# For what trading strategies is the tax payment stream of infinite variation ?\*

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#### Abstract

For an Itô asset price process and under quite mild structural assumptions, we show that the accumulated payments of a linear tax on trading gains are of infinite variation if the quadratic covariation of the trading strategy and the asset price is negative. By contrast, if the strategy is a smooth function of the asset price and some finite variation processes with positive partial derivative with respect to the price variable, then accumulated tax payments are of finite variation.

An interesting example are Constant Proportion Portfolio Insurance (CPPI) strategies which we extend to models with capital gains taxes. The associated tax payment stream is of finite variation if the tax-adjusted constant multiple of the cushion which is invested in the risky asset is bigger or equal to one. Otherwise, it is of infinite variation.

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# 1 Introduction

In most countries, trading gains have to be taxed. The modeling is complicated by the rule that gains on assets are taxed when assets are sold and not when gains actually occur. This means that an investor can influence the timing of her tax payments, i.e., she holds a timing option.

Dybvig and Koo [7] model the so-called *exact tax basis* or *specific share identification method* in discrete time, which corresponds, e.g., to the tax legislation in the US and seems economically the most reasonable tax basis. Here, an investor who wants to reduce her position, say, the amount of Apple stocks, can freely choose which of the Apple stocks in her portfolio are relevant for taxation. Though all Apple stocks possess the same market price, they have in general different purchasing prices if bought at different points in time. Thus, their book profits, i.e., the difference between current market price and purchasing price that are *linearly* taxed at the liquidation time are different. To capture this, strategies in Dybvig and Koo [7] are double-indexed: they specify the number of identical shares which are purchased at some time s and kept in the portfolio until some time t. Other tax bases are the first-in-first-out rule and an average of past purchasing prices (for continuous time models, see Jouini, Koehl, and Touzi [11, 12] and Ben Tahar, Soner, and Touzi [3], respectively).

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As observed in Dybvig and Koo [7], for a nonnegative interest rate, it is pathwise optimal to wash sale every share with negative book profit and, when reducing the total position, to sell the shares with the lowest book profits first (see also Subsection 3.1 of Constantinides [6]). A wash sale is a sale with the aim to declare a loss to the tax office, but the security is immediately re-purchased. By following a procedure which consists of "automatic" wash sales and the rule to sell shares with the lowest book profits first, one can get rid of the above mentioned doubleindexing, and a strategy can be identified with a one-dimensional predictable process which specifies only the time-dependent total number of identical stocks that the investor holds in her portfolio (see Appendix A of Kühn and Ulbricht [14]). Taxes are either triggered by wash or "real" sales. Wash sales always lead to negative tax payments (tax credits). On the other hand, by the wash sales, unrealized book profits are always nonnegative, and thus real sales always lead to nonnegative tax payments. In continuous time, the tax payment stream for elementary trading strategies can be uniquely extended to all adapted and càglàd strategies (see [14]).

Of course, the above mentioned simplification relies on many idealized assumptions. It fails if there are, for example, different tax rates for short and long term capital. In this case, wash sales need not be optimal anymore since they reduce the residence time of the shares in the portfolio and may force the investor later on to pay the usually higher short term rate. In practice, there are also legal wash sales restrictions. E.g., in the US, a realized loss can only be claimed to the tax office if the same asset is not repurchased within 30 days, see [1]. To get a better view on the nature of capital gains taxes, it may nevertheless be helpful to abstract from special restrictions which are different in each country and need not remain forever.

From a theoretical point of view, it is important to note that, in contrast to models with proportional transaction costs, in models with taxes, reasonable strategies are not restricted to processes of finite variation. The reason for this is that taxes are proportional to the product of price movements and changes of the position.

In this article, we investigate the fine structure of the tax payment stream. We are interested in the basic mathematical question for what trading strategies typically appearing in continuous time finance the tax process is of (in)finite variation. Put differently: for what strategies do total tax payments, positive or negative, explode when passing to the continuous time limit ? In doing so, we focus on Itô asset price processes.

If gains *were* taxed immediately when they occur (i.e., based on a marking to market of the asset), the answer would be obvious because accumulated taxes would then just be the tax rate times accumulated trading gains. Trading gains are modeled by stochastic integrals that are of infinite variation if the integrator is a diffusion process and the integrand does not vanish. Actually, the (not necessarily positive) settlement payments of *futures* are taxed immediately. Thus, even when futures are traded in discrete time, their continuous settlement payments trigger tax payments of infinite variation.

However, in the case of stocks, by the deferment of tax payments, the question becomes tricky. It turns out that the key quantity which decides whether it is of (in)finite variation is the quadratic covariation of the stock price and the strategy (this process exists for the strategies we consider). Roughly speaking, if this quadratic covariation is negative, the investor reduces her position after the asset price goes up (profit-taking), and she is thus be forced to realize some of her book profits. After the price goes down, the position is increased, but there might be negative tax payments by wash sales. Surprisingly, it turns out that this is sufficient to produce infinite variation of the tax payment stream – even though the quadratic covariation of asset price and strategy, which may be seen as a proxy of tax payments when ignoring book profits from the past, is finite. By contrast, if the strategy depends positively on the price, the position is more frequently reduced after the asset price goes down. This requires less realizations of positive gains, and thereby less positive tax payments. As a result, the variation of the tax payment stream is finite. The conditions under which we show the (in)finite variation property cover a wide range of reasonable trading strategies. An interesting example with a plausible economic motivation are Constant Proportion Portfolio insurance (CPPI) strategies.

The second main goal of the article is to incorporate capital gains taxes into the concept of Constant Proportion Portfolio insurance. CPPI strategies were first studied by Perold [16], Black and Jones [4], and Black and Perold [5] in a Black-Scholes framework. They invest a constant multiple  $m \in (0,\infty)$  of some cushion in a risky asset (or index). For  $m \ge 1$ , this leads to a superlinear participation in upward price movements while guaranteeing a given part of the invested capital, even if the cushion gets completely lost. First, we extend the standard definition of CPPI strategies for frictionless markets to models with capital gains taxes. This is possible because the tax payment stream can be constructed for all càglàd trading strategies, including strategies of infinite variation which appear in the CPPI concept. The portfolio value process of the CPPI strategy is given by the unique solution of a similar SDE as in the tax-free case, but with a coefficient that depends on the past history of the portfolio value process and not only on the current portfolio value. The deferred taxes have a major impact on the portfolio value and thereby on the investment strategy. By contrast, proportional transaction costs, another major market friction, can only be incorporated into discrete time CPPI strategies, because they would explode otherwise (see Section 5 of Balder, Brandl, and Mahayni 2 for calculations of the gap risk of discrete time CPPI strategies with transaction costs). For empirical studies on the impact of capital gain taxes on the performance of CPPI products, we refer the reader to Gregory, Knox, and Ewald [8].

It can be derived from the main results of the current article that the tax payment stream of CPPI strategies is of finite variation if  $m \ge 1/(1-\alpha)$  and of infinite variation otherwise. Here,  $\alpha \in (0,1)$  denotes the tax rate. The main reason for this is that, after an upward price movement, the asset position is increased if  $m > 1/(1-\alpha)$  and reduced if  $m < 1/(1-\alpha)$ .  $m(1-\alpha)$  can be seen as the tax-adjusted multiple of the cushion that is invested in the risky asset. It is also discussed how the (in)finite variation property of the tax payment process is related to the (im)possibility to defer taxes while following a CPPI strategy. We think that the analysis of CPPI strategies provides some good intuition about the nature of a linear tax on trading gains, especially by looking at the impact of taxes on risky and riskless investments.

Finally, we analyze the tax effect of the taming of a trading strategy of infinite variation. It turns out that possible benefits by an increase of deferred tax payments are quite moderate. Based on this result, we make a conjecture about the total variation of the optimal strategy in a typical continuous time utility maximization problem.

The article is organized as follows. In Section 2, we introduce the tax payment process and state the main results of the article (Theorem 2.3 and Theorem 2.4). In Section 3, we discuss the extension of CPPI strategies to models with taxes (Theorem 3.3). The strategies satisfy the conditions of Theorem 2.3 which yields Corollary 3.7. In Section 5, we analyze tamed strategies (Theorem 5.1). The proofs of the main theorems can be found in Section 4, and the article ends with a conclusion.

# 2 Statement of main results

Throughout the article, we fix a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, P)$  satisfying the usual conditions, and S denotes the stochastic price process of the risky asset. A sequence of optional processes  $(X^n)_{n \in \mathbb{N}}$  converges to an optional process X uniformly on compacts in probability iff for every  $t \in \mathbb{R}_+$ ,  $\sup_{s \in [0,t]} |X_s^n - X_s|$  converges to 0 in probability. For a random variable Y, we set  $Y^+ := \max(Y, 0)$  and  $Y^- := \max(-Y, 0)$ .

For the convenience of the reader, we briefly repeat the construction of the continuous time tax payment process from [14]. Motivated by pathwise optimality in discrete time, it is implicitly assumed that, when reducing her position in the risky asset, the investor liquidates the (identical) shares with the lowest book profits and, when book profits become negative, shares are wash sold (i.e., sold and immediately re-purchased). By these implicit assumptions, the strategy becomes a one-dimensional process, denoted by  $\varphi$ , that only specifies the total number of identical shares that the investor holds – but not their purchasing times.

Let  $\varphi \in \mathbb{L}_+$  with  $\varphi_0 = 0$ , where  $\mathbb{L}_+$  denotes the set of nonnegative adapted processes with càglàd paths (i.e.,  $\varphi_{0+} \ge 0$  is interpreted as the initial position built-up by a purchase at price  $S_0$ ). The tax rate is denoted by  $\alpha \in (0, 1)$ . For every t, the stocks are sorted by the time spending in the portfolio and labeled by x: the bigger x the longer the residence time in the portfolio. The procedure described above is in line with defining the purchasing time of the xth share by

$$\tau_{t,x} := \sup\{u \in \mathbb{R}_+ \mid (u \le t \text{ and } \varphi_u \le \varphi_t - x) \text{ or } (u < t \text{ and } \varphi_{u+} \le \varphi_t - x)\}, \quad x \in [0, \varphi_t]. (2.1)$$

Based on  $\tau_{t,x}$ , the book profit function:  $F: \Omega \times [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$  is defined by

$$F_{\omega}(t,x) := S_t(\omega) - \inf_{\tau_{t,x}(\omega) \le u \le t} S_u(\omega) \quad \text{for } x \in [0,\varphi_t(\omega)]$$
(2.2)

and zero otherwise.  $\tau_{t,x}$  is interpreted as the purchasing time of share x ignoring later wash sells. Actually, share x is lastly wash sold at the point at which the infimum in (2.2) is attained. As a consequence of the wash sales, the shares with the shortest residence time in the portfolio possess the lowest book profit, i.e.,  $x \mapsto F_{\omega}(t, x)$  is nondecreasing.

For a nonnegative elementary strategy  $\varphi$ , i.e.,  $\varphi = \sum_{i=1}^{k} H_{i-1} \mathbb{1}_{]\!]\kappa_{i-1},\kappa_{i}]\!]$ , where  $0 = \kappa_{0} \leq \kappa_{1} \leq \ldots \leq \kappa_{k} = T$  are stopping times and  $H_{i-1}$  is  $\mathcal{F}_{\kappa_{i-1}}$ -measurable, accumulated tax payments are defined by

$$\Pi_{t}(\varphi) := \alpha \sum_{i=1}^{k} \mathbb{1}_{(\kappa_{i-1} < t)} \int_{0}^{(H_{i-1} - H_{i-2})^{-}} F(\kappa_{i-1}, x) \, dx + \alpha \sum_{i=1}^{k} \mathbb{1}_{(\kappa_{i-1} < t)} \int_{0}^{\varphi_{t}} \left( F(\kappa_{i-1} + , x) + \inf_{\kappa_{i-1} \le u \le t \land \kappa_{i}} (S_{u} - S_{\kappa_{i-1}}) \right) \land 0 \, dx,$$
(2.3)

where  $H_{-1} := 0$ . The first sum are the positive tax payments triggered by real sales. Since F is nondecreasing in x, one sells the shares with the lowest book profits. The second sum are the negative tax payments (tax credits) caused by wash sales.

Assumption 2.1. Let the asset price be a positive Itô process given by

$$S_t = s_0 + \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dB_s, \qquad s_0 > 0, \tag{2.4}$$

where  $\mu$  is a locally bounded optional process,  $\sigma$  is a locally bounded, continuous process, and B is a standard Brownian motion.

**Definition 2.2** (cf. [14]). The tax payment stream  $\varphi \mapsto \Pi(\varphi)$  is the unique continuous extension of (2.3) to  $\mathbb{L}_+$ , where continuity is understood w.r.t. the convergence "uniformly on compacts in probability" (see Theorem 2.7 of [14] for the required continuity of operator  $\Pi$  in (2.3)).

 $\Pi(\varphi)$  is optional and làglàd (i.e., paths possess finite left and right limits). We define its variation process  $\operatorname{Var}(\Pi(\varphi))$  by setting  $\operatorname{Var}(\Pi(\varphi))_t$  equal to the variation of  $\Pi(\varphi)$  on [0, t], which can be infinite. For continuous  $\varphi$  and S, which we consider here, one knows by (2.8) and (2.9) in [14] that  $\Pi(\varphi)$  is also continuous. Consequently, we have for  $\Pi = \Pi(\varphi)$ 

$$\operatorname{Var}(\Pi)_t = \sup_{n \in \mathbb{N}} \sum_{k=1}^{\infty} \left| \Pi_{t \wedge \frac{k}{n}} - \Pi_{t \wedge \frac{k-1}{n}} \right|.$$

One says that  $\Pi$  is of finite variation iff  $P(\operatorname{Var}(\Pi)_t < \infty) = 1$  for all  $t \in \mathbb{R}_+$  and of infinite variation iff  $P(\operatorname{Var}(\Pi)_t = \infty) = 1$  for some  $t \in \mathbb{R}_+$ .

The following theorems are the main results of the article. The conditions are satisfied by CPPI strategies that are constructed in Section 3.

**Theorem 2.3.** Let  $\varphi_t = g(S_t, A_t)$  for all t > 0, where  $g \in C^{2,1}(\mathbb{R}_+ \times \mathbb{R}^d)$ ,  $d \in \mathbb{N}$ , is a nonnegative function and A is an  $\mathbb{R}^d$ -valued finite variation process that is absolutely continuous with a locally bounded rate and initial value  $a_0 \in \mathbb{R}^d$ .

- (i) If  $\partial_1 g(s_0, a_0) < 0$  and the volatility process  $\sigma$  satisfies  $\sigma_0 > 0$ , then  $\Pi(\varphi)$  is of infinite variation.
- (ii) If  $\inf_{(s,a)\in K} \partial_1 g(s,a) > 0$  for all compact sets  $K \subset (0,\infty) \times \mathbb{R}^d$ , then  $\Pi(\varphi)$  is of finite variation.

