Price Setting of Market Makers: A Filtering Problem with Endogenous Filtration

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Abstract

This paper studies the price-setting problem of market makers under risk neutrality and perfect competition in continuous time. The classic approach of Glosten-Milgrom (1985) is followed. Bid and ask prices are defined as conditional expectations of a true value of the asset given the market makers' partial information that includes the customers' trading decisions. The true value is modeled as a Markov process that can be observed by the customers with some noise at Poisson times.

A mathematically rigorous analysis of the price-setting problem is carried out, solving a filtering problem with endogenous filtration that depends on the bid and ask price processes quoted by the market maker. The existence and uniqueness of the bid and ask price processes is shown under some conditions.

Keywords: market making, bid-ask spread, stochastic filtering, point processes

JEL classification: G12, G14.

Mathematics Subject Classification (2000): 60G35, 91G80, 60G55.

1 Introduction

In specialist markets one or several market makers (also called specialists) provide liquidity by offering to buy or to sell a given asset at any time. They quote both a bid price at which they commit themselves to buy and a higher ask price at which they sell. By doing so, market makers face certain risks for which they are compensated by the bid-ask spread.

The risk can be broken down into two primary components: inventory and information risk. Inventory risk describes the risk that market makers or other liquidity providers accumulate large positive or negative inventories in the asset, and then the price moves against them. This issue was first studied by Ho and Stoll (1981) in a continuous time framework. Recent developments based

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We would like to thank the editor, an anonymous associate editor, and two anonymous referees for their valuable comments and suggestions. The second named author acknowledges the financial support by the Deutsche Forschungsgemeinschaft (DFG).

on optimal stochastic control are provided by Avellaneda and Stoikov (2008), Guilbaud and Pham (2013), Veraart (2010), and Cartea and Jaimungal (2012), among others.

The other risk market makers take is information risk, i.e. the risk that at least some of the customers have superior (or inside) information about the hidden true value of the asset and trade strategically to their advantage and therefore to the disadvantage of the market maker. Thus, the market maker faces an adverse selection problem. Although the nature of the two types of risk is quite different, their effects are similar. Namely, if a customer buys assets, the market maker will most likely raise both his bid and his ask price – on the one hand because he wants to avoid further buying and to stimulate the sell-side to control his inventory, and on the other hand because he believes that the customer's purchase has conveyed some good news about the true value of the asset. In this article, we focus on the information risk. Thus, it seems instructive to assume risk neutrality, as otherwise the different effects described above overlap. The information risk was first studied by Copeland and Galai (1983) and more generally and in continuous time by Glosten and Milgrom (1985) who describe the prices as expectations of a hidden true value. This zero expected profit condition can be explained by risk neutrality and perfect competition among market makers. It leads to tractable models and may still be used as a benchmark for more involved situations. An alternative approach has been taken by Kyle (1985) (developed further by Back (1992)) who not only modeled how the market makers handle the information flow from customers, but who also considered a strategic insider, optimally using his knowledge to his advantage. However, in contrast to Glosten-Milgrom, Kyle models a single price process and hence cannot explain the bid-ask spread. A connection to the Glosten-Milgrom model was established by Krishnan (1992) and generalized by Back and Baruch (2004).

Even showing or disproving the existence or the uniqueness of Glosten-Milgrom prices in a static model is a non-trivial issue, and there are few substantial contributions of this kind. Bagnoli et al. (2001) derive necessary and sufficient conditions for the existence of a linear equilibrium in a one-period model with several insiders behaving strategically. Linearity means that, after observing the size of the arriving market order, the market maker quotes a price per share which is affine linear, but not constant, in the order size. The market maker can draw conclusions from the order size about the type of the trader submitting the order. It turns out that linear equilibria only exist in special cases. Back and Baruch (2004) derive (in)equalities under which they prove the existence of an equilibrium in the continuous time Gloston-Milgrom model with a strategic insider and two possible states of the true asset value. Then, it is shown numerically that the (in)equalities have a solution and an equilibrium is constructed. The decision making in our model is very similar to Das (2005, 2008), who provides methods to simulate the Glosten-Milgrom price process in a discrete time model, and numerically examines some statistical properties of the prices in the market model.

