# BRANCHING PROCESSES AND THEIR APPLICATIONS: LECTURE 1: Classification of Galton-Watson processes

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## 0.1 Basic Notions

**Historical introduction**.

Malthus: Essay on the Principle of Population: 379 out of 487 bourgeois families in the city Berne died out between 1583 and 1783 and 379/487 approx. 0.75 (!)

**Definition**. A Galton-Watson process is a Markov chain $\{Z(n), n = 0, 1, 2, ..\}$ on nonnegative integers. Its transition function is specified by a probability law $\{p_k, k = 0, 1, ...\}$, $p_k \geq 0, \sum p_k = 1$ with

$$P_{ij} = P\left\{Z(n+1) = j | Z(n) = i\right\} = \begin{cases} p_j^{*i} & \text{if} \quad i \geq 1, j \geq 0 \\ \delta_{0j} & \text{if} \quad i = 0, j \geq 0. \end{cases}$$

where

$$p_j^{*i} = \sum_{j_1+...+j_i=j} p_{j_1} p_{j_2} ... p_{j_i}.$$

**Generating functions**.

It is usually denoted by $f$ and is viewed as a function of a real variable $s \in [0, 1]$:

$$f(s) = \mathbb{E}[s^{\xi}] = \sum_{k=0}^{\infty} \mathbb{P}(\xi = k) s^k = \sum_{k=0}^{\infty} p_k s^k, 0 \leq s \leq 1, \tag{1}$$

in terms of a random variable $\xi$ giving the offspring of an individual, or in terms of its distribution $p_0, p_1, p_2, \ldots$.For geometric offspring size distribution we have

$$f(s) = \sum_{k=0}^{\infty} qp^k s^k = \frac{q}{1-ps}.$$

It is not difficult to understand that

$$Z(n+1) = \xi_1^{(n)} + ... + \xi_{Z(n)}^{(n)},$$

where $\xi_i^{(n)} \stackrel{d}{=} \xi$ are iid. Iterations

$$f_0(s) = s, f_{n+1}(s) = f_n(f(s)).$$

Example $f(s) = q + ps^2$.

In particular, given $Z(0) = 1$

$$\begin{aligned} F(n+1,s) \quad : &= Es^{Z(n+1)} = E\left[E\left[s^{Z(n+1)}|Z(n)\right]\right] = E\left[E\left[s^{\xi_1^{(n)}+...+\xi_{Z(n)}^{(n)}}\right] Z(n)\right] \\ &= E\left(Es^{\xi}\right)^{Z(n)} = F(n, f(s)) = ... = f_{n+1}(s). \end{aligned}$$

**The space of elementary events**.

Initial particle has label (0), its $i-$th offspring by $(i)$ the individuals of the $n$-th generation by

$$(0i_1 i_2 ... i_n).$$

History of a family

$$\omega = (\xi_0, \xi_{01}, \xi_{02}, ...)$$

and this is our space of elementary events $\Omega$.

The tree of the family: $(0), (01), ..., (0\xi_0), ...(011), (012), ..., (021), ....$

is a PLANE ROOTED TREE.

For each $\omega \in \Omega$ we define $I_0(\omega), I_1(\omega), ..., I_n(\omega), ...$ where $I_n(\omega)$ is the set of particles constituting $n$-th generation.

**The generation** $I_0(\omega) = \{(0)\}, I_1(\omega) = \{(01), ..., (0\xi_0)\}$ , and $I_n(\omega)$ consists of all the sequences $(0i_1...i_n)$ such that $(0i_1...i_{n-1}) \in I_{n-1}(\omega)$ and $\xi_{0i_1...i_{n-1}}(\omega) \geq i_n$.

**Probability measure** $P$**:** on cylindric sets of $\Omega$ using

$$P\left(\xi_{0i_1...i_n} = k\right) = p_k.$$

and

$$\begin{aligned} & P\left((\xi_0, \xi_{01}, \xi_{02}, ..., \xi_{0i_1...i_n}) = k_0, k_{01}, k_{02}, ..., k_{0i_1...i_n}\right) \\ = \quad & p_{k_0} p_{k_{01}} \cdots p_{k_{0i_1...i_n}}. \end{aligned}$$

**Probability space** $(\Omega, \mathcal{F}, P)$ can be constructed in an obvious way.

## 0.2 Classification

$$A = E\xi = EZ(1) = f^{'}(1).$$

The process is called subcritical if $A < 1$, critical, if $A = 1$ and supercritical, if $A > 1$.