Theorem 2.4. Let

$$\varphi_t = \varphi_{0+} + \int_0^t H_s \, dS_s + \int_0^t G_s \, ds \ge 0 \qquad \text{for all } t > 0, \tag{2.5}$$

where H is a continuous process with  $H_0 < 0$  and G is a locally bounded process. Assume that  $\sigma_0 > 0$ . Then,  $\Pi(\varphi)$  is of infinite variation.

**Remark 2.5.** Theorem 2.4 is a slightly stronger version of Theorem 2.3(i). This can easily be seen by applying Itô's formula to the function g and the processes S and A.

If the filtration is generated by a one-dimensional Brownian motion, (2.5) is essentially equivalent to the property that the quadratic covariation of the strategy and the asset price is negative. By contrast, if  $\partial_1 g(s_0, a_0) > 0$ , this quadratic covariation is positive (for the intuition behind these conditions, we refer to the introduction).

**Remark 2.6.** Theorems 2.3 tells us under what conditions total tax payments explode if the frequency at which tax liabilities are settled tends to infinity. Since the theorem is stated for continuous time strategies, it is not immediate what it implies for the case that the frequencies of portfolio turnovers and tax payments explode simultaneously. To provide some clarification, we state the following consequences of Theorems 2.3:

(i) Let  $\varphi$  satisfy the assumptions in Theorems 2.3(i). Let  $(\varphi^n)_{n\in\mathbb{N}}$  be a sequence of elementary strategies converging to  $\varphi$  uniformly in probability, and  $(T_k^n)_{k=0,\ldots,k_n}$ ,  $n\in\mathbb{N}$ , be a refining sequence of grids on [0,1] with stopping times  $0 = T_0^n \leq T_1^n \leq \ldots \leq T_{k_n}^n = 1$  and  $\max_{k=1,\ldots,k_n} |T_k^n - T_{k-1}^n| \to 0$ , P-a.s. Then, one has

$$\sum_{k=1}^{k_n} \left| \Pi_{T_k^n}(\varphi^n) - \Pi_{T_{k-1}^n}(\varphi^n) \right| \to \infty \quad in \ probability, \quad n \to \infty.$$

Indeed, by Theorem 2.3(i) and the continuity of  $\Pi(\varphi)$ , for every  $\varepsilon > 0$ , there is an  $n_1 \in \mathbb{N}$  s.t.

$$P\left(\sum_{k=1}^{k_{n_1}} \left| \Pi_{T_k^{n_1}}(\varphi) - \Pi_{T_{k-1}^{n_1}}(\varphi) \right| \le 1/\varepsilon + 1 \right) \le \varepsilon/2.$$

From  $\varphi^n \to \varphi$  and thus  $\Pi(\varphi^n) \to \Pi(\varphi)$  both uniformly in probability, we conclude that there exists an  $n_2 \in \mathbb{N}$  s.t.

$$P\left(\sum_{k=1}^{k_{n_1}} \left| \Pi_{T_k^{n_1}}(\varphi^n) - \Pi_{T_{k-1}^{n_1}}(\varphi^n) \right| \le 1/\varepsilon \right) \le \varepsilon \quad \forall n \ge n_2$$

Since the sequence of grids is refining the latter probability is bigger or equal to  $P\left(\sum_{k=1}^{k_n} \left| \prod_{T_k^n} (\varphi^n) - \prod_{T_{k-1}^n} (\varphi^n) \right| \le 1/\varepsilon \right)$  for all  $n \ge n_1 \lor n_2$ , and we are done.

(Furthermore, for fixed n, the continuous time wash sales of the elementary strategy  $\varphi^n$ , cf. definition (2.3), can be approximated pointwise by realizing looses along a discrete time grid.)

(ii) Now let  $\varphi$  satisfy the assumptions in Theorems 2.3(ii). Since the limit of the total variations of a sequence of processes exceeds, in general, the total variation of the pointwise limiting process, we cannot estimate the variation from above for arbitrary approximating  $\varphi^n$ . Instead, we take the particular approximating sequence

$$\varphi^n := \sum_{l=1}^{\infty} \varphi_{T_{l-1}^n} + \mathbf{1}_{(T_{l-1}^n, T_l^n]}, \quad n \in \mathbb{N},$$

where  $T_0^n := 0$  and  $T_l^n := \inf\{t > T_{l-1}^n \mid |\varphi_t - \varphi_{T_{l-1}^n} + | = 1/n\}, \ l \ge 1.$  One has

$$P\left(\liminf_{n \to \infty} \sum_{k=1}^{n} \left| \Pi_{k/n}(\varphi^n) - \Pi_{(k-1)/n}(\varphi^n) \right| < \infty \right) = 1.$$
(2.6)

(2.6) follows along the lines of the proof of Theorem 2.3(ii). Namely, in the calculations for the variation of  $\Pi(\varphi)$  along the grid  $\{0, 1/n, 2/n, \ldots, 1\}$ , the strategy  $\varphi$  can be replaced by  $\varphi^n$ , and the expectation of the additional error term is of order  $1/\sqrt{n}$ . We leave this exercise to the reader.

### **3** CPPI for models with taxes

The goal of Constant Proportion Portfolio Insurance (CPPI) strategies is to participate in upward price movements of an asset (or index) in a superlinear way, while guaranteeing a given part of the invested capital. For this, a constant multiple (usually much bigger than one) of the difference between the current portfolio value and some floor is invested in the risky asset. Since the portfolio value process itself is influenced by investor's tax liability, the standard definition of a CPPI strategy for frictionless markets has to be extended to models with taxes.

Besides the risky asset with price process S, there is a riskless bank account paying interest rate  $r = (r_t)_{t\geq 0}$ , where r is a nonnegative, locally bounded predictable process. For simplicity, we assume that gains from the bank account are taxed immediately (like dividends). Formally, the bank account is an asset with price 1 and dividend process

$$D_t = \int_0^t r_s \, ds, \quad t \ge 0,$$

i.e., the after-tax interest rate is  $(1 - \alpha)r_t$ .

**Definition 3.1.** Let  $\eta$  be a predictable process modeling the number of monetary units in the bank account, and  $\varphi \in \mathbb{L}_+$  with  $\varphi_0 = 0$  models the number of risky assets the investor holds. The portfolio value process of  $(\eta, \varphi)$  is defined as the liquidation value

$$V_t := \eta_t + \varphi_t S_t - \alpha \int_0^{\varphi_t} F(t, x) \, dx, \qquad (3.1)$$

and  $(\eta, \varphi)$  is called self-financing iff

$$V_t = V_0 + (1 - \alpha)\eta \bullet D_t + (1 - \alpha)\varphi \bullet S_t, \quad t \ge 0,$$
(3.2)

where • denotes the stochastic integral.

**Remark 3.2.**  $\alpha \int_0^{\varphi_t} F(t, x) dx$  are the deferred tax payments, i.e., tax rate times unrealized book profits that the investor would have to pay if the stock position was liquidated. By implicit wash sales, this term is always nonnegative. Thus, the liquidation value is a conservative valuation. Like in illiquid markets, there is no canonical valuation: since taxes are actually payed later and the interest rate is nonnegative, this obligation could be valued lower. The most aggressive portfolio valuation would just ignore this obligation and define the portfolio value by  $\tilde{V}_t := \eta_t + \varphi_t S_t$  (of course, with another self-financing condition).

The construction of the CPPI strategy depends on the valuation by the cushion from which a given multiple is risked. But, we think that the liquidation value V is economically the most reasonable one as it takes unavoidable future tax payments into account. This holds in particular for CPPI strategies, which may dictate the investor to liquidate her risky position quite quickly if the market goes down.

For elementary strategies, it can be seen that the self-financing condition (3.2) is equivalent to the assumption that portfolio re-groupings do not involve costs. For this, one uses the identity

$$\alpha \varphi \bullet S_t = \Pi_t + \alpha \int_0^{\varphi_t} F(t, x) \, dx \tag{3.3}$$

(see Proposition 4.1 of [14]). Since actual tax payments flow away from the portfolio and deferred tax payments are subtracted from the portfolio value by definition, only fraction  $1 - \alpha$ 

of the trading gains enters in (3.2). Nevertheless, the model is different from a tax-free model with interest rate  $(1-\alpha)r$  and stock price  $s_0\mathcal{E}((1-\alpha)(1/S) \cdot S)$ , where  $\mathcal{E}$  denotes the stochastic exponential, since the money that is only reserved but not yet spent for taxes earns some interest.

Let us now define CPPI strategies. At time zero, the amount  $\lambda V_0$  is reserved for the riskless investment, where  $\lambda \in [0, 1)$ . Since the bank account is immediately taxed, this leads to the wealth  $\lambda V_0 \exp\left((1-\alpha)\int_0^t r_s ds\right)$  at time t. In addition, multiple m of the cushion

$$C_t := V_t - \lambda V_0 \exp\left((1-\alpha) \int_0^t r_s \, ds\right) \tag{3.4}$$

is invested in the risky asset, i.e.,

$$\varphi_t = \frac{mC_t}{S_t}.\tag{3.5}$$

**Theorem 3.3.** For given  $v_0 > 0$ , there exists a unique self-financing strategy  $(\eta, \varphi)$  s.t. (3.4) and (3.5) hold with  $V_0 = v_0$ , where V is the portfolio value process of  $(\eta, \varphi)$ .  $\varphi$  can be written as

$$\varphi_t = g(S_t, A_t), \quad t > 0, \tag{3.6}$$

where

$$g(s,a) = \frac{m(1-\lambda)v_0}{s_0^{(1-\alpha)m}} s^{(1-\alpha)m-1} \exp(a)$$

and A is a one-dimensional process of finite variation with locally bounded rate.

**Remark 3.4.** For  $\alpha = 0$ , it is well-known that  $\varphi$  can be written as a smooth function of S and a process of finite variation which can be determined explicitly, see Theorem 5 of Schied [18], who considers a very general probability-free CPPI model. For  $\alpha > 0$ , the process A is not given explicitly, but with representation (3.6), one can apply Theorem 2.3.

**Remark 3.5.** One may interpret

$$\widetilde{m} := (1 - \alpha)m$$

as the effective or tax-adjusted fraction of the cushion which is invested in the risky asset. Namely, by tax payments and credits, only fraction  $1 - \alpha$  of the price movements enters into the portfolio value as defined in (3.1). Thus, in the short run or for r = 0, choosing multiplier m in the model with taxes leads to the same portfolio value process as the smaller multiplier  $\tilde{m}$ for a tax-exempt investor. But, in general, the relation is more complicated. One reason is that the deferred tax payments, for which the portfolio value is already adjusted, earn some interest. Furthermore, the bigger after-tax interest rate of the tax-exempt investor influences the cushion, cf. (3.12). These effects can only be ignored for short time intervals for which fluctuations of the risky asset dominate.

Proof of Theorem 3.3. Step 1: By

$$\varphi_t = \frac{mV_t - m\lambda v_0 \exp\left((1 - \alpha) \int_0^t r_s \, ds\right)}{S_t} \tag{3.7}$$

and

$$\eta_t = V_t - \varphi_t S_t + \alpha \int_0^{\varphi_t} F(t, x) dx$$
  
=  $(1 - m)V_t + m\lambda V_0 \exp\left((1 - \alpha) \int_0^s r_u du\right) + \alpha \int_0^{\varphi_t} F(t, x) dx,$ 

the self-financing condition (3.2) leads to the following path-dependent SDE for V:

$$V_{t} = v_{0} + (1-\alpha) \int_{0}^{t} \left( (1-m)V_{s} + m\lambda V_{0} \exp\left((1-\alpha) \int_{0}^{s} r_{u} \, du\right) + \alpha \int_{0}^{\varphi_{s} \vee 0} F^{\varphi \vee 0}(s,x) \, dx \right) r_{s} \, ds + (1-\alpha) \int_{0}^{t} \frac{m \left(V_{s} - \lambda V_{0} \exp\left((1-\alpha) \int_{0}^{s} r_{u} \, du\right)\right)}{S_{s}} \, dS_{s},$$
(3.8)

where  $\varphi$  still appearing in  $\int_0^{\varphi_s \vee 0} F^{\varphi \vee 0}(s, x) dx$  depends on V through (3.7).  $s \mapsto \int_0^{\varphi_s \vee 0} F^{\varphi \vee 0}(s, x) dx$  is a làglàd process, but in (3.8) it can of course be replaced by its leftcontinuous verification without changing anything. The right-continuous verification of  $s \mapsto \int_0^{\varphi_s \vee 0} F^{\varphi \vee 0}(s, x) dx$  is the outcome G(V) of an operator  $G : \mathbb{D} \to \mathbb{D}$ , where  $\mathbb{D}$  denotes the set of càdlàg adapted processes. Namely, it depends on the paths of  $\varphi$  and S, where  $\varphi$  depends on V, S, and r by (3.7). This means that the mapping G itself depends on the exogenous processes Sand r, but its single argument is V. The maximum with 0 is taken because F is only defined for nonnegative strategies. Later on, it is seen that the solution is nonnegative anyway. For two strategies  $\varphi^1$  and  $\varphi^2$ , one has

$$\left| \int_{0}^{\varphi_{s}^{1} \vee 0} F^{\varphi^{1} \vee 0}(s,x) \, dx - \int_{0}^{\varphi_{s}^{2} \vee 0} F^{\varphi^{2} \vee 0}(s,x) \, dx \right| \leq 3 \sup_{0 \leq u \leq s} |\varphi_{u}^{1} - \varphi_{u}^{2}| \left( \sup_{0 \leq u \leq s} S_{u} - \inf_{0 \leq u \leq s} S_{u} \right)$$

(see Lemma 3.1 of [14]). If  $V^1$  and  $V^2$  are the corresponding portfolio value processes, the RHS is smaller or equal to

$$\frac{3m\sup_{0\leq u\leq s}|V_u^1-V_u^2|}{\inf_{0\leq u\leq s}S_u}\left(\sup_{0\leq u\leq s}S_u-\inf_{0\leq u\leq s}S_u\right).$$

Thus, the operator G is functional Lipschitz in V in the sense of Protter [17], page 250, and by Theorem V.7 of [17], (3.8) has a unique strong solution (note that the theorem is stated under Assumption 2.1). A solution of (3.8) with  $\varphi \geq 0$  induces a CPPI strategy and vice versa. By (3.4), C and V only differ by an exogenous process of finite variation. One has a completely analogue path-dependent SDE for C instead of V which reads

$$C_t = C_0 + (1 - \alpha) \int_0^t \left( (1 - m)C_s + \alpha \int_0^{\varphi_s \vee 0} F^{\varphi \vee 0}(s, x) \, dx \right) r_s \, ds + (1 - \alpha) \int_0^t \frac{mC_s}{S_s} \, dS_s \tag{3.9}$$

and which possesses again a unique strong solution. Since  $r_s \int_0^{\varphi_s \vee 0} F^{\varphi \vee 0}(s, x) dx \ge 0$ , one has by standard comparison arguments that

$$\inf_{s \le t} C_s > 0 \qquad \text{for all } t \in \mathbb{R}_+, \tag{3.10}$$

i.e., there is no gap risk. This already shows that  $\varphi \in \mathbb{L}_+$  for  $\varphi_t = mC_t/S_t$  for all t > 0.