In the current paper, a mathematically rigorous continuous time Glosten-Milgrom model is developed by solving a filtering problem with endogenous filtration, and existence and uniqueness of the price processes are shown under some conditions. The bid and ask prices of the market maker are determined by the zero profit condition, given his information about the time-dependent true value of the asset. However, this information, i.e. the filtration, depends again on the bid and ask prices set. Thus, the market maker influences the learning, leading to a fixed point problem. If, for example, the market maker sets a very large spread, there will be only a small number of trades on which he can base his estimation of the true value. Mathematically this means that the filtering problem is stated with respect to an endogenous filtration. The filtration depends on the bid and ask price processes, which in turn have to be predictable with respect to the filtration. This represents a marked difference relative to other filtering problems in market microstructure models with an unobservable true asset value. However, in these models, point processes are used as well, see e.g. Zeng (2003). We show that Glosten-Milgrom bid and ask price processes are fixed points of certain functionals acting on the set of stochastic processes, and they are given by some deterministic functions of the conditional probabilities of the true value process (under the resulting partial information of the market maker). The conditional probabilities can be obtained as the solution of a system of SDEs.

Filtering problems with endogenous filtration have already appeared in several articles on the Kyle model, see Back (1992), Back and Baruch (2004), Lasserre (2004), Aase et al. (2012), and Biagini et al. (2012), among others. In the Kyle model, a rational price process is characterized as the conditional expectation of the true value of the asset under the filtration of the market maker, which itself depends on the price process through the demand of the insider. However, the inherent fixed point problem, which is solved in a Brownian setting, differs fundamentally from the problem we consider. Namely, in the model we consider, there are only finitely many trades on finite time intervals, as opposed to the continuous accumulation of buys and sells in a Kyle-style model. In addition, the Kyle model cannot explain the bid-ask spread as it models a single price process at which both buy and sell orders are executed.

The paper is organized as follows. In Section 2, the continuous time model is introduced and the main result (Theorem 2.3) is stated. Section 3 considers the static case. Under certain conditions, we prove an existence and uniqueness result (Theorem 3.6). In Section 4, we prove Theorem 2.3 using the results in Section 3. In Section 5, the conditions needed for existence and uniqueness in the dynamic model are discussed. Especially, it is shown that the uniqueness of Glosten-Milgrom prices in a family of static models does not, in general, imply uniqueness of Glosten-Milgrom price processes in dynamic versions of the models.

2 The model and the main result

In the following, a continuous time model for a specialist market will be developed, i.e. a market where a market maker or specialist offers to buy or sell at any point in time to the bid and ask prices he quotes.

All random variables that are introduced live on the probability space $(\Omega, \mathcal{F}, P)$, whereas different filtrations are considered. We assume that the càdlàg process $X = (X_t)_{t\geq 0}$, interpreted as the time-dependent true value of the asset, is a time-homogeneous Markov process with finite state space $\{x_1, \ldots, x_n\} \subset \mathbb{R}$, $n \geq 2$, where $x_{\min} = x_1 < \ldots < x_n = x_{\max}$, and has transition kernel

(2.1)
$$q(i,j) := \lim_{t\to 0} \frac{1}{t} P[X_t = j \mid X_0 = i] \quad \text{for } i \neq j \quad \text{and} \quad q(i,i) := -\sum_{j\neq i} q(i,j).$$

The market maker knows the distribution of X but does not know the actual value. The only source of information that is available to the market maker is the trades that take place at the prices he sets.

To model the customer flow, let N be a Poisson process with rate $\lambda > 0$. We denote the ordered jump times of N by $\tau_1 < \tau_2 < \tau_3, \ldots$ We assume that at these times potential customers arrive at the market (unobserved by the market maker). The customers have some disturbed information about the true value of the asset that is given by $X_{\tau_i} + \epsilon_i$ for the i-th customer, where $(\epsilon_i)_{i\in\mathbb{N}}$ is a sequence of i.i.d. $[-\infty, \infty]$-valued random variables. We assume that X, N, and $(\epsilon_i)_{i\in\mathbb{N}}$ are independent of each other. Note that N models the times at which customers *may* trade, but not the actual trading times. The latter are endogenous in the model and depend, apart from special cases, on the movements of the true value X.