The expacted number of individuals and the second factorial moment for the number of particles at the $n$-th generation can be calculated by

$$\begin{aligned} EZ(n) &= \left(Es^{Z(n)}\right)^{'}|_{s=1} = (f_n(s))'|_{s=1} \\ &= f'(f_{n-1}(s))\,(f_{n-1}(s))'|_{s=1} \\ &= \prod_{k=0}^{n-1} f'(f_k(s))|_{s=1} = \left(f^{'}(1)\right)^n = A^n \end{aligned}$$

and

$$f_n''(s) = f'(f_{n-1}(s))f_{n-1}''(s) + f''(f_{n-1}(s))(f'(f_{n-1}(s)))^2, \tag{2}$$

giving

$$\mathbb{E}[Z(n)(Z(n)-1)] = A\mathbb{E}[Z(n-1)(Z(n-1)-1)] + f''(1)A^{2(n-1)}.$$

Hence, given $Z(0) = 1$

$$\begin{aligned} \mathbb{E}[Z(n)(Z(n)-1)] &= \sum_{k=1}^{n} f''(1)A^{2(n-k)}A^{k-1} \\ &= f''(1)A^{n-1}\sum_{k=1}^{n} A^{n-k} \end{aligned}$$

Simplifying this we get

$$\mathbb{E}[Z(n)(Z(n)-1)] = f''(1)\frac{A^{n-1}(A^n-1)}{A-1},$$

if $A \neq 1$ and $\mathbb{E}[Z(n)(Z(n)-1)] = f''(1)n$ in the critical case. Consequently for $A \neq 1$,

$$\mathbb{E}[Z^2(n)] = f''(1)\frac{A^{n-1}(A^n-1)}{A-1} + A^n. \tag{3}$$

$$Var[Z(n)] = f''(1)\frac{A^{n-1}(A^n-1)}{A-1} + A^n - A^{2n}. \tag{4}$$

But since $\sigma^2 = Var[\xi] = f''(1) - A(A-1)$ with $Z(0) = 1$ it follows that

$$Var[Z(n)] = \begin{cases} \sigma^2 \frac{A^{n-1}(A^n-1)}{A-1} & if \;\; A \neq 1, \\ \sigma^2 n & if \;\; A = 1. \end{cases} \tag{5}$$

Coefficient $CV$ of variation of a population is defined as

$$CV = \frac{\sqrt{Var[Z(n)]}}{EZ(n)}.$$

For supercritical populations

$$CV = \frac{\sigma}{\sqrt{A(A-1)}}\sqrt{1-A^{-n}} \approx \frac{\sqrt{Var[Z(1)]}}{EZ(1)}$$

for large $A$ and stabilizes quickly.

Calculation of iterations for the pure geometric reproduction law

$$f(s) = \sum_{k=0}^{\infty} qp^k s^k = \frac{q}{1-ps}.$$

Clearly, $f^{'}(1) = A = p/q$. Further we have

$$1 - f(s) = \frac{p(1-s)}{1-ps}$$

and

$$\begin{aligned} & \frac{1}{1-f(s)} - \frac{1}{A(1-s)} \\ = & \frac{1-ps}{p(1-s)} - \frac{q}{p(1-s)} = 1. \end{aligned}$$

Thus,

$$\frac{1}{1-f_n(s)} - \frac{1}{A(1-f_{n-1}(s))} = \frac{1}{1-f(f_{n-1}(s))} - \frac{1}{A(1-f_{n-1}(s))} = 1$$

or

$$\frac{1}{1-f_n(s)} = 1 + \frac{1}{A(1-f_{n-1}(s))} = 1 + \frac{1}{A} + \frac{1}{A^2(1-f_{n-2}(s))} = \ldots.$$

The end of this is a simple closed form,

$$\begin{aligned} \frac{1}{1-f_n(s)} &= 1 + (1/A) + (1/A)^2 + \ldots + (1/A)^{n-1} + 1/A^n(1-s) \\ &= \begin{cases} \frac{A^n-1}{A^{n-1}(A-1)} + \frac{1}{A^n(1-s)} & \text{if} \quad A \neq 1 \\ n + \frac{1}{1-s} & \text{if} \quad A = 1. \end{cases} \end{aligned}$$

Therefore,if $A \neq 1$ then

$$1 - f_n(s) = \frac{A^n(A-1)(1-s)}{A(A^n-1)(1-s) + A - 1}. \tag{6}$$

and if $A = 1$ then

$$1 - f_n(s) = \frac{1}{n + (1-s)^{-1}}.$$

Survival probability: if $A = p/q \neq 1$ then

$$\begin{aligned} P\left(Z(n) > 0\right) &= 1 - f_n(0) \\ &= \frac{A^n(A-1)}{A(A^n-1)+A-1} = \frac{A^{n+1}(1-1/A)}{A^{n+1}-1} \\ &= \frac{\left(\frac{p}{q}\right)^n (1-\frac{p}{q})}{1-\left(\frac{p}{q}\right)^{n+1}}, \end{aligned}$$

if $A = 1$ then

$$P\left(Z(n) > 0\right) = \frac{1}{n+1}.$$

In particular, if $A > 1$ then

$$\begin{aligned} \lim_{n\to\infty} P\left(Z(n) > 0\right) &= \lim_{n\to\infty} \frac{A^{n+1}(1-1/A)}{A^{n+1}-1} \\ &= 1 - \frac{1}{A}. \end{aligned}$$