Step 2: It remains to prove that  $\varphi$  is of the form given in the theorem. With (3.9), this could be deduced from the corresponding property in the tax-free model with interest rate  $(1-\alpha)r$  and stock price  $s_0 \mathcal{E} ((1-\alpha)(1/S) \cdot S)$  by the Yoeurp-Yor formula for inhomogeneous linear SDEs (see, e.g., (5) in [10]). Alternatively, applying Itô's formula, one obtains

$$\ln(C_t) = \ln(C_0) + \int_0^t \frac{1}{C_s} dC_s - \frac{1}{2} \int_0^t \frac{1}{C_s^2} d[C, C]_s$$
  
=  $\ln((1-\lambda)v_0) + (1-\alpha) \int_0^t \left(1 - m + \frac{\alpha \int_0^{\varphi_s} F^{\varphi}(s, x) dx}{C_s}\right) r_s ds$   
+ $(1-\alpha)m \int_0^t \frac{1}{S_s} dS_s - \frac{1}{2}(1-\alpha)^2 m^2 \int_0^t \frac{1}{S_s^2} d[S, S]_s.$  (3.11)

For  $\alpha > 0$ , (3.11) is not explicit, but one can nevertheless apply the same arguments as in Schied [18] because on the RHS, C enters only in the finite variation part. Namely, the infinite variation process  $\int_0^t \frac{1}{S_s} dS_s$  appearing in (3.11) is not a function of  $S_t$ , but, by Itô's formula, it can be written as the sum of  $\ln(S_t)$  and a process of finite variation. This yields

$$C_{t} = (1 - \lambda)v_{0} \exp\left((1 - \alpha)m\ln(S_{t}) - (1 - \alpha)m\ln(s_{0}) + \frac{(1 - \alpha)m}{2}\int_{0}^{t}\frac{1}{S_{s}^{2}}d[S, S]_{s} - \frac{(1 - \alpha)^{2}m^{2}}{2}\int_{0}^{t}\frac{1}{S_{s}^{2}}d[S, S]_{s} + (1 - \alpha)\int_{0}^{t}\left(1 - m + \frac{\alpha\int_{0}^{\varphi_{s}}F^{\varphi}(s, x)\,dx}{C_{s}}\right)r_{s}\,ds\right)$$
  
$$= (1 - \lambda)v_{0}\frac{S_{t}^{(1 - \alpha)m}}{s_{0}^{(1 - \alpha)m}}\exp\left(\frac{(1 - \alpha)m(1 - (1 - \alpha)m)}{2}\int_{0}^{t}\sigma_{s}^{2}\,ds + (1 - \alpha)(1 - m)\int_{0}^{t}r_{s}\,ds\right)$$
  
$$\times \exp\left((1 - \alpha)\alpha\int_{0}^{t}\frac{\int_{0}^{\varphi_{s}}F^{\varphi}(s, x)\,dx\,r_{s}}{C_{s}}\,ds\right).$$
(3.12)

Thus, one has

$$\varphi_t = \frac{mC_t}{S_t} = g(S_t, A_t),$$

where

$$g(s,a) = \frac{m(1-\lambda)v_0}{s_0^{(1-\alpha)m}} s^{(1-\alpha)m-1} \exp(a)$$

and

$$A_t = \frac{(1-\alpha)m(1-(1-\alpha)m)}{2} \int_0^t \sigma_s^2 \, ds + (1-\alpha)(1-m) \int_0^t r_s \, ds \\ + (1-\alpha)\alpha \int_0^t \frac{\int_0^{\varphi_s} F^{\varphi}(s,x) \, dx \, r_s}{C_s} \, ds.$$

By (3.10), A possesses a locally bounded rate.

By

$$\partial_1 g(s,a) = \frac{m(1-\lambda)v_0((1-\alpha)m-1)}{s_0^{(1-\alpha)m}} s^{(1-\alpha)m-2} \exp(a) \qquad \text{for } m \neq 1/(1-\alpha),$$

it follows that for  $m < 1/(1 - \alpha)$ ,  $\varphi$  satisfies the assumptions of Theorem 2.3(i) and for  $m > 1/(1 - \alpha)$ , the assumptions of Theorem 2.3(ii) are satisfied. It remains to consider the case  $m = 1/(1 - \alpha)$ .

For  $m = 1/(1 - \alpha)$ , i.e.,  $\tilde{m} = 1$ , the effective fraction of the cushion which is risked is one, and the CPPI strategy satisfies

$$\varphi_t = \frac{(1-\lambda)v_0}{(1-\alpha)s_0} \exp\left(-\alpha \int_0^t r_s \, ds + (1-\alpha)\alpha \int_0^t \frac{\int_0^{\varphi_s} F^{\varphi}(s,x) \, dx \, r_s}{C_s} \, ds\right). \tag{3.13}$$

This means that it is *not* buy-and-hold, as in the case without taxes, but, in contrast to the case  $\tilde{m} \neq 1$ , the strategy is of finite variation. Compared to the case  $\tilde{m} = 1$  without taxes, there appear two new effects: Since the "nominal" fraction of the cushion that is risked is bigger than one, there is leverage, i.e., a part of the risky investments has to be financed by a short position in the bank account earning negative interest. In addition, there is the already mentioned effect with the deferred tax payments  $\alpha \int_0^{\varphi_s} F^{\varphi}(s, x) dx$ , which can be considered as a reserve for which the portfolio value is adjusted, but which is nevertheless available for investment. Now, the negative interest caused by leverage and the positive interest for the reserve let the wealth decrease or increase. The additional wealth is invested or disinvested in stocks. Of course, for r = 0, both effects disappear, and the strategy is again buy-and-hold.

For  $\varphi$  from (3.13) the associated tax payment stream is of finite variation. More generally, one has:

**Proposition 3.6.** For every trading strategy of finite variation, the corresponding tax payment stream is also of finite variation.

Proof. Let  $\varphi \in \mathbb{L}_+$  with  $\varphi_0 = 0$  be of finite variation. There exists a sequence of nonnegative elementary strategies  $(\varphi^n)_{n\in\mathbb{N}}$  with  $\varphi^n \to \varphi$  uniformly in probability for  $n \to \infty$  and  $\operatorname{Var}(\varphi^n) \leq \operatorname{Var}(\varphi)$  for all  $n \in \mathbb{N}$  (take the construction in the proof of Theorem II.10 of [17]). It follows immediately from the definition of  $\Pi$  for elementary strategies, see (2.3), that  $(\Pi_{t_2}(\varphi^n) - \Pi_{t_1}(\varphi^n))^+ \leq \alpha (\operatorname{Var}(\varphi^n)_{t_2} - \operatorname{Var}(\varphi^n)_{t_1}) \left( \sup_{u \in [0,t]} S_u - \inf_{u \in [0,t]} S_u \right)$  for all  $t_1 \leq t_2 \leq t$ . By  $|x| = 2x^+ - x$ , this implies that  $\operatorname{Var}(\Pi(\varphi^n))_t \leq 2\alpha \operatorname{Var}(\varphi^n)_t \left( \sup_{u \in [0,t]} S_u - \inf_{u \in [0,t]} S_u \right) - \Pi_t(\varphi^n)$ . Since, by construction,  $\Pi(\varphi^n) \to \Pi(\varphi)$  uniformly in probability, one arrives at

$$\operatorname{Var}(\Pi(\varphi))_t \leq \liminf_{n \to \infty} \operatorname{Var}(\Pi(\varphi^n))_t \leq 2\alpha \operatorname{Var}(\varphi)_t \left( \sup_{u \in [0,t]} S_u - \inf_{u \in [0,t]} S_u \right) - \Pi_t(\varphi), \quad P\text{-a.s.}$$

where the first inequality holds by Proposition A.1 of Guasoni, Lépinette, and Rásonyi [9] that captures processes with double-jumps (the proof in [9] also works if the limiting process is not ex ante of finite variation, but only làglàd).

We conclude:

**Corollary 3.7** (Corollary of Theorem 2.3). The tax process of a CPPI strategy is of infinite variation if  $m \in (0, 1/(1 - \alpha))$  and of finite variation if  $m \in [1/(1 - \alpha), \infty)$ .

**Remark 3.8** (Tax-efficieny). Whereas it is always optimal to realize losses immediately by wash sales, following a strategy  $\varphi$  may require premature realizations of positive gains. Intuitively,  $\varphi$  seems to be "tax-efficient" if positive taxes can be deferred to a large extent.

For  $m < 1/(1 - \alpha)$ , the CPPI strategy is rather tax-inefficient because the stock position is reduced after its price goes up which requires the realization of positive book profits.

For  $m > 1/(1 - \alpha)$ , the CPPI strategy is more tax-efficient. First note that there is leverage which implies that the investor immediately earns negative taxes on borrowing costs. In addition, the asset position is reduced after the price goes down. Finally, new shares are purchased after the price goes up. As a result, even when the asset price lies much above its initial price, the investor may still declare losses to the tax office for some of her newly purchased shares. This is in contrast to buy-and-hold strategies where positive tax payments can perfectly be deferred but where the possibility to declare losses disappears after a positive market development. Since in practice  $m(1 - \alpha)$  is much bigger than one, this can be seen as good news for CPPI. If the strategy were an increasing function of the asset price alone (and not also depending on a finite variation process as in Theorem 3.3), there would only be negative tax payments prior to maturity (see Section 7 of [14]).

Thus, for CPPI strategies, the (im)possibility to defer taxes is related to the (in)finite variation property of the tax payment process (cf. Corollary 3.7). Namely, since trading losses are always realized immediately by wash sells, infinite variation occurs if the strategy forces frequent realizations of positive book profits. However, this relationship between tax-efficiency and the variation of the tax process is not robust, i.e., in general, it does not hold beyond the family of CPPI strategies.

#### 4 Proofs of Theorems 2.3 and 2.4

For both proofs we need the following technical lemma.

**Lemma 4.1.** Let  $s, t \in \mathbb{R}_+$  with  $s \leq t$  and  $\varphi \in \mathbb{L}_+$  be nonincreasing on the interval [s, t]. Then, the associated tax payment process  $\Pi$  and the book profit function F satisfy

$$\Pi_t - \Pi_s = \alpha \int_0^{\varphi_s - \varphi_t} \left( F(s, x) + S_{\varphi^{-1}(\varphi_s - x)} - S_s \right) dx + \alpha \int_{\varphi_s - \varphi_t}^{\varphi_s} \left( F(s, x) + \inf_{s \le u \le t} S_u - S_s \right) \wedge 0 \, dx$$
(4.1)

where  $\varphi^{-1}(y) := \sup\{u \in [s,t] \mid \varphi_u > y\} \lor s$ .

The first integral covers the total profits of the shares which are liquidated between s and t in order to reduce the stock position – including realized losses from wash sales of these shares between s and t. The second integral are the realized losses from those wash sales between s and t for which the re-bought shares are still in the portfolio at time t. Of course, such a decomposition of the tax process is only available if the strategy is monotone on [s, t].

Intuitively, the assertion appears to be obvious, but nevertheless, it needs a verification which only makes use of the formal definition of the tax payment process and the book profit function.

*Proof. Step 1:* Let us first prove the assertion for nonincreasing elementary strategies, i.e.,  $\varphi = H_{-1} \mathbb{1}_{\{\kappa_0\}} + \sum_{i=1}^k H_{i-1} \mathbb{1}_{\{\kappa_{i-1},\kappa_i\}}$  on [s,t], where  $s = \kappa_0 \leq \kappa_1 \leq \ldots \leq \kappa_k = t$  and  $H_{-1} \geq H_0 \geq t$ 

 $\ldots \ge H_{k-1}$  with  $H_{-1} := \varphi_s$ . By definition of the tax payment process, see (2.3), one has

$$\Pi_t - \Pi_s = \alpha \sum_{i=1}^k \int_0^{H_{i-2} - H_{i-1}} F(\kappa_{i-1}, x) \, dx + \alpha \sum_{j=1}^k \int_0^{H_{j-1}} \left( F(\kappa_{j-1} + , x) + \inf_{\kappa_{j-1} \le u \le \kappa_j} (S_u - S_{\kappa_{j-1}}) \right) \wedge 0 \, dx$$

We split the integrals of the second sum into

$$\int_{0}^{H_{j-1}} = \int_{0}^{H_{j-1}-H_{j}} + \int_{H_{j-1}-H_{j}}^{H_{j-1}-H_{j+1}} + \dots + \int_{H_{j-1}-H_{k-2}}^{H_{j-1}-H_{k-1}} + \int_{H_{j-1}-H_{k-1}}^{H_{j-1}}$$

and define  $I_{j,i} := \int_{H_{j-1}-H_{i-1}}^{H_{j-1}-H_{i-1}}$  for  $j = 1, \ldots, k$ ,  $i = j + 1, \ldots, k$ ,  $I_j := \int_{H_{j-1}-H_{k-1}}^{H_{j-1}}$ .  $I_{j,i}$  are the negative tax payments triggered by wash sales between  $\kappa_{j-1}$  and  $\kappa_j$  for the shares which are liquidated at time  $\kappa_{i-1}$ .  $I_j$  are the corresponding negative tax payments for shares which are still in the portfolio at t (note that purchases "at time t" enter into  $\varphi$  not before t+). One has

$$\Pi_{t} - \Pi_{s} = \alpha \sum_{i=1}^{k} \int_{0}^{H_{i-2} - H_{i-1}} F(\kappa_{i-1}, x) \, dx + \alpha \sum_{j=1}^{k} \sum_{i=j+1}^{k} I_{j,i} + \alpha \sum_{j=1}^{k} I_{j}$$
$$= \alpha \sum_{i=1}^{k} \left( \int_{0}^{H_{i-2} - H_{i-1}} F(\kappa_{i-1}, x) \, dx + \sum_{j=1}^{i-1} I_{j,i} \right) + \alpha \sum_{j=1}^{k} I_{j}, \tag{4.2}$$

where  $\sum_{j=1}^{0} \ldots = 0$ . Let us show that

$$\int_{0}^{H_{i-2}-H_{i-1}} F(\kappa_{i-1}, x) \, dx = \int_{H_{-1}-H_{i-2}}^{H_{-1}-H_{i-1}} F(\kappa_0, x) \, dx + (H_{i-2}-H_{i-1})(S_{\kappa_{i-1}}-S_{\kappa_0}) - \sum_{j=1}^{i-1} I_{j,i} \quad (4.3)$$

for i = 1, ..., k. (4.3) says that the book profits of the shares liquidated at  $\kappa_{i-1}$  are their initial book profits plus price change minus the sum of realized (negative) profits in the periods in between. By definition of F, one has

$$\int_{H_0-H_{i-2}}^{H_0-H_{i-1}} F(\kappa_0+,x) \, dx = \int_{H_0-H_{i-2}}^{H_0-H_{i-1}} F(\kappa_0,x+H_{-1}-H_0) \, dx = \int_{H_{-1}-H_{i-2}}^{H_{-1}-H_{i-1}} F(\kappa_0,x) \, dx, \quad (4.4)$$

i.e., by real sells, shares are shifted to the left, and by Lemma 4.2 of [14],

$$\int_{H_0-H_{i-2}}^{H_0-H_{i-1}} F(\kappa_1, x) \, dx = \int_{H_0-H_{i-2}}^{H_0-H_{i-1}} F(\kappa_0+, x) \, dx + (H_{i-2}-H_{i-1})(S_{\kappa_1}-S_{\kappa_0}) - I_{1,i}.$$
(4.5)