It is further assumed that the market maker sets a pair of prices according to an $\mathcal{F} \otimes \mathcal{B}([0,\infty))$-measurable mapping $S : \Omega \times [0,\infty) \to \mathbb{R}^2$. We write $S = (\overline{S}, \underline{S})$ to denote ask and bid prices, and we only admit prices with $\overline{S}_t(\omega) \geq \underline{S}_t(\omega)$ for all (ω, t). To be economically meaningful, the strategy S has to satisfy some predictability condition that is given in Definition 2.1.

A potential customer buys one asset if $X_{\tau_i} + \epsilon_i \geq \overline{S}_{\tau_i}$ and sells one asset if $X_{\tau_i} + \epsilon_i \leq \underline{S}_{\tau_i}$. He does nothing if his valuation is within the spread.

In the decision making of the customers we follow Das (2005). In the original paper by Glosten and Milgrom (1985) a buy occurs, say, if $\rho_t E[X|\mathcal{A}] \geq \overline{S}_t$, where ρ_t is an independent random variable which represents time-preference and plays the role of ϵ_i in our model. $\rho_t \gg 1$ means that an impatient buyer arrives, and $\rho_t \ll 1$ stands for an impatient seller. The sigma-algebra $\mathcal{A}$ represents the partial information of the insider. For $\mathcal{A} = \sigma(X)$, the models, including possible interpretation of ϵ_i and ρ_t, are similar. Further, the behavior of the customers is not rational. A rational exploitation of the given information would be to buy iff $E[X_{\tau_i}|X_{\tau_i} + \epsilon_i] \geq \overline{S}_{\tau_i}$. A high realization of $X_{\tau_i} + \epsilon_i$ could simply mean that ϵ_i is large, which the customer may be well aware of if he knows the distributions of X_{τ_i} and ϵ_i separately. It was shown by Milgrom and Stokey (1982) that there has to be some irrational behavior for a price to exist. Very often in information-based models (for example in the famous Kyle model, cf. Kyle (1985)), this irrational behavior is introduced by the assumption that there are two types of traders: those who trade on superior information, called insiders (with $\epsilon = 0$), and those who trade for liquidity reasons, sometimes called noise traders (with $\epsilon = \pm\infty$). This describes a limiting case of the model we consider here, where customers' valuation depends on all kinds of noise or preferences.

As the volume of each trade is assumed to be one, we ignore any volume effect.

It is a disputed question among economists whether the volume of a trade has any information content (cf. O'Hara (2007), p.160 ff.).

Let $B_0 = C_0 = 0$. We introduce the sequence of random times of actual buys by

$$B_i := \inf\{\tau_j | \tau_j > B_{i-1}, X_{\tau_j} + \epsilon_j \geq \overline{S}_{\tau_j}\}, \qquad i \geq 1, \tag{2.2}$$

and a sequence of actual sells by

$$C_i := \inf\{\tau_j | \tau_j > C_{i-1}, X_{\tau_j} + \epsilon_j \leq \underline{S}_{\tau_j}\}, \qquad i \geq 1. \tag{2.3}$$

In addition, we define the counting processes of actual buys and sells by

$$N_t^B := \sum_{i \geq 1} 1_{\{B_i \leq t\}} \qquad \text{and} \qquad N_t^C := \sum_{i \geq 1} 1_{\{C_i \leq t\}}. \tag{2.4}$$

The filtration of the market maker is given by $\mathbb{F}^S = (\mathcal{F}_t^S)_{t \geq 0}$, where

$$\mathcal{F}_t^S := \sigma(\{B_i \leq s\}, \{C_i \leq s\}, s \leq t, i \in \mathbb{N}) = \sigma\left(N_s^B, N_s^C, s \leq t\right). \tag{2.5}$$

Since $\mathbb{F}^S$ is generated by counting processes, it is right-continuous (see Theorem I.25 in Protter (2004)). However, it does not in general satisfy the usual conditions, since the null sets are not necessarily included.

