customer is distributed according to $B_p(x)$ while the distribution of the service time of non-permanent custiomers is $B(x)$. Initially only the permanent customer is in the queue and its service starts. When the service is ended the premanent customer joins the queue consisting of the customers coming during the its service time and becomes the last one in the queue. The service dicipline is FIFO - first-in-first-out.

Let $Y(n)$ be the number of nonpermanent customers in the queue just after the moment when the nth service of the permanent customer is finished. Then

$$Y(n+1) = \xi_1^{(n)} + ... + \xi_{Y(n)}^{(n)} + \eta^{(n)}$$

where $\xi_i^{(n)}-$ is the number of customers arriving during the service time of the $i-$th nonpermanent customer being in the queue at the end of the $(n-1)$-th service of the permananet customer and $\eta^{(n)}$ the number of customers arriving during the $n-$th service of the permanent customer.

Thus, at these moments we have a Galton-Watson branching process with immigration. Its ingredients are specified by the Poisson flow of intensity Λ.

Let $\mu(u)$ be the number of batches of customers arriving within the interval $[0, u]$. Then

$$\mathbf{P}(\mu(u) = k) = \frac{(\Lambda u)^k}{k!} e^{-\Lambda u}, \quad k = 0, 1, ...$$

Hence, the total amount $M(u)$ customers arriving within this interval has the probability generating function

$$\begin{aligned} \mathbf{E} s^{M(u)} &= \sum_{k=0}^{\infty} \mathbf{P}(\mu(u) = k) \mathbf{E} \left[s^{M(u)} | \mu(u) = k \right] \\ &= \sum_{k=0}^{\infty} \frac{(\Lambda u)^k}{k!} e^{-\Lambda u} h^k(s) = e^{\Lambda u (h(s)-1)}. \end{aligned}$$

Thus, the offspring probability generating function $f(s)$ for the number of new customers arriving during the service time l of a nonpermanent customer is

$$\begin{aligned} f(s) &= \mathbf{E}s^{\xi} = \int_0^{\infty} \mathbf{E}\left[s^{\xi}|l = u\right] dB(u) \\ &= \int_0^{\infty} \sum_{k=0}^{\infty} e^{-\Lambda u} \frac{(\Lambda u)^k}{k!} h^k(s) dB(u) \\ &= \int_0^{\infty} e^{\Lambda u(h(s)-1)} dB(u) \end{aligned}$$

and the offspring probability generating function $g(s)$ for the number of new customers arriving during the service time l_p of the permanent customer is

$$\begin{aligned} g(s) &= \mathbf{E}s^{\eta} = \int_0^{\infty} \mathbf{E}\left[s^{\eta}|l_p = u\right] dB_p(u) \\ &= \int_0^{\infty} e^{\Lambda u(h(s)-1)} dB_p(u). \end{aligned}$$

And if $g'(1) = \Lambda h'(1) \int_0^{\infty} u dB_p(u) < \infty$ and $A = f'(1) = \Lambda h'(1) \int_0^{\infty} u dB(u) < 1$ we have a stationary distribution for the size of queue at the moments of the end of the service of the permanent customer. Besides, we have transition phenomena for this process as $\Lambda h'(1) \int_0^{\infty} u dB(u) \uparrow 1$.