By iterating the calculations in (4.4)/(4.5) up to time  $\kappa_{i-1}$ , one arrives at (4.3). Summing up over all *i* yields

$$\alpha \sum_{i=1}^{k} \left( \int_{0}^{H_{i-2}-H_{i-1}} F(\kappa_{i-1}, x) \, dx + \sum_{j=1}^{i-1} I_{j,i} \right) = \alpha \int_{0}^{\varphi_s - \varphi_t} \left( F(s, x) + S_{\varphi^{-1}(\varphi_s - x)} - S_s \right) \, dx,$$

using that  $S_{\varphi^{-1}(\varphi_s-x)} = \kappa_{i-1}$  for  $x \in (H_{-1} - H_{i-2}, H_{-1} - H_{i-1}]$ . To complete the proof for elementary strategies, by (4.2), it remains to show that

$$\int_{H_{-1}-H_{k-1}}^{H_{-1}} \left( F(\kappa_0, x) + \inf_{\kappa_0 \le u \le \kappa_k} S_u - S_{\kappa_0} \right) \wedge 0 \, dx = \sum_{j=1}^k I_j.$$
(4.6)

(4.6) tells us how negative tax payments by wash sells accumulate over different periods. By definition of F, one has

$$\int_{H_{-1}-H_{k-1}}^{H_{-1}} \left( F(\kappa_0, x) + \inf_{\kappa_0 \le u \le \kappa_k} S_u - S_{\kappa_0} \right) \wedge 0 \, dx$$
  
= 
$$\int_{H_0-H_{k-1}}^{H_0} \left( F(\kappa_0 +, x) + \inf_{\kappa_0 \le u \le \kappa_k} S_u - S_{\kappa_0} \right) \wedge 0 \, dx.$$
(4.7)

For  $x \in (H_0 - H_{k-1}, H_0]$  fixed, we distinguish two cases: For  $F(\kappa_0 +, x) + \inf_{\kappa_0 \le u \le \kappa_1} S_u - S_{\kappa_0} \ge 0$ , one has

$$\left( F(\kappa_0 +, x) + \inf_{\kappa_0 \le u \le \kappa_k} S_u - S_{\kappa_0} \right) \wedge 0 = \left( F(\kappa_0 +, x) + \inf_{\kappa_1 \le u \le \kappa_k} S_u - S_{\kappa_0} \right) \wedge 0$$
$$= \left( F(\kappa_0 +, x) + S_{\kappa_1} - S_{\kappa_0} + \inf_{\kappa_1 \le u \le \kappa_k} S_u - S_{\kappa_1} \right) \wedge 0$$

and for  $F(\kappa_0+, x) + \inf_{\kappa_0 \le u \le \kappa_1} S_u - S_{\kappa_0} < 0$ ,

$$\left( F(\kappa_0 +, x) + \inf_{\kappa_0 \le u \le \kappa_k} S_u - S_{\kappa_0} \right) \wedge 0 = F(\kappa_0 +, x) + \inf_{\kappa_0 \le u \le \kappa_1} S_u - S_{\kappa_0} + \left( \inf_{\kappa_1 \le u \le \kappa_k} S_u - \inf_{\kappa_0 \le u \le \kappa_1} S_u \right) \wedge 0.$$

Putting together, one arrives at

$$\left( F(\kappa_0+,x) + \inf_{\kappa_0 \le u \le \kappa_k} S_u - S_{\kappa_0} \right) \wedge 0$$

$$= \left( F(\kappa_0+,x) + \inf_{\kappa_0 \le u \le \kappa_1} S_u - S_{\kappa_0} \right) \wedge 0$$

$$+ \left( \max \left( F(\kappa_0+,x) + S_{\kappa_1} - S_{\kappa_0}, S_{\kappa_1} - \inf_{\kappa_0 \le u \le \kappa_1} S_u \right) + \inf_{\kappa_1 \le u \le \kappa_k} S_u - S_{\kappa_1} \right) \wedge 0.$$

Again by Lemma 4.2 of [14], it follows that  $\max (F(\kappa_0 +, x) + S_{\kappa_1} - S_{\kappa_0}, S_{\kappa_1} - \inf_{\kappa_0 \le u \le \kappa_1} S_u) = F(\kappa_1, x)$  and thus

$$\int_{H_0-H_{k-1}}^{H_0} \left( F(\kappa_0+,x) + \inf_{\kappa_0 \le u \le \kappa_k} S_u - S_{\kappa_0} \right) \wedge 0 \, dx$$
  
=  $\underbrace{\int_{H_0-H_{k-1}}^{H_0} \left( F(\kappa_0+,x) + \inf_{\kappa_0 \le u \le \kappa_1} S_u - S_{\kappa_0} \right) \wedge 0 \, dx}_{=I_1}$   
+  $\int_{H_0-H_{k-1}}^{H_0} \left( F(\kappa_1,x) + \inf_{\kappa_1 \le u \le \kappa_k} S_u - S_{\kappa_1} \right) \wedge 0 \, dx.$  (4.8)

Iterating the calculations in (4.7)/(4.8) up to time  $\kappa_k$  yields (4.6).

Step 2: Now we proceed to general strategies. Let  $\varphi \in \mathbb{L}_+$  be nonincreasing on [s, t]. There exists a sequence  $(\varphi^n)_{n \in \mathbb{N}}$  of elementary strategies that are nonincreasing on the interval [s, t] s.t.  $\varphi^n \to \varphi$  uniformly in probability (cf., e.g., the construction in the proof of Theorem II.10 in [17]). Then, on both sides of (4.1), the terms associated with the strategies  $\varphi^n$  converge in probability to those associated with  $\varphi$ . On the LHS, the convergence holds just by definition of  $\Pi(\varphi)$ . On the RHS, one can estimate the integrands against each other with slightly shifted values for the variable x. Namely, the book profit function is bounded by  $\sup_{u \in [0,t]} S_u - S_0 < \infty$ . By a shift in x, one can ensure that the liquidation times  $\varphi^{-1}(\varphi_s - x)$  of the limiting strategy are approximated from the right by those of the elementary strategies, and then one can use the right-continuity of the paths of S. To estimate the book profit functions against each other, one can use (3.2) of [14].

Proof of Theorem 2.4. Outline of the proof: The main idea of the proof is the following. At each grid point (k-1)/n, k = 2, ..., n, one considers the time reversal of the trading strategy, i.e., the process  $t \mapsto \varphi_{(k-1)/n-t} - \varphi_{(k-1)/n}$ . By continuity of  $\sigma$  and H, after a random time change, this process is essentially a standard Brownian motion  $\widetilde{B}$  on a small time scale. Thus, the purchasing time  $\tau_{(k-1)/n,1/\sqrt{n}}$  of share  $1/\sqrt{n}$  is (k-1)/n minus the first time  $\widetilde{B}$  hits the level  $-1/\sqrt{n}$ . If H were a negative constant  $H_0$ , the book profit of this share would essentially be

$$F\left(\frac{k-1}{n}, \frac{1}{\sqrt{n}}\right) = \sup_{\substack{\tau_{(k-1)/n, 1/\sqrt{n}} \le u \le (k-1)/n}} (S_{(k-1)/n} - S_u) \\ \approx -\frac{1}{H_0} \sup_{\substack{\tau_{(k-1)/n, 1/\sqrt{n}} \le u \le (k-1)/n}} (\varphi_u - \varphi_{(k-1)/n}) \\ \approx \frac{1}{|H_0|} \sup_{0 \le t \le \inf\{s \ge 0 \mid \tilde{B}_s = -1/\sqrt{n}\}} \tilde{B}_t.$$
(4.9)

By the scaling property of Brownian motion, the last line coincides in distribution with

$$\frac{1}{|H_0|\sqrt{n}} \sup_{0 \le t \le \inf\{s \ge 0 \mid \widetilde{B}_s = -1\}} \widetilde{B}_t =: \frac{1}{|H_0|\sqrt{n}} Z.$$
(4.10)

For every  $a \in \mathbb{R}_+$ , one has P(Z > a) = 1/(1+a). This means that the book profit of share  $1/\sqrt{n}$  is of order  $1/\sqrt{n}$  with a nonintegrable prefactor. Since the reduction of the asset position on the interval ((k-1)/n, k/n] is of order  $1/\sqrt{n}$  (and shares with the lowest book profit are sold first), this leads to positive tax payments of order 1/n with a nonintegrable prefactor. Since the tax payments in different periods (that are of course not stochastically independent) are strongly mixing, it can be shown that this leads to  $\sum_{k=1}^{n} (\prod_{k/n} (\varphi) - \prod_{(k-1)/n} (\varphi))^+ \to \infty$ . It is crucial that  $H_0 < 0$ . Otherwise, the supremum in the second line of (4.9) would turn into an infimum, and the approximation of  $F((k-1)/n, 1/\sqrt{n})$  would be bounded by  $1/(|H_0|\sqrt{n})$ . The tax payments described above would still be of order 1/n, but now with integrable prefactor which leads to finite variation (cf. the proof of Theorem 2.3(ii)).

So far, this is only heuristics. In the following steps, we work out this vague idea in detail. Of course, there appear several difficulties. First, the time reversal of the integral  $t \mapsto \varphi_{0+} + \int_0^t H_s \sigma_s \, dB_s \approx \varphi_t$  is in general no time-changed Brownian motion, unless the integrand is not deterministic. But, using the continuity of  $\sigma$  and H, it can be shown that

the process behaves similar to Brownian motion on a small time scale. In addition, H is not a constant, while the non-integrability of (4.10) is caused by large values of the first hitting times of the level -1. Furthermore, book profits are not realized at grid points but continuously, and they fluctuate also between (k-1)/n and k/n (this is analyzed by using the decomposition in Step 1). Most importantly, between two grid points, there is an overlapping with negative tax payments triggered by wash sells. It has to be shown that given some time scale 1/n, there are enough intervals ((k-1)/n, k/n],  $k = 1, \ldots, n$  in which realizations of large book profits dominate the overall tax payments. Of course,  $\sum_{k=1}^{n} (\prod_{k/n}(\varphi) - \prod_{(k-1)/n}(\varphi))^+ \to \infty$  implies that  $\sum_{k=1}^{n} (\prod_{k/n}(\varphi) - \prod_{(k-1)/n}(\varphi))^- \to \infty$  because  $\prod_{1}(\varphi)$  is known to be finite. Thus, there would also be intervals with dominating negative tax payments triggered by wash sales.

Step 1: We start with a decomposition of the tax payment process which turns out to be useful for both the current proof and that of Theorem 2.3(i). This step holds without any restriction on the strategy  $\varphi \in \mathbb{L}_+$ .

Let  $n \in \mathbb{N}$  and  $k \in \{1, ..., n\}$ . For each period ((k-1)/n, k/n], we decompose  $\varphi$  into the number of "old shares" that are already in the portfolio at time (k-1)/n and "new shares" that are purchased afterwards:

$$\varphi_t^{\text{old}} := \begin{cases} \varphi_t & \text{for } t \le (k-1)/n \\ \inf_{(k-1)/n \le u \le t} \varphi_u & \text{for } t > (k-1)/n \end{cases} \quad \text{``old shares'', purchased before } \frac{k-1}{n} \quad (4.11)$$

and

$$\varphi_t^{\text{new}} := \begin{cases} 0 & \text{for } t \le (k-1)/n \\ \varphi_t - \inf_{(k-1)/n \le u \le t} \varphi_u & \text{for } t > (k-1)/n \end{cases}$$
 "new shares", purchased after  $\frac{k-\frac{\nu}{n}}{n}$ 4.12)

The precise mathematical meaning of this property is the following: For all  $t \ge (k-1)/n$ ,  $x \le \varphi_t$ , the equivalence

$$x \le \varphi_t^{\text{new}} \Leftrightarrow \tau_{t,x} \ge (k-1)/n$$
 (4.13)

holds, i.e., shares with label smaller than  $\varphi_t^{\text{new}}$  are purchased after (k-1)/n and shares with bigger label prior to that. Thus,  $\varphi_t^{\text{new}}$  is the Lebesgue-measure of the set  $\{x \mid \tau_{t,x} \ge (k-1)/n\}$ . One has that  $\varphi^{\text{old}}, \varphi^{\text{new}} \ge 0$  and  $\varphi^{\text{old}} + \varphi^{\text{new}} = \varphi$ . Denote the corresponding book profit

One has that  $\varphi^{\text{old}}, \varphi^{\text{new}} \ge 0$  and  $\varphi^{\text{old}} + \varphi^{\text{new}} = \varphi$ . Denote the corresponding book profit functions by  $F^{\text{old}}$  and  $F^{\text{new}}$  and the purchasing times from (2.1) by  $\tau^{\text{old}}$  and  $\tau^{\text{new}}$ . We have that  $\varphi^{\text{old}}$  is nonincreasing on [(k-1)/n, k/n] (which allows us to apply Lemma 4.1) and  $F^{\text{new}}((k-1)/n, \cdot) = 0$ .

Let us show that

$$\Pi(\varphi) = \Pi(\varphi^{\text{old}}) + \Pi(\varphi^{\text{new}}) \tag{4.14}$$

(Note that  $\Pi$  is in general only subadditive in the strategy, see Proposition 2.13 in [14]). For  $t \leq (k-1)/n$ , (4.14) is obvious. Let t > (k-1)/n and  $x \leq \varphi_t^{\text{new}}$ . By (4.13), the share with label x in the book profit function F is purchased not before (k-1)/n. In addition, it is purchased at the same time as the share with label x in  $F^{\text{new}}$ , i.e.,  $\tau_{t,x} = \tau_{t,x}^{\text{new}}$ . Indeed, by definition,  $\tau_{t,x}$  is not smaller than the biggest u (or u+) at which  $\inf_{(k-1)/n \leq u \leq t} \varphi_u$  is attained. But, between this u and t, the increments of  $\varphi$  and  $\varphi^{\text{new}}$  coincide. By construction, this implies  $\tau_{t,x} = \tau_{t,x}^{\text{new}}$  and thus  $F(t, x) = F^{\text{new}}(t, x)$ .