From an economic viewpoint, pricing strategies of market makers make sense only if they are $\mathbb{F}^S$-predictable, as $\mathbb{F}^S$ is the information flow of the market maker.

Definition 2.1. *S is an admissible pricing strategy iff it is $\mathbb{F}^S$-predictable and $x_{\max} \geq \overline{S}_t(\omega) \geq \underline{S}_t(\omega) \geq x_{\min}$ for all $(\omega, t) \in \Omega \times \mathbb{R}_+$.*

We impose the restriction that the prices lie between $x_{\min}$ and $x_{\max}$, because otherwise there would be either arbitrage opportunities or no trades at all. The definition is implicit, since the filtration $\mathbb{F}^S$ depends itself on S. Economically, this means that the strategy uses only the information that is generated by past trades. Predictability is assumed since customers react to prices that are published by the market maker in advance. For a trade taking place at time t, the corresponding quotes are fixed under the information $\mathcal{F}_{t-}$. However, by the special form of the filtration $\mathbb{F}^S$, we would obtain the same results with *optional* strategies. This is because the market maker can quote different prices for buyers and sellers, and $\mathbb{F}^S$ contains no other information than buying and selling times. To determine the ask (bid) price, the market maker can implicitly assume that the next customer is a buyer (seller) even if his strategy has to be predictable.

The model stated above gives a natural, though complex, framework within which to examine the price setting of market makers. We now proceed to consider a certain type of price setting that involves the Glosten-Milgrom idea of risk neutrality and perfect competition between market makers.

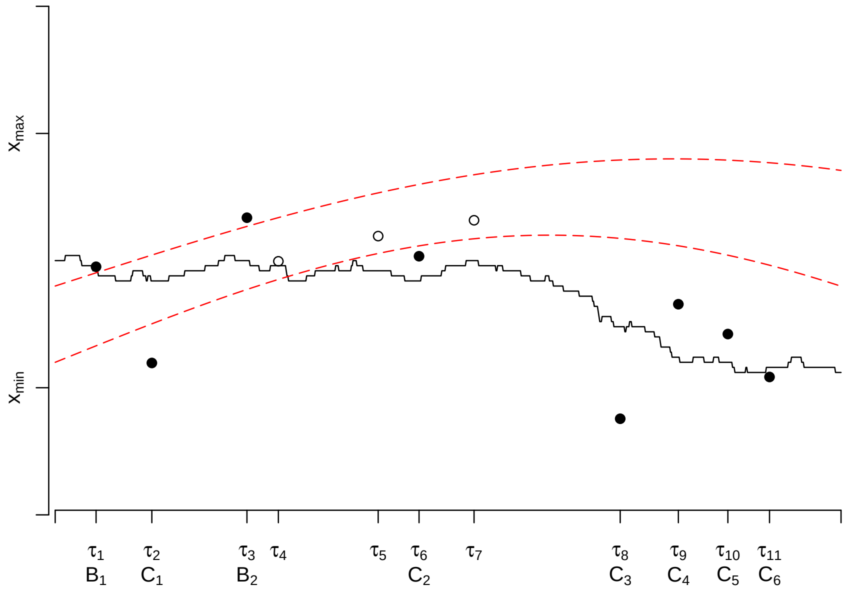

Figure 1: The black line represents the true value X, and some quoted prices $\overline{S} \geq \underline{S}$ (here not at equilibrium) are given by the dotted red lines. All potential trades $X_{\tau_i} + \epsilon_i$ are given by the bullets, which are filled if a trade takes place at $\overline{S}$ or $\underline{S}$.

Definition 2.2. *An admissible pricing strategy S is a Glosten-Milgrom pricing strategy (GMPS) iff*

$$E\left[\sum_{B_i\le\tau}(\overline{S}_{B_i}-X_{B_i})\right]=0 \text{ and } E\left[\sum_{C_i\le\tau}(\underline{S}_{C_i}-X_{C_i})\right]=0 \tag{2.6}$$

for every bounded $\mathbb{F}^S$-stopping time τ.