Now let t > (k-1)/n and  $x > \varphi_t^{\text{new}}$ . The share x of F is purchased before (k-1)/n and at the same time as the share  $x - \varphi_t^{\text{new}}$  of  $F^{\text{old}}$ , i.e.,  $\tau_{t,x} = \tau_{t,x-\varphi_t^{\text{new}}}^{\text{old}}$  (cf. again the definition of the

purchasing time and note that  $\varphi_u = \varphi_u^{\text{old}}$  for all  $u \leq (k-1)/n$  and  $x - \varphi_t = x - \varphi_t^{\text{new}} - \varphi_t^{\text{old}}$ . Thus  $F(t, x) = F^{\text{old}}(t, x - \varphi_t^{\text{new}})$ . Putting together, one obtains

$$F(t,x) = 1_{(x \le \varphi_t^{\text{new}})} F^{\text{new}}(t,x) + 1_{(x > \varphi_t^{\text{new}})} F^{\text{old}}(t,x - \varphi_t^{\text{new}}), \quad t > (k-1)/n, \ x \in (0,\varphi_t]$$

Thus, the deferred tax payments split into

$$\alpha \int_0^\infty F(t,x) \, dx = \alpha \int_0^\infty F^{\text{old}}(t,x) \, dx + \alpha \int_0^\infty F^{\text{new}}(t,x) \, dx. \tag{4.15}$$

By  $\varphi \bullet S_t = \varphi^{\text{old}} \bullet S_t + \varphi^{\text{new}} \bullet S_t$  and  $\Pi_t(\psi) = \alpha \psi \bullet S_t - \alpha \int_0^\infty F^{\psi}(t, x) dx$  (see Proposition 4.1 in [14], which holds by continuity not only for elementary strategies, but for all strategies from  $\mathbb{L}_+$ ), we arrive at (4.14).

Since  $\varphi_{(k-1)/n}^{\text{new}} = 0$ , we have the trivial estimate  $\int_0^\infty F^{\text{new}}(k/n, x) dx \leq \varphi_{k/n}^{\text{new}}(S_{k/n} - \inf_{(k-1)/n \leq u \leq k/n} S_u)$  which yields, again by Proposition 4.1 in [14],

$$\Pi_{\frac{k}{n}}(\varphi^{\text{new}}) - \Pi_{\frac{k-1}{n}}(\varphi^{\text{new}}) \ge -\alpha\varphi_{k/n}^{\text{new}}(S_{k/n} - \inf_{(k-1)/n \le u \le k/n}S_u) + \alpha \int_{((k-1)/n, k/n]} \varphi_t^{\text{new}} \, dS_t.$$
(4.16)

Let

$$A_{n,k} := \left\{ F\left(\frac{k-1}{n}, \frac{1}{\sqrt{n}}\right) \ge \frac{1}{\sqrt{n}}, \quad \inf_{(k-1)/n \le u \le k/n} S_u - S_{\frac{k-1}{n}} \ge -\frac{1}{\sqrt{n}}, \quad \varphi_{\frac{k}{n}}^{\text{old}} - \varphi_{\frac{k-1}{n}}^{\text{old}} \le -\frac{2}{\sqrt{n}} \right\} \subset \Omega.$$

On the set  $A_{n,k}$ , one can easily estimate the negative tax payments of  $\varphi^{\text{old}}$  between (k-1)/nand k/n, while it turns out that there are still enough positive tax payments when restricting to intervals ((k-1)/n, k/n], k = 1, ..., n with  $\omega \in A_{n,k}$ . Especially, on  $A_{n,k}$ , there is a large enough reduction of old shares.

By Lemma 4.1 applied to  $\tilde{\varphi} = \varphi^{\text{old}}$  and by the monotonicity of  $x \mapsto F^{\text{old}}((k-1)/n, x)$ , one has that

$$\Pi_{k/n}(\varphi^{\text{old}}) - \Pi_{(k-1)/n}(\varphi^{\text{old}}) \ge \underbrace{\alpha \left( F^{\text{old}} \left( \frac{k-1}{n}, \frac{1}{\sqrt{n}} \right) - \frac{1}{\sqrt{n}} \right) \frac{1}{\sqrt{n}}}_{\text{lower bound for taxes triggered by "real" sales}} - \frac{\alpha}{n} \quad \text{on } A_{n,k}(4.17)$$

Namely, on the set  $A_{n,k}$ , shares possessing labels greater than  $1/\sqrt{n}$  at time (k-1)/n are not affected at all by wash sales, and tax rebates of other shares are bounded by  $\alpha/\sqrt{n}$  per share. This gives  $\alpha/n$  as an upper bound for tax rebates of  $\varphi^{\text{old}}$ . Putting (4.14), (4.16), and (4.17) together and using  $F^{\text{old}}((k-1)/n, \cdot) = F((k-1)/n, \cdot)$ , one obtains

$$(\Pi_{k/n}(\varphi) - \Pi_{(k-1)/n}(\varphi))^{+} \geq 1_{A_{n,k}}(\Pi_{k/n}(\varphi) - \Pi_{(k-1)/n}(\varphi))$$
$$\geq 1_{A_{n,k}}\left(\alpha F\left(\frac{k-1}{n}, \frac{1}{\sqrt{n}}\right)\frac{1}{\sqrt{n}} - \frac{2\alpha}{n} - \alpha \varphi_{k/n}^{\operatorname{new}}(S_{k/n} - \inf_{(k-1)/n \leq u \leq k/n}S_{u}) + \alpha \int_{((k-1)/n, k/n]} \varphi_{t}^{\operatorname{new}} dS_{t}\right).$$
(4.18)

This lower estimate for the positive part of the difference between positive and negative tax payments on the interval ((k-1)/n, k/n] is used in the next steps. It turns out that

 $1_{A_{n,k}} \alpha F((k-1)/n, 1/\sqrt{n})/\sqrt{n}$ , i.e., tax payments triggered by realizations of "old" book profits, is the dominating term on the RHS of (4.18).

Step 2: We want to prove that  $\Pi(\varphi)$  is of infinite variation already "shortly" after time zero. For this, we estimate the sum over k of the RHS of (4.18) from below. We need properties like that the local volatilities of the stock price and the strategy are bounded and bounded away from zero. Under the assumptions of the theorem, these properties do not hold globally. Moreover, there does not even exist a nonnegative trading strategy, i.e., a strategy for which the tax payment stream is declared, which satisfies these properties globally. However, one can do some localization that is based on the observation that near to zero the properties hold with high probability. In the following, we work out these arguments in detail.

Let  $\sigma_{\min} := \sigma_0/2$ ,  $\sigma_{\max} := 2\sigma_0$ ,  $h_{\max} := H_0/2$ ,  $h_{\min} := 2H_0$ , and

$$C_{\delta} := \{ \omega \mid \varphi_t(\omega) \ge \varphi_{0+}/2, \ \sigma_t(\omega) \in [\sigma_{\min}, \sigma_{\max}], \ H_t(\omega) \in [h_{\min}, h_{\max}] \quad \forall t \in (0, \delta] \}, \quad \delta > 0.$$

Under the assumptions of the theorem, one has  $\varphi_{0+} > 0$ . Thus, by continuity of  $\varphi$ ,  $\sigma$ , and H, one has  $P(C_{\delta}) \to 1$  for  $\delta \to 0$ . In addition, for every  $\eta > 0$ , there exist a mapping  $\xi : \mathbb{N} \to \mathbb{R}_+ \setminus \{0\}$  and  $K \in \mathbb{R}_+$  s.t.

$$P(D_{\eta}) \geq 1 - \eta$$
, where

$$D_{\eta} := \{ |\mu| \lor |G| \le K \} \cap \bigcap_{j \in \mathbb{N}} \{ |\sigma_s - \sigma_t| \lor |H_s - H_t| \le 1/j \quad \forall s, t \in [0, 1], \ |s - t| \le \xi(j) \}.$$

Thus, it is sufficient to prove that for all  $\delta, \eta > 0$ 

$$P\left(\{\operatorname{Var}(\Pi(\varphi))_{\delta} < \infty\} \cap C_{\delta} \cap D_{\eta}\right) = 0.$$

$$(4.19)$$

To prove (4.19), given  $\delta, \eta$ , we can modify  $\sigma$ , H,  $\mu$ , and G outside  $(C_{\delta} \cap D_{\eta}) \times [0, \delta]$  in a predictable manner s.t. the *modified* quantities satisfy

(I)  $0 < \sigma_{\min} \le \sigma \le \sigma_{\max} < \infty$ 

(II) 
$$-\infty < h_{\min} \le H \le h_{\max} < 0$$

(III) 
$$|\sigma_s - \sigma_t| \vee |H_s - H_t| \le 1/j \quad \forall s, t \in [0, 1], \ |s - t| \le \xi(j)$$

(IV) 
$$|\mu| \lor |G| \le K$$

everywhere, for some mapping  $\xi : \mathbb{N} \to \mathbb{R}_+ \setminus \{0\}$  and some  $K \in \mathbb{R}_+ \setminus \{0\}$ .

Now, we fix some  $\delta, \eta \in (0, 1)$ . In the following, we denote by  $\sigma$ ,  $\mu$ , S, H, G, and  $\varphi$  the modified quantities that satisfy Properties (I) to (IV) on  $\Omega \times [0, 1]$ . The modifications coincide with the original quantities on  $(C_{\delta} \cap D_{\eta}) \times [0, \delta]$ . Outside this set, the strategy  $\varphi$  may become negative, but the following estimates still make sense because  $F^{\varphi}(t, x)$  from (2.2) is declared for all t, x with  $0 \leq x \leq \varphi_t$ .

Step 3: By (2.4) and (2.5), the strategy can be written as

$$\varphi_t = \varphi_{0+} + \int_0^t H_s \sigma_s \, dB_s + \int_0^t \left( H_s \mu_s + G_s \right) \, ds, \qquad t > 0. \tag{4.20}$$

Define

$$M_{n,k,l}^{1} := \left\{ \sup_{(k-1-l)/n \le u \le (k-1)/n} \left| \sigma_{(k-1-l)/n} (B_{(k-1)/n} - B_{u}) - (S_{(k-1)/n} - S_{u}) \right| > \frac{1}{\sqrt{n}} \right\}$$
for  $k = l+1, \dots, n$ 

and

$$M_{n,k,l}^2 := \left\{ \sup_{(k-1-l)/n \le u \le (k-1)/n} \left| H_{(k-1-l)/n} \sigma_{(k-1-l)/n} (B_{(k-1)/n} - B_u) - (\varphi_{(k-1)/n} - \varphi_u) \right| > \frac{1}{2\sqrt{n}} \right\}$$
for  $k = l+1, \dots, n$ .

This means that for each triple (n, k, l), one considers piecewise constant approximations of the integrands  $\sigma$  and H in (2.4) and (4.20). The larger l, the larger one has to choose n to obtain good approximations of the price process and the strategy.

We want to express the book profit  $F((k-1)/n, 1/\sqrt{n})$  appearing in (4.18) by the time reversed process

$$\widehat{B}_t := B_{(k-1)/n} - B_{(k-1)/n-t}, \quad t \in [0, (k-1)/n],$$

that is again a Brownian motion. Throughout the proof, a triple (n,k,l) satisfies  $1 \leq l \leq k-1 \leq n-1.$  One has

$$F\left(\frac{k-1}{n},\frac{1}{\sqrt{n}}\right) = \sup_{\tau_{(k-1)/n,1/\sqrt{n}} \le u \le (k-1)/n} \left(S_{(k-1)/n} - S_{u}\right)$$

$$\geq \sup_{\tau_{(k-1)/n,1/\sqrt{n}} \lor (k-1-l)/n \le u \le (k-1)/n} \left(S_{(k-1)/n} - S_{u}\right)$$

$$\geq 1_{\Omega \setminus M_{n,k,l}^{1}} \sigma_{(k-1-l)/n} \sup_{\tau_{(k-1)/n,1/\sqrt{n}} \lor (k-1-l)/n \le u \le (k-1)/n} \left(B_{(k-1)/n} - B_{u}\right) - \frac{1}{\sqrt{n}}$$

$$\geq 1_{\Omega \setminus M_{n,k,l}^{1}} \sigma_{(k-1-l)/n} \sup_{0 \le t \le ((k-1)/n - \tau_{(k-1)/n,1/\sqrt{n}}) \land l/n} \widehat{B}_{t} - \frac{1}{\sqrt{n}}$$

$$\geq 1_{\Omega \setminus (M_{n,k,l}^{1} \cup M_{n,k,l}^{2})} \sigma_{(k-1-l)/n} \sup_{0 \le t \le \inf\{s \ge 0 \mid \widehat{B}_{s} = 1/(2H_{(k-1-l)/n}\sigma_{(k-1-l)/n}\sqrt{n})\} \land l/n} \widehat{B}_{t} - \frac{1}{\sqrt{n}}$$

$$\geq 1_{\Omega \setminus (M_{n,k,l}^{1} \cup M_{n,k,l}^{2})} \sigma_{\min} \sup_{0 \le t \le \inf\{s \ge 0 \mid \widehat{B}_{s} = 1/(2H_{\min}\sigma_{\max}\sqrt{n})\} \land l/n} \widehat{B}_{t} - \frac{1}{\sqrt{n}}$$
on the set  $\{\varphi_{(k-1)/n} \ge \frac{1}{\sqrt{n}}\}.$  (4.21)

By the scaling property of Brownian motion, the process

$$\widehat{B}_t^n := 2|h_{\min}|\sigma_{\max}\sqrt{n} \ \widehat{B}_{t/(4h_{\min}^2\sigma_{\max}^2n)}$$

is again a standard Brownian motion, and we conclude

$$F\left(\frac{k-1}{n},\frac{1}{\sqrt{n}}\right) \geq 1_{\Omega \setminus (M_{n,k,l}^1 \cup M_{n,k,l}^2)} \frac{\sigma_{\min}}{2|h_{\min}|\sigma_{\max}\sqrt{n}} \sup_{0 \le t \le \inf\{s \ge 0 \mid \hat{B}_s^n = -1\} \wedge 4h_{\min}^2 \sigma_{\max}^2 l} \widehat{B}_t^n - \frac{1}{\sqrt{n}}$$
on the set  $\{\varphi_{(k-1)/n} \ge \frac{1}{\sqrt{n}}\}.$ 

Define

$$M_{n,k}^3 := \left\{ \sup_{(k-1)/n \le u \le k/n} \left| \sigma_{(k-1)/n} (B_u - B_{(k-1)n}) - (S_u - S_{(k-1)/n}) \right| > \frac{1}{2\sqrt{n}} \right\}$$

and

$$M_{n,k}^4 := \left\{ \sup_{(k-1)/n \le u \le k/n} \left| H_{(k-1)/n} \sigma_{(k-1)/n} (B_u - B_{(k-1)/n}) - (\varphi_u - \varphi_{(k-1)/n}) \right| > \frac{1}{\sqrt{n}} \right\}.$$