Each summand in (2.6) is bounded by $x_{\max}-x_{\min}$. Moreover, the sequences $(B_i)_{i\in\mathbb{N}}$ and $(C_i)_{i\in\mathbb{N}}$ are included in the Poisson times. This yields integrability of the sums. The definition implies that neither the buy-side nor the sell-side makes a profit, not only the overall business. We assume that in no stochastic time interval is it possible to make a gain in expectation. Through perfect competition among market makers, it is not possible to offset a loss in order to obtain zero overall profits.

The following theorem is the main result of the article. It guarantees the existence and uniqueness of Glosten-Milgrom pricing strategies under the condition that the "observation error" ϵ_i is volatile, i.e. it has a flat density, in relation to the range of values X can take which should not be too large. In particular, the support of ϵ_i has to be larger than twice the support of X.

Theorem 2.3. *Let $C := x_{\max}-x_{\min}$ and $\Phi(y) := P[\epsilon_1 \ge y]$, $y\in\mathbb{R}$. Assume that Φ is differentiable (i.e. the distribution of ϵ_1 has density $-\Phi'$) on $[-C,C]$, $1>\Phi(0)>0$, and*

$$-\Phi'(y)\le \frac{K}{C}\min\{\Phi(y),1-\Phi(y)\}$$

for all $y\in[-C,C]$ and a constant $K<1$. Then, there exists a Glosten-Milgrom pricing strategy, and it is unique up to a $(P\otimes\lambda)$-null set, where λ denotes the Lebesgue measure on $\mathbb{R}_+$.

Note that ϵ_1 can be $\pm\infty$ with positive probability (i.e. there can be pure noise traders), but $P(\epsilon_1=0)=0$ (no perfect insiders). The theorem is proven in Section 4.

3 Glosten-Milgrom prices in a static model

As a first step to prove Theorem 2.3, we consider a static version of the dynamic Glosten-Milgrom model introduced in Section 2, which also illustrates the idea of Glosten-Milgrom prices. It examines the situation at a time when a potential customer arrives at the market in the continuous time model.

In this section, let X be a real-valued random variable that represents the true value of some asset. We assume that X is unknown to all market participants, but the customer has a disturbed valuation given by $X+\epsilon$, where ϵ represents some observation error or time preference and is independent of X. For the rest of the section, we consider only ask prices because bid and ask prices can be determined independently, and bid prices are developed completely analogously. The independence of the price-setting problems is in contrast to the dynamic case, where some interdependency occurs as the filtration contains the information of both buys and sells. We assume that a potential customer buys if his valuation is higher than the ask price s. Thus, the profit of the market maker is given by $(s-X)1_{\{X+\epsilon\geq s\}}$. Again, we assume that the price setting must satisfy a zero expected profit condition.

Definition 3.1. *$s \in \mathbb{R}$ is a static Glosten-Milgrom ask price iff*

$$E[(s-X)1_{\{X+\epsilon\geq s\}}] = 0. \tag{3.1}$$

The question is now whether solutions to (3.1) exist and if so, whether they are unique. Roughly speaking, a Glosten-Milgrom price exists and is unique if the tails of ϵ are heavy enough compared with those of X. Let us start with simple counterexamples for existence and uniqueness.

Example 3.2. *Let $\epsilon = 0$ and assume that X is not essentially bounded from above. Then, there exists no $s \in \mathbb{R}_+$ with $E[(s-X)1_{\{X\geq s\}}] = 0$, since the integrand is always non-positive and negative with positive probability. For $\epsilon = 0$ and X essentially bounded by $x_{\max}$, $s = x_{\max}$ is the unique (trivial) solution in $(-\infty, x_{\max}]$.*

Example 3.3. *Let ϵ assume values 1 and -1 with probability $1/2$, and let X be equal to 1 with probability $3/4$ and 3 with probability $1/4$. Then, $9/5$ and 3 are solutions of (3.1), i.e. the static Glosten-Milgrom ask price is not unique.*

Lemma 3.4. *Let X be bounded between $x_{\min}$ and $x_{\max}$ a.s. and define $C := x_{\max} - x_{\min}$. Let $\Phi(y) := P[\epsilon \geq y]$ be the inverse distribution function of ϵ. If Φ is differentiable (i.e. the distribution of ϵ has density $-\Phi'$) on $[-C, C]$, $\Phi(0) > 0$ and*

$$-\Phi'(y) \leq \frac{K}{C}\Phi(y) \tag{3.2}$$

for all $y \in [-C, C]$ and a constant $K < 1$, then we have $\Phi(C) > 0$, which implies that $P[X + \epsilon \geq s] > 0$, i.e. the probability that a buy occurs is strictly larger than 0 for all prices $s \leq x_{\max}$.