We conclude

$$\begin{split} &1_{A_{n,k}} \frac{1}{\sqrt{n}} F\left(\frac{k-1}{n}, \frac{1}{\sqrt{n}}\right) \\ &\geq 1_{\Omega \setminus (M_{n,k,l}^1 \cup M_{n,k}^2 \cup M_{n,k}^3)} \frac{\sigma_{\min}}{2|h_{\min}|\sigma_{\max}n} 1_{D_{n,k}} \sup_{0 \leq t \leq \inf\{s \geq 0 \ | \ \widehat{B}_s^n = -1\} \wedge 4h_{\min}^2 \sigma_{\max}^2 l} \widehat{B}_t^n - \frac{2}{n} \end{split}$$

on  $\{\varphi_{(k-1)/n} \ge 1/\sqrt{n}\}$ , where  $A_{n,k}$  is defined in Step 1 and

$$D_{n,k} = \left\{ \inf_{(k-1)/n \le u \le k/n} B_u - B_{(k-1)/n} \ge -\frac{1}{2\sigma_{\max}\sqrt{n}}, \\ \sup_{(k-1)/n \le u \le k/n} B_u - B_{(k-1)/n} \ge \frac{3}{|h_{\max}|\sigma_{\min}\sqrt{n}} \right\}.$$

Let

$$I_{1}^{n,k,l} := \frac{\sigma_{\min}}{2|h_{\min}|\sigma_{\max}} \mathbb{1}_{D_{n,k}} \sup_{0 \le t \le \inf\{s \ge 0 \mid \widehat{B}_{s}^{n} = -1\} \land 4h_{\min}^{2} \sigma_{\max}^{2} l} \widehat{B}_{t}^{n}$$
(4.22)

and

$$I_2^{n,k,l} := I_1^{n,k,l} \mathbf{1}_{M_{n,k,l}^1 \cup M_{n,k,l}^2 \cup M_{n,k}^3 \cup M_{n,k}^4}, \tag{4.23}$$

 ${\rm i.e.},$ 

$$1_{A_{n,k}} \frac{1}{\sqrt{n}} F\left(\frac{k-1}{n}, \frac{1}{\sqrt{n}}\right) \ge \frac{I_1^{n,k,l} - I_2^{n,k,l} - 2}{n} \quad \text{on } \left\{\varphi_{(k-1)/n} \ge \frac{1}{\sqrt{n}}\right\}.$$
(4.24)

Define

$$I_{3}^{n,k} := -\varphi_{k/n}^{\text{new}} \left( S_{k/n} - \inf_{(k-1)/n \le u \le k/n} S_{u} \right) + \int_{((k-1)/n, k/n]} \varphi_{t}^{\text{new}} \, dS_{t}.$$

Putting (4.18) and (4.24) together, one obtains that on the set  $\{\omega \mid \varphi_t(\omega) \ge 0, \forall t \in [0,1]\}$ , where the process  $\Pi(\varphi)$  is declared,

$$(\Pi_{k/n}(\varphi) - \Pi_{(k-1)/n}(\varphi))^{+} \ge \frac{\alpha}{n} \left( I_{1}^{n,k,l} - I_{2}^{n,k,l} \right) \mathbf{1}_{\left\{ \varphi_{(k-1)/n} \ge \frac{1}{\sqrt{n}} \right\}} - \frac{4\alpha}{n} + \alpha \mathbf{1}_{A_{n,k}} I_{3}^{n,k}.$$
(4.25)

Step 4: As already mentioned in the outline of the proof, one has  $E\left(\sup_{0\leq t\leq \inf\{s\geq 0 \mid \widehat{B}_{s}^{n}=-1\}}\widehat{B}_{t}^{n}\right) = \infty$ , which implies

$$\lim_{l \to \infty} E\left(\sup_{0 \le t \le \inf\{s \ge 0 \mid \widehat{B}^n_s = -1\} \land 4h^2_{\min}\sigma^2_{\max}l} \widehat{B}^n_t\right) \to \infty, \quad l \to \infty.$$

Since the event  $D_{n,k}$  is independent of  $\widehat{B}^n$  and possesses a positive probability, one arrives at

$$E(I_1^{2l,2l,l}) \to \infty, \quad l \to \infty.$$
 (4.26)

Note that the distribution of  $I_1^{n,k,l}$  does not depend on n and k. This means that by truncating the first hitting time of  $\hat{B}^n$  at  $4h_{\min}^2 \sigma_{\max}^2 l$ , the random factor in (4.22) becomes integrable, but by choosing l large, its expectation can be made arbitrary large. Then, given some  $l \in \mathbb{N}$ , it is shown that the expectation of  $I_2^{n,k,l}$ , the error term introduced in (4.23), tending to get bigger for bigger l, becomes arbitrary small for  $n \to \infty$ .

Note that if H were positive, the supremum in the sixth line of (4.21) would be bounded from above by  $1/(2|H_{(k-1-l)/n}|\sigma_{(k-1-l)/n}\sqrt{n})$ , which is smaller than  $1/(2|h_{\max}|\sigma_{\min}\sqrt{n})$ .

By construction,  $I_1^{n,k_1,l}$  and  $I_1^{n,k_2,l}$  are independent for  $|k_2 - k_1| \ge 4h_{\min}^2 \sigma_{\max}^2 l + 1$ . Thus, for l fixed, the triangular array  $(I_1^{n,k,l})_{n\in\mathbb{N},k\in\{l+1,\dots,n\}}$  is obviously strongly mixing. Since it has even bounded second moments, it satisfies the weak law of large numbers, i.e.,

$$\lim_{n \to \infty} \frac{\sum_{k=l+1}^{\lfloor \delta n \rfloor} I_1^{n,k,l}}{n} \to \delta E(I_1^{2l,2l,l}) \quad \text{in probability}, \quad n \to \infty$$
(4.27)

(this follows, e.g., from Theorem 2.1 of Peligrad [15]).

By Doob's maximal quadratic inequality and Itô's isometry, and Property (III), one has

$$E\left(\sqrt{n}\sup_{(k-1-l)/n \le u \le (k-1)/n} \left| \sigma_{(k-1-l)/n} (B_{(k-1)/n} - B_u) - (S_{(k-1)/n} - S_u) \right| \right)^2$$

$$\leq 3nE\left(\int_{(k-1-l)/n}^{(k-1)/n} (\sigma_{(k-1-l)/n} - \sigma_s) \, dB_s\right)^2$$

$$+ 3nE\left(\sup_{(k-1)/n \le u \le (k-1)/n} \int_{(k-1-l)/n}^{u} (\sigma_{(k-1-l)/n} - \sigma_s) \, dB_s\right)^2 + 3K^2 l^2/n$$

$$\leq \frac{15l}{j^2} + \frac{3K^2 l^2}{n} \quad \text{for } l/n \le \xi(j). \tag{4.28}$$

For l fixed, the last line of (4.28) can be made arbitrary small by choosing j big. We conclude: for every l and  $\varepsilon > 0$ , we find  $n_{l,\varepsilon}$  s.t.

$$P(M_{n,k,l}^1) \le \varepsilon, \quad \forall n \ge n_{l,\varepsilon}, \ k = l+1,\ldots,n.$$

By the same arguments, the assertion holds for  $M_{n,k,l}^2$  and by similar but simpler arguments for  $M_{n,k}^3$  and  $M_{n,k}^4$ . This means: for every l and  $\varepsilon > 0$ , one finds  $n_{l,\varepsilon}$  s.t.

$$P(M_{n,k,l}^1 \cup M_{n,k,l}^2 \cup M_{n,k}^3 \cup M_{n,k}^4) \le \varepsilon, \quad \forall n \ge n_{l,\varepsilon}, \ k = l+1, \dots, n.$$

Since  $I_1^{2l,2l,l}$  is integrable and the distribution of  $I_1^{n,k,l}$  does not depend on n, k, one has

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall A \in \mathcal{F} \; \forall n, k \quad P(A) \le \delta \implies E\left(I_1^{n,k,l} \mathbf{1}_A\right) \le \epsilon$$

(see, e.g., Theorem 7.37 of [13]). We arrive at: for every l and  $\varepsilon > 0$ , we find  $n_{l,\varepsilon}$  s.t.

$$E(I_2^{n,k,l}) = E\left(I_1^{n,k,l} \mathbf{1}_{M_{n,k,l}^1 \cup M_{n,k,l}^2 \cup M_{n,k}^3 \cup M_{n,k}^4}\right) \le \varepsilon, \quad \forall n \ge n_{l,\varepsilon}, \ k = l+1,\dots,n.$$
(4.29)

It remains to estimate  $I_3^{n,k}$ , i.e., taxes of new shares. By (2.4), Cauchy-Schwarz's inequality, Itô's isometry, and Doob's maximal quadratic inequality, one has

$$E\left(S_{k/n} - \inf_{(k-1)/n \le u \le k/n} S_u\right)^2$$
  

$$\le 3E\left(\int_{(k-1)/n}^{k/n} \sigma_s \, dB_s\right)^2 + 3E\left(\inf_{(k-1)/n \le u \le k/n} \int_{(k-1)/n}^u \sigma_s \, dB_s\right)^2 + 3K^2/n^2$$
  

$$\le \frac{15\sigma_{\max}^2}{n} + 3K^2/n^2.$$
(4.30)

By (4.20), an analogue estimate holds for  $E\left(\left(\varphi_{k/n} - \inf_{(k-1)/n \leq u \leq k/n} \varphi_u\right)\right)$  and by Cauchy-Schwarz's inequality, one arrives at

$$E\left(\left(\varphi_{k/n} - \inf_{(k-1)/n \le u \le k/n} \varphi_u\right) \left(S_{k/n} - \inf_{(k-1)/n \le u \le k/n} S_u\right)\right) \le \frac{\widehat{K}}{n} \quad \text{for some } \widehat{K} \in \mathbb{R}_+ \quad (4.31)$$

and all  $n, k \in \mathbb{N}$  with  $k \leq n$ . Again by Itô's isometry,

$$E\left(\left|\int_{\left(\frac{k-1}{n},\frac{k}{n}\right]} (\varphi_t - \inf_{\frac{k-1}{n} \le u \le t} \varphi_u) dS_t\right|\right)$$

$$\leq K\left(E\left|\int_{\left(\frac{k-1}{n},\frac{k}{n}\right]} (\varphi_t - \inf_{\frac{k-1}{n} \le u \le t} \varphi_u) dt\right| + \sigma_{\max} \sqrt{E\left(\int_{\left(\frac{k-1}{n},\frac{k}{n}\right]} (\varphi_t - \inf_{\frac{k-1}{n} \le u \le t} \varphi_u)^2 dt\right)}\right)$$

$$\leq \frac{K}{n} E\left(\sup_{\frac{k-1}{n} \le u \le \frac{k}{n}} \varphi_u - \inf_{\frac{k-1}{n} \le u \le \frac{k}{n}} \varphi_u\right) + \frac{\sigma_{\max}}{\sqrt{n}} \sqrt{E\left(\sup_{\frac{k-1}{n} \le u \le \frac{k}{n}} \varphi_u - \inf_{\frac{k-1}{n} \le u \le \frac{k}{n}} \varphi_u\right)^2}$$

$$\leq \frac{\hat{K}}{n} \quad \text{for some } \hat{K} \in \mathbb{R}_+$$
(4.32)

and all  $n, k \in \mathbb{N}$  with  $k \leq n$ . For the last estimate, we use that  $E\left(\sup_{(k-1)/n \leq u \leq k/n} \varphi_u - \inf_{(k-1)/n \leq u \leq k/n} \varphi_u\right)^2 \leq 24h_{\min}^2 \sigma_{\max}^2/n + 3(|h_{\min}| + 1)K/n^2$ . (4.31) and (4.32) imply that  $E(\sum_{k=1}^n |I_3^{n,k}|)$  is bounded in  $n \in \mathbb{N}$ . Especially, there exists a  $\widetilde{K} \in \mathbb{R}_+$  s.t.

$$E\left(\sum_{k=1}^{n} \left(I_3^{n,k}\right)^{-}\right) \le \widetilde{K} \quad \text{for all } n \in \mathbb{N}.$$
(4.33)

Step 5: Now, we complete the proof by combining the estimates of the previous step. Remember that we have to estimate the total variation of  $\Pi$  on the subinterval  $[0, \delta] \subset [0, 1]$  and given that  $\omega \in C_{\delta} \cap D_{\eta}$ . Let  $L, \varepsilon > 0$ . For  $\widetilde{K}$  from (4.33), one has that

$$P\left(\sum_{k=1}^{\lfloor \delta n \rfloor} \left(I_3^{n,k}\right)^- \ge 2\widetilde{K}/\varepsilon\right) \le \varepsilon/2.$$
(4.34)

By (4.26)/(4.27), one can find *l* s.t.

$$P\left(\frac{\sum_{k=l+1}^{\lfloor \delta n \rfloor} I_1^{n,k,l}}{n} \ge \frac{L}{\alpha} + \frac{2\widetilde{K}}{\varepsilon} + 6\right) \to 1, \quad n \to \infty.$$
(4.35)

Fixing l, by (4.29), one has

$$P\left(\frac{\sum_{k=l+1}^{\lfloor \delta n \rfloor} I_2^{n,k,l}}{n} \le 1\right) \to 1, \qquad n \to \infty,$$
(4.36)

Now let  $\varphi$  be again the original trading strategy. Putting (4.25), (4.34), (4.35), and (4.36) together, we arrive at

$$P\left(1_{C_{\delta}\cap D_{\eta}}\sum_{k=l+1}^{\lfloor \delta n \rfloor} (\Pi_{k/n}(\varphi) - \Pi_{(k-1)/n}(\varphi))^{+} \leq L\right) \leq \varepsilon \quad \text{for } n \text{ large enough},$$

where we use that  $C_{\delta} \subset \{\varphi_{(k-1)/n} \geq 1/\sqrt{n} \text{ for } k = 1, \dots, \lfloor \delta n \rfloor\}$  if  $n \geq (\varphi_{0+}/2)^{-2}$  and  $\varphi, S$  have only be modified outside the set  $(C_{\delta} \cap D_{\eta}) \times [0, \delta]$ . Since  $L, \varepsilon > 0$  are arbitrary chosen,  $\sum_{k=1}^{\lfloor \delta n \rfloor} |\Pi_{k/n}(\varphi) - \Pi_{(k-1)/n}(\varphi)|$  converges to infinity in probability for  $n \to \infty$  on the set  $C_{\delta} \cap D_{\eta}$ . Since  $\lim_{n\to\infty} \sum_{k=1}^{\lfloor \delta n \rfloor} |\Pi_{k/n}(\varphi) - \Pi_{(k-1)/n}(\varphi)|$  exists pointwise, one has

$$P(\operatorname{Var}(\Pi(\varphi))_{\delta} = \infty) \ge P(C_{\delta} \cap D_{\eta}) \ge P(C_{\delta}) - \eta \quad \text{for all } \delta \in (0, 1), \ \eta > 0.$$

This implies

$$P(\operatorname{Var}(\Pi(\varphi))_{\delta_0} = \infty) \ge \limsup_{\delta \downarrow 0} P(\operatorname{Var}(\Pi(\varphi))_{\delta} = \infty) \ge \limsup_{\delta \downarrow 0} P(C_{\delta}) = 1 \quad \text{for all } \delta_0 > 0.$$

Lemma 4.2. One has

$$\sum_{k=1}^{n} \int_{0}^{\varphi_{(k-1)/n} - \inf_{(k-1)/n \le u \le k/n} \varphi_u} \left(\frac{k-1}{n} - \tau_{(k-1)/n,x}\right) \, dx \le \sup_{0 \le u \le 1} \varphi_u. \tag{4.37}$$

Proof. Define

$$M_k := \left\{ (t,y) \in [0,1] \times \mathbb{R}_+ \mid \inf_{(k-1)/n \le u \le k/n} \varphi_u < y < \varphi_{(k-1)/n}, \ \tau_{(k-1)/n,\varphi_{(k-1)/n}-y} < t < \frac{k-1}{n} \right\}.$$

Let l < k and assume that  $(t, y) \in M_l \cap M_k$ . One obtains that  $\inf_{(l-1)/n \leq u \leq l/n} \varphi_u < y < \varphi_{(k-1)/n}$  and thus  $\tau_{(k-1)/n,\varphi_{(k-1)/n}-y} > (l-1)/n$ , which is a contradiction to  $(l-1)/n > t > \tau_{(k-1)/n,\varphi_{(k-1)/n}-y}$ . Thus, we have that  $M_l \cap M_k = \emptyset$  implying that  $\lambda^2(\cup_{k=1,\dots,n}M_k) = \sum_{k=1}^n \lambda^2(M_k)$ , where  $\lambda^2$  denotes the Lebesgue measure on  $\mathbb{R}^2$ . Since  $\lambda^2(\cup_{k=1,\dots,n}M_k) \leq \sup_{0 < u < 1} \varphi_u$  and

$$\begin{split} \lambda^2(M_k) &= \int_{\inf_{(k-1)/n}}^{\varphi_{(k-1)/n}} \left( \frac{k-1}{n} - \tau_{(k-1)/n,\varphi_{(k-1)/n}-y} \right) \, dy \\ &= \int_0^{\varphi_{(k-1)/n} - \inf_{(k-1)/n} \leq u \leq k/n \, \varphi_u} \left( \frac{k-1}{n} - \tau_{(k-1)/n,x} \right) \, dx, \end{split}$$

this implies the assertion.

**Remark 4.3.** One may interpret Lemma 4.2 and its proof as follows: Consider the reparametrization  $y := \varphi_t - x$  of shares which are in the portfolio at t. The bigger y, the shorter is the residence time. With this parametrization, a share keeps its label until it is liquidated by a real sale. Thus,  $(t, y) \in M_k$ , t < (k - 1)/n means that share y being in the portfolio at t is liquidated between (k-1)/n and k/n. Since every shares can be liquidated only once,  $M_k$  and  $M_l$  are disjunct for  $l \neq k$ . Thus,  $\lambda^2(\bigcup_{k=1,\dots,n}M_k) = \sum_{k=1}^n \lambda^2(M_k)$ . It coincides with the accumulated residence times of all shares which are liquidated until time 1, but without including the time in the period the share is sold. It is bounded by the product of time horizon and the maximal total number of shares.

Proof of Theorem 2.3(ii). We turn again to the decomposition (4.11)/(4.12) of "old shares" and "new shares".  $\varphi^{\text{old}}$ , the number of shares being "old" in the *k*th period, is nonincreasing on ((k-1)/n, k/n], which allows us to apply Lemma 4.1 to the corresponding tax payment process and to obtain

$$\left(\Pi_{k/n}(\varphi^{\text{old}}) - \Pi_{(k-1)/n}(\varphi^{\text{old}})\right)^+$$
(4.38)

$$\leq \alpha \int_0^{\varphi_{(k-1)/n}^{\text{old}} - \varphi_{k/n}^{\text{old}}} F\left(\frac{k-1}{n}, x\right) \, dx + \alpha (\varphi_{(k-1)/n}^{\text{old}} - \varphi_{k/n}^{\text{old}}) \left(\sup_{(k-1)/n \leq u \leq k/n} S_u - S_{(k-1)/n}\right).$$

For  $\varphi^{\text{new}}$ , the number of "new" shares in the kth period, one has the estimate

$$\Pi_{k/n}(\varphi^{\text{new}}) - \Pi_{(k-1)/n}(\varphi^{\text{new}}) \le \alpha \int_{((k-1)/n, k/n]} \varphi_t^{\text{new}} \, dS_t, \tag{4.39}$$

using that  $F^{\text{new}}((k-1)/n, \cdot) = 0$ ,  $F^{\text{new}}(k/n, \cdot) \ge 0$ , and (3.3). By (4.14), one can put (4.38) and (4.39) together and obtains

$$\left( \Pi_{k/n}(\varphi) - \Pi_{(k-1)/n}(\varphi) \right)^{+} \leq \alpha \int_{0}^{\varphi_{(k-1)/n}^{\text{old}} - \varphi_{k/n}^{\text{old}}} F\left(\frac{k-1}{n}, x\right) dx + \alpha (\varphi_{(k-1)/n}^{\text{old}} - \varphi_{k/n}^{\text{old}}) \left( \sup_{(k-1)/n \le u \le k/n} S_u - S_{(k-1)/n} \right) \\ + \alpha \left( \int_{((k-1)/n, k/n]} \varphi_t^{\text{new}} dS_t \right)^{+} =: J_1^{n,k} + J_2^{n,k} + J_3^{n,k}.$$

$$(4.40)$$

By scaling properties, it is sufficient to prove the finite variation property on the time interval [0, 1]. In addition, the sequence

$$T_m := \inf \{t \ge 0 \mid S_t \notin [1/m, m], \ |\mu_t| > m, \ \sigma_t > m, \\ \text{or } A^i_t \notin [-m, m] \text{ for some } i = 1, \dots, d\} \land 1, \quad m \in \mathbb{N}.$$

is localizing by Assumption 2.1, and one only needs to show that the stopped tax processes  $\Pi^{T_m}$ ,  $m \in \mathbb{N}$ , are of finite variation. Thus, given m, we can modify g outside the compact set  $[1/m, m] \times [-m, m]^d \subset (0, \infty) \times \mathbb{R}^d$  s.t. the *modified* function, still denoted by g, satisfies

- (I)  $\partial_1 g$  is positive, bounded, and bounded away from zero on  $\mathbb{R}_+ \times \mathbb{R}^d$ ,
- (II)  $\partial_2 g$  and  $\partial_{11} g$  are bounded on  $\mathbb{R}_+ \times \mathbb{R}^d$ ,

and one still has

$$\varphi_t^{T_m} = g\left(S_t^{T_m}, A_t^{T_m}\right), \quad t>0.$$

In the following, for notational convenience and w.l.o.g., we put d = 1 and  $A_t = t$ . In addition, the stopped processes are still denoted by  $\mu$ ,  $\sigma$ , S, and  $\varphi$ . By the implicit function theorem, it follows that  $S_t = f(\varphi_t, t), t > 0$ , for some smooth function f with  $\partial_1 f > 0$  and  $\partial_1 f, \partial_2 f$  bounded on  $\mathbb{R}_+ \times [0, 1]$ . For the later, one needs that  $\partial_1 g$  is bounded away from zero.

Let us first estimate  $\sum_{k=1}^{n} J_1^{n,k}$ . For all  $k \ge 2$  and  $x \in (0, \varphi_{(k-1)/n})$ , one has

$$F\left(\frac{k-1}{n},x\right) = f\left(\varphi_{(k-1)/n},\frac{k-1}{n}\right) - \inf_{\tau_{(k-1)/n,x} < u \le (k-1)/n} f\left(\varphi_{u},u\right)$$

$$\leq f\left(\varphi_{(k-1)/n},\frac{k-1}{n}\right) - \inf_{\tau_{(k-1)/n,x} < u \le (k-1)/n} f\left(\varphi_{u},\frac{k-1}{n}\right)$$

$$+ \left(\frac{k-1}{n} - \tau_{(k-1)/n,x}\right) \sup_{(p,t) \in \mathbb{R}_{+} \times [0,1]} |\partial_{2}f(p,t)| \qquad (4.41)$$

$$\leq x \sup_{(p,t) \in \mathbb{R}_{+} \times [0,1]} \partial_{1}f(p,t) + \left(\frac{k-1}{n} - \tau_{(k-1)/n,x}\right) \sup_{(p,t) \in \mathbb{R}_{+} \times [0,1]} |\partial_{2}f(p,t)|.$$

For the equality, one uses that  $S_0 = f(\varphi_{0+}, 0)$  and thus  $\inf_{\tau_{(k-1)/n,x} \le u \le (k-1)/n} S_u = \inf_{\tau_{(k-1)/n,x} \le u \le t} f(\varphi_u, u)$ . For the crucial last inequality one needs that f is increasing in its first argument, which leads to

$$\inf_{\substack{\tau_{(k-1)/n,x} < u \le (k-1)/n}} f\left(\varphi_u, \frac{k-1}{n}\right) = f\left(\inf_{\substack{\tau_{(k-1)/n,x} < u \le (k-1)/n}} \varphi_u, \frac{k-1}{n}\right) \\
= f\left(\left(\varphi_{(k-1)/n} - x\right) \lor \inf_{0 < u \le (k-1)/n} \varphi_u, \frac{k-1}{n}\right) \\
\ge f\left(\varphi_{(k-1)/n} - x, \frac{k-1}{n}\right).$$

Putting (4.41) and Lemma 4.2 together, one obtains

$$\sum_{k=1}^{n} J_{1}^{n,k} \leq \frac{\alpha}{2} \sup_{(p,t)\in\mathbb{R}_{+}\times[0,1]} \partial_{1}f(p,t) \sum_{k=1}^{n} \left(\varphi_{(k-1)/n}^{\text{old}} - \varphi_{k/n}^{\text{old}}\right)^{2} + \alpha \sup_{(p,t)\in\mathbb{R}_{+}\times[0,1]} |\partial_{2}f(p,t)| \sup_{0\leq u\leq 1} \varphi_{u} \\
\leq M_{1} \sum_{k=1}^{n} \left(\varphi_{(k-1)/n}^{\text{old}} - \varphi_{k/n}^{\text{old}}\right)^{2} + M_{2}$$
(4.42)

for some  $M_1, M_2 \in \mathbb{R}_+$  which do not depend on n. By Itô's formula,  $\varphi$  satisfies

$$d\varphi_t = \partial_1 g(S_t, t) \, dS_t + \partial_2 g(S_t, t) \, dt + \frac{1}{2} \partial_{11} g(S_t, t) \, d[S, S]_t.$$
(4.43)

From Properties (I), (II),  $S \leq m$ ,  $|\mu| \leq m$ , and  $\sigma \leq m$ , one concludes that both the drift rates and the rates of the continuous quadratic variation processes of S and  $\varphi$  are bounded. Thus, by (4.42) and the same estimates as in (4.30) and (4.32), one has that

$$\sum_{k=1}^{n} E(J_1^{n,k}), \quad \sum_{k=1}^{n} E(J_2^{n,k}), \quad \sum_{k=1}^{n} E(J_3^{n,k}) \quad \text{are bounded in } n \in \mathbb{N}.$$
(4.44)

Putting (4.40) and (4.44) together yields that

$$E\left(\sum_{k=1}^{n} \left(\Pi_{k/n}(\varphi) - \Pi_{(k-1)/n}(\varphi)\right)^{+}\right) \text{ is bounded in } n \in \mathbb{N}.$$

By the Lemma of Fatou, this implies that

$$E\left(\liminf_{n\to\infty}\sum_{k=1}^n \left(\Pi_{k/n}(\varphi) - \Pi_{(k-1)/n}(\varphi)\right)^+\right) < \infty.$$

As  $|x| = 2x^+ - x$  and  $\Pi_1(\varphi)$  is almost surely finite, one concludes that

$$P\left(\liminf_{n\to\infty}\sum_{k=1}^n \left|\Pi_{k/n}(\varphi) - \Pi_{(k-1)/n}(\varphi)\right| < \infty\right) = 1.$$

By continuity of the tax payment process,  $Var(\Pi(\varphi))_1$  is attained by any sequence of grids with vanishing mesh, and we are done.

# 5 Tax effect of taming strategies and a conjecture

In this section, we argue why we belief that for a risky asset price following geometric Brownian motion and preferences given by CRRA-utility ("constant relative risk aversion") from terminal wealth, the optimal strategy with the exact tax basis is of infinite variation if the parameters are suitably chosen. This would mean that it behaves like in a frictionless market and different from a market with proportional transaction costs. In the latter, the optimal strategy consists of a no-trading region. Since the portfolio optimization problem with the exact tax basis is not solved (and maybe it is even far away from being tractable), the considerations about the optimal strategy are pure heuristics that may stimulate further research (see Remark 5.3). They are however based on a rigorous result on the tax effect of the taming of a strategy which follows Brownian motion with drift (Theorem 5.1) that is interesting in itself.

To make an educated guess about the path properties of the optimal strategy, we are interested in strategies of the form  $\varphi_t = g(S_t, A_t), \ \partial_1 g > 0$ , i.e., we focus on leverage, and want to compare them with their tamed approximations  $\varphi^{\varepsilon}, \varepsilon > 0$ , where

$$\varphi_t^{\varepsilon} := \varphi_s \quad \text{with} \quad s := \sup \left\{ u < t \mid \varphi_u \in \{0, \pm \varepsilon, \pm 2\varepsilon, \ldots\} \right\}, \quad t > 0$$

and  $\varphi_0^{\varepsilon} = 0$ , i.e.,  $\varphi^{\varepsilon}$  is piecewise constant and  $|\varphi^{\varepsilon} - \varphi| \leq \varepsilon$ .

For simplicity, we restrict to the case that  $\varphi_t = 1_{(t>0)}g(S_t)$  for some increasing function g with  $g' \geq g'_{\min} > 0$  and  $S_t = g^{-1}(a + \mu t + B_t)$ , i.e.,  $\varphi_t = 1_{(t>0)}(a + \mu t + B_t)$ , with some  $a \in \mathbb{R}_+ \setminus \{0\}, \mu \in \mathbb{R}$ , and a standard Brownian motion B. To avoid irrelevant problems, we only consider  $\varepsilon$  with  $\varepsilon = a/k_0$  for some  $k_0 \in \mathbb{N}$ , which implies that  $\varphi_{0+}^{\varepsilon} = \varphi_{0+} = a P$ -a.s.

**Theorem 5.1.** Under the above conditions, one has

$$E\left(1_{\{\varphi_t^{\varepsilon}\geq 0\}}\int_0^{\varphi_t^{\varepsilon}} F^{\varepsilon}(t,x)\,dx\right) \leq E\left(1_{\{\varphi_t\geq 0\}}\int_0^{\varphi_t} F(t,x)\,dx\right) + \frac{3\varepsilon^2}{2g'_{\min}},\tag{5.1}$$

where  $F^{\varepsilon}$  denotes the book profit function for strategy  $\varphi^{\varepsilon}$  ( $F^{\varepsilon}(t, \cdot)$  and  $F(t, \cdot)$  are declared on  $\{\varphi_t \geq 0\}$  and  $\{\varphi_t \geq 0\}$ , resp.).