Proof. Remember that Φ is $[0, 1]$-valued and decreasing. We have

$$\begin{aligned}\Phi(C) &= \int_0^C \Phi'(t)dt + \Phi(0) \geq \frac{K}{C}\int_0^C -\Phi(t)dt + \Phi(0) \\ &\geq -\frac{K}{C}C\Phi(0) + \Phi(0) \geq (1-K)\Phi(0) > 0,\end{aligned}$$

since $K < 1$ and $\Phi(0) > 0$. Furthermore, we have for all $s \leq x_{\max}$

$$P[X + \epsilon \geq s] = P[\epsilon \geq s - X] \geq P[\epsilon \geq x_{\max} - x_{\min}] = \Phi(C) > 0. \qquad \square$$

Under the assumptions of Lemma 3.4, we make the following definition.

Definition 3.5. *The distribution of X is indicated by π. For $s \in [x_{\min}, x_{\max}]$ we define*

$$g(s,\pi) := E[X|X+\epsilon \geq s] := \frac{E\left[X1_{\{X+\epsilon \geq s\}}\right]}{P\left[X+\epsilon \geq s\right]} = \frac{E[X\Phi(s-X)]}{E[\Phi(s-X)]}.$$

Lemma 3.4 ensures that g is well-defined for every π. Now, the zero profit condition (3.1) translates to

$$g(s,\pi) = s.$$

Thus, for given π, the question of existence and uniqueness of a Glosten-Milgrom ask price is the same as the existence and uniqueness of a fixed point of g, which indicates a way of proving the following theorem.

Theorem 3.6. *Let all assumptions of Lemma 3.4 be fulfilled. Then, there exists a unique static Glosten-Milgrom ask price in $[x_{\min}, x_{\max}]$.*

For parametric families of distributions of ϵ, (e.g. the normal distribution), condition (3.2) can usually be guaranteed by choosing parameters such that the variance is high. In economic terms, this corresponds to customers whose information is less precise or who are impatient. We still allow that ϵ takes the values $\pm\infty$.

Observe that we do not impose any assumption on the distribution of X apart from the boundedness but explicit assumptions on the distribution of ϵ. The fact that we have existence and uniqueness for all distributions of X with compact support is essential for the continuous time model as the distribution changes over time.

Proof. Since π is fixed, we omit it. We consider the derivative of

$$g(s) = E[X|X+\epsilon \geq s] = \frac{E[X\Phi(s-X)]}{E[\Phi(s-X)]}$$

for $x_{\min} \leq s \leq x_{\max}$ which is given by

$$
\begin{aligned}
g'(s) &= \frac{E_X[X\Phi'(s-X)]E_Z[\Phi(s-Z)] - E_Z[Z\Phi(s-Z)]E_X[\Phi'(s-X)]}{(E_X[\Phi(s-X)])^2} \\
&= \frac{E_X[E_Z[X\Phi'(s-X)\Phi(s-Z) - Z\Phi(s-Z)\Phi'(s-X)]]}{E_X[E_Z[\Phi(s-X)\Phi(s-Z)]]} \\
&= \frac{E_X[E_Z[-\Phi'(s-X)\Phi(s-Z)(Z-X)]]}{E_X[E_Z[\Phi(s-X)\Phi(s-Z)]]} \\
&\leq \frac{E_X[E_Z[-\Phi'(s-X)\Phi(s-Z)|Z-X|]]}{E_X[E_Z[\Phi(s-X)\Phi(s-Z)]]} \\
&\leq C\frac{E_X[E_Z[-\Phi'(s-X)\Phi(s-Z)]]}{E_X[E_Z[\Phi(s-X)\Phi(s-Z)]]} \\
&\leq C\frac{K}{C}\frac{E_X[E_Z[\Phi(s-X)\Phi(s-Z)]]}{E_X[E_Z[\Phi(s-X)\Phi(s-Z)]]} \\
&= K,
\end{aligned}
$$