This means that for  $\varepsilon \to 0$ , the taming of the strategy can lead, if at all, only to a moderate increase of deferred tax payments. The proof of Theorem 5.1 is based on the following lemma.

**Lemma 5.2.** Let X be a continuous supermartingale with  $X_0 = 0$  and  $[X, X]_{\infty} = \infty$ , Pa.s.,  $\varepsilon > 0$ ,  $\sigma_1 := \inf\{s \ge 0 \mid X_s = -\varepsilon\}$ ,  $\sigma_2 := \sup\{s \ge 0 \mid X_s = 0 \text{ and } s < \sigma_1\}$ , and  $H := \inf_{0 \le s \le \sigma_2} X_s$ . Then,  $P(H \le -b) \le 1 - b/\varepsilon$  for all  $b \in (0, \varepsilon)$ . In the case that X is a martingale, the inequality holds with equality, i.e., H is uniformly distributed on  $(-\varepsilon, 0)$ .

Proof of Lemma 5.2. The event  $\{H \leq -b\}$ ,  $b \in (0, \varepsilon)$ , occurs if and only if after the first hitting time of -b, X hits 0 before it hits  $-\varepsilon$ . By the supermartingale property, the probability p of this event satisfies  $pb + (b - \varepsilon)(1 - p) \leq 0$ .

*Proof of Theorem 5.1.* One has the following decompositions of the book profits:

$$\int_{0}^{\varphi_{t}^{\varepsilon} \vee 0} F^{\varepsilon}(t,x) \, dx = \varepsilon \mathbf{1}_{\{\varphi_{t} \ge k\varepsilon\}} \sum_{k=1}^{\infty} F^{\varepsilon}(t,(\varphi_{t}^{\varepsilon}-k\varepsilon)+) + \varepsilon \mathbf{1}_{\{\varphi_{t}^{\varepsilon} > \varphi_{t}\}} F^{\varepsilon}(t,0+), \tag{5.2}$$

where  $F^{\varepsilon}(t, (\varphi_t^{\varepsilon} - k\varepsilon) +) = \lim_{x \downarrow \varphi_t^{\varepsilon} - k\varepsilon} F^{\varepsilon}(t, x)$ , and

$$\int_{0}^{\varphi_t \vee 0} F(t,x) \, dx \ge \int_{0}^{\lfloor \varphi_t / \varepsilon \rfloor \varepsilon \vee 0} F(t,x) \, dx = \sum_{k=1}^{\infty} \mathbb{1}_{\{\varphi_t \ge k\varepsilon\}} \int_{(k-1)\varepsilon}^{k\varepsilon} F(t,\varphi_t - y) \, dy. \tag{5.3}$$

Let us fix some  $k \in \mathbb{N}$  and compare the book profit  $F^{\varepsilon}(t, (\varphi_t^{\varepsilon} - k\varepsilon) +)$  of the tamed strategy with the average book profit  $\int_{(k-1)\varepsilon}^{k\varepsilon} F(t, \varphi_t - y) \, dy/\varepsilon$  of the original strategy. One has

$$F^{\varepsilon}(t,(\varphi_t^{\varepsilon}-k\varepsilon)+) = F^{\varepsilon}(t,\varphi_t^{\varepsilon}-(k-1)\varepsilon) = g^{-1}(\varphi_t) - g^{-1}\left(\inf_{\widetilde{\tau} \le u \le t} \varphi_{u+}\right)$$
(5.4)

for  $\inf_{0 < u \leq t} \varphi_u \leq k\varepsilon \leq \varphi_t^{\varepsilon}$ , where

$$\widetilde{\tau} := \inf\{u > 0 \mid \varphi_u = k\varepsilon \text{ and } \varphi_s > (k-1)\varepsilon \ \forall s \in (u,t]\}$$

and

$$F(t,\varphi_t - y) = S_t - \inf_{\tau_{t,\varphi_t} - y \le u \le t} S_u = g^{-1}(\varphi_t) - g^{-1}(y) \quad \text{for} \quad \inf_{0 < u \le t} \varphi_u \le y \le \varphi_t, \tag{5.5}$$

see (7.1) of [14]. We firstly condition on the event  $\{\varphi_t \geq k\varepsilon\} \cap \{\inf_{0 \leq s \leq t} \varphi_s \leq (k-1)\varepsilon\}$ . Then,  $\sigma \leq \tau \leq t$  for sure, where  $\sigma := \inf\{u > 0 \mid \varphi_u = (k-1)\varepsilon\}$  and  $\tau := \tau_{t,\varphi_t-k\varepsilon}$ . Given  $\sigma$  and  $\tau$ , the time reversed process  $u \mapsto \varphi_{\tau-u} - k\varepsilon$  is distributed like a Brownian bridge  $(\tilde{B}_u)_{u \in [0, \tau-\sigma]}$  with  $\tilde{B}_0 = 0$  and  $\tilde{B}_{\tau-\sigma} = -\varepsilon$ . Especially, the (conditional) law of the process  $(\tilde{B}_u)_{u \in [0, \tau-\sigma]}$  does not depend on  $\mu$ , and under the filtration  $\tilde{\mathcal{F}}_u = \sigma(\tau, \sigma, \tilde{B}_{s \wedge (\tau-\sigma)}, s \in [0, u])$ , it possesses the drift rate  $(-\varepsilon - \tilde{B}_{u \wedge (\tau-\sigma)})/(\tau - \sigma - u)$ . Thus,  $\tilde{B}$  stopped at the first time it hits  $-\varepsilon$  is a supermartingale. This allows us to apply Lemma 5.2 (where it is of course sufficient to assume that the stopped process  $X^{\sigma_1}$  is a supermartingale), and we conclude that

$$P\left(\inf_{\widetilde{\tau} \le s \le t} \varphi_s \le k\varepsilon - b \mid \varphi_t \ge k\varepsilon, \inf_{0 < s \le t} \varphi_s \le (k-1)\varepsilon\right) \le 1 - b/\varepsilon, \quad b \in (0,\varepsilon).$$

Since  $g^{-1}$  is increasing, this yields

$$E\left(g^{-1}\left(\inf_{\widetilde{\tau}\leq u\leq t}\varphi_u\right) \mid \varphi_t\geq k\varepsilon, \ \inf_{0(5.6)$$

On the other hand, one has the pathwise identity

$$\varepsilon F^{\varepsilon}(t, (\varphi_t^{\varepsilon} - k\varepsilon) +) = \varepsilon \left( S_t - \inf_{0 \le u \le t} S_u \right) = \int_{(k-1)\varepsilon}^{k\varepsilon} F(t, \varphi_t - y) \, dy \quad \text{on } \left\{ \inf_{0 < s \le t} \varphi_s > k\varepsilon \right\}.$$
(5.7)

Putting (5.6) and (5.7) together, one has, in view of (5.4) and (5.5), that

$$\varepsilon E\left(F^{\varepsilon}(t,(\varphi_{t}^{\varepsilon}-k\varepsilon)+)\mid\varphi_{t}\geq k\varepsilon,\inf_{0< s\leq t}\varphi_{s}\notin((k-1)\varepsilon,k\varepsilon]\right)$$

$$\leq E\left(\int_{(k-1)\varepsilon}^{k\varepsilon}F(t,\varphi_{t}-y)\,dy\mid\varphi_{t}\geq k\varepsilon,\inf_{0< s\leq t}\varphi_{s}\notin((k-1)\varepsilon,k\varepsilon]\right),\quad k\in\mathbb{N}.$$
(5.8)

Book profits of shares in the layer  $((k-1)\varepsilon, k\varepsilon]$  of the pathwise infimum have to be considered separately, and we use the rough estimate

$$\varepsilon E\left(F^{\varepsilon}(t,(\varphi_{t}^{\varepsilon}-k\varepsilon)+) \mid \varphi_{t} \geq k\varepsilon, \ (k-1)\varepsilon < \inf_{0 < s \leq t} \varphi_{s} \leq k\varepsilon\right) \\ -E\left(\int_{(k-1)\varepsilon}^{k\varepsilon} F(t,\varphi_{t}-y) \, dy \mid \varphi_{t} \geq k\varepsilon, \ (k-1)\varepsilon < \inf_{0 < s \leq t} \varphi_{s} \leq k\varepsilon\right) \\ = E\left(\int_{\inf_{0 < s \leq t} \varphi_{s}}^{k\varepsilon} \left(g^{-1}(y) - g^{-1}\left(\inf_{0 < s \leq t} \varphi_{s}\right)\right) \, dy \mid \varphi_{t} \geq k\varepsilon, \ (k-1)\varepsilon < \inf_{0 < s \leq t} \varphi_{s} \leq k\varepsilon\right) \\ \leq \int_{(k-1)\varepsilon}^{k\varepsilon} \left(g^{-1}(y) - g^{-1}\left((k-1)\varepsilon\right)\right) \, dy \leq \frac{\varepsilon^{2}}{2g'_{\min}}.$$
(5.9)

The difference to the argumentation in (5.6) is that for  $\inf_{0 \le s \le t} \varphi_s > (k-1)\varepsilon$ , the original strategy does not purchase the shares of the layer  $((k-1)\varepsilon, k\varepsilon]$  earlier than the tamed one. Thus, book profits triggered by wash sells of the tamed strategy cannot be compensated. In addition, we cannot apply Lemma 5.2 as, without the condition  $\inf_{0 \le s \le t} \varphi_s \le (k-1)\varepsilon$ , the process  $u \mapsto \varphi_{\tau-u} - k\varepsilon$  has in general no negative drift rate at u = 0. Thus, we have to compute the worst case that  $\inf_{\tilde{\tau} \le s \le t} \varphi_s \approx (k-1)\varepsilon$ .

For newly purchased shares of the tamed strategy, we also use a rough estimate, namely

$$1_{\{\varphi_t^{\varepsilon} > \varphi_t\}} F^{\varepsilon}(t, 0+) \le 1_{\{\varphi_t^{\varepsilon} > \varphi_t\}} \left( g^{-1}(\varphi_t) - g^{-1}(\varphi_t^{\varepsilon} - \varepsilon) \right) \le \frac{\varepsilon}{g'_{\min}}.$$
(5.10)

Putting (5.2), (5.3), (5.8), (5.9), and (5.10) together yields (5.1).

**Remark 5.3.** In view of Theorem 5.1, we conjecture that for the exact basis, a risky asset price following geometric Brownian motion and preferences given by CRRA-utility from terminal wealth, the optimal trading strategy is of infinite variation if the parameters are suitably chosen.

Let  $u(x) = x^{1-R}$ ,  $R \in (0, \infty) \setminus \{1\}$ , be the utility function, V the liquidation value from Definition 3.1, and  $\pi_t = \varphi_t S_t / V_t$  the fraction invested in the risky asset. By Itô's formula, one has

$$du(V_t) = u'(V_t)(1-\alpha)\pi_t V_t \mu \, dt + u'(V_t)(1-\alpha)\pi_t V_t \sigma \, dB_t + u'(V_t)(1-\alpha)(1-\pi_t)V_t r \, dt + \frac{1}{2}u''(V_t)(1-\alpha)^2 \pi_t^2 V_t^2 \sigma^2 \, dt + u'(V_t)\alpha \int_0^{\varphi_t} F(t,x) \, dxr \, dt.$$
(5.11)

If trading gains were taxed immediately, the last summand would disappear. Then, there would exist a value function v, depending on time t and liquidation value x only, with v(t, x) = f(t)u(x). The optimal strategy would be given by

$$\varphi_t = \frac{\pi^* V_t}{S_t}, \quad where \ \pi^* = \frac{\mu - r}{(1 - \alpha)R\sigma^2}.$$
(5.12)

This means that the optimal Merton fraction of a tax-exempt investor is just divided by  $1 - \alpha$ , and if  $\mu - r \neq (1-\alpha)R\sigma^2$ ,  $\varphi$  is of infinite variation as in frictionless markets. Furthermore, when following a strategy which differs from the optimal one "on average" by  $\varepsilon$ , the utility-loss in (5.11) without the last summand is at least of order  $\varepsilon^2$ . On the other hand, by Theorem 5.1, interest losses by frequent realizations of book profits are bounded in expectation by  $\mathcal{O}(\varepsilon^2)$ . Provided that this holds similarly for strategies of the form  $\varphi_t = g(S_t, A_t)$ , where we have leverage in mind, i.e.,  $\partial_1 g > 0$ , one can choose  $\alpha > 0$  small enough such that at least for some  $(\omega, t)$ , the utilityloss term which ignores interest on deferred tax payments  $\alpha \int_0^{\varphi_t} F(t, x) dxr dt$ , dominates. This indicated that the taming of a strategy reduces expected utility, and the optimal strategy is in general not of finite variation (although one should of course expect that it differs from (5.12), even for small  $\alpha$ ). It is evident that this is only a conjecture with outstanding points at many places.

# 6 Conclusion

In this article, we investigate the fine structure of the payment stream of a linear tax on trading gains when using the so-called exact tax basis. Roughly speaking, for an Itô asset price process, we show that the tax payment process is of infinite variation if the quadratic covariation of trading strategy and asset price is negative, while the tax process is of finite variation if there is a positive dependency between strategy and price. A reason for this is that in the first case, there is more profit-taking, which triggers positive tax payments, as the position is more frequently reduced after the asset price goes up. On the other hand, losses are *always* realized immediately by wash sells.

To prove that the tax payment stream is of infinite variation, we estimate book profits of "recently" purchased shares from below. For the estimation, it is crucial that the pathwise supremum of a standard Brownian motion killed when hitting -1 has infinite expectation. For small time intervals, the order of the reduction of the asset position is square root of the length of the interval, and it follows that the positive part of the increment of the tax payment process is of the order of the length of the interval, but with a nonintegrable random prefactor. Together with a strong mixing property, this yields infinite variation on almost all paths.

In addition, we provide a definition of Constant Proportion Portfolio Insurance (CPPI) strategies that incorporates taxes. The deferment of tax payments by unrealized book profits has a major impact on the portfolio value which controls the investment in the risky asset. CPPI strategies are an important application of [14] where the tax payment stream is constructed for all càglàd trading strategies – including strategies of infinite variation. We obtain the CPPI portfolio value process as the unique solution of a SDE with a coefficient functional acting on the whole past history of the process. The number of risky assets the investor holds is increasing (decreasing) in the asset price if the tax-adjusted multiple of the cushion which is invested in the risky asset is bigger (smaller) than one. In the first case, the strategy requires less profit-taking which means that profits are better deferred.

Finally, we show that the tax benefits of the taming of a strategy of infinite variation are quite moderate which provides some intuition how an optimal continuous time strategy may look like.

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