for K from (3.2), where Z is an independent copy of X. Hence, $0 \leq g'(s) \leq K < 1$ for all $s \in [x_{\min}, x_{\max}]$ and therefore

$$
|g(s) - g(t)| \leq K|s-t|. \tag{3.3}
$$

This means that g is a contraction which has a unique fixed point by the Banach fixed point theorem. ☐

We have already seen that g is Lipschitz continuous in s with parameter $K < 1$. If the law of X is discrete, we further obtain Lipschitz continuity in the distribution π (which we use in the proof of Theorem 2.3).

Lemma 3.7. *Let all assumptions of Lemma 3.4 be satisfied. In addition, assume that X takes only finitely many values, i.e. there exist $x_{\min} = x_1 < \ldots < x_n = x_{\max}$ and $\pi = (\pi_1, \ldots, \pi_n)$ such that $P[X = x_i] = \pi_i$ for all i and $\sum_{i=1}^n \pi_i = 1$. Then, we have*

$$
|g(s,\pi) - g(\widetilde{s},\widetilde{\pi})| \leq K|s-\widetilde{s}| + L\sum_{i=1}^n |\pi_i - \widetilde{\pi}_i|
$$

for K from (3.2), $L = 2x_{\max}/\Phi(C)^2 < \infty$, all $s, \widetilde{s} \in [x_{\min}, x_{\max}]$, and all distributions $\pi, \widetilde{\pi}$.

Proof. First, we see that

$$
\begin{aligned}
|g(s,\pi) - g(\widetilde{s},\widetilde{\pi})| &= |g(s,\pi) - g(s,\widetilde{\pi}) + g(s,\widetilde{\pi}) - g(\widetilde{s},\widetilde{\pi})| \\
&\leq |g(s,\pi) - g(s,\widetilde{\pi})| + K|s-\widetilde{s}|
\end{aligned} \tag{3.4}
$$

by (3.3). It remains to show that

$$
|g(s,\pi) - g(s,\widetilde{\pi})| \leq L\sum_{i=1}^n |\pi_i - \widetilde{\pi}_i|.
$$

To shorten the notation, we write

$$\alpha(f(X), \pi) := E[f(X)\Phi(s - X)] = \sum_{i=1}^{n} \pi_i f(x_i)\Phi(s - x_i).$$

Hence, we have

$$g(s, \pi) = \frac{\alpha(X, \pi)}{\alpha(1, \pi)}$$

and

$$\begin{aligned}
|g(s,\pi) - g(s,\widetilde{\pi})| &= \left| \frac{\alpha(X,\pi)}{\alpha(1,\pi)} - \frac{\alpha(X,\widetilde{\pi})}{\alpha(1,\widetilde{\pi})} \right| \\
&= \frac{|\alpha(X,\pi)\alpha(1,\widetilde{\pi}) - \alpha(X,\widetilde{\pi})\alpha(1,\pi)|}{\alpha(1,\pi)\alpha(1,\widetilde{\pi})} \\
&\leq \frac{|\alpha(X,\pi)\alpha(1,\widetilde{\pi}) - \alpha(X,\pi)\alpha(1,\pi)|}{\alpha(1,\pi)\alpha(1,\widetilde{\pi})} \\
&\quad + \frac{|\alpha(X,\pi)\alpha(1,\pi) - \alpha(X,\widetilde{\pi})\alpha(1,\pi)|}{\alpha(1,\pi)\alpha(1,\widetilde{\pi})} \\
&= \frac{|\alpha(X,\pi)\sum_{i=1}^{n}(\widetilde{\pi}_i - \pi_i)\Phi(s - x_i)|}{\alpha(1,\pi)\alpha(1,\widetilde{\pi})} \\
&\quad + \frac{|\alpha(1,\pi)\sum_{i=1}^{n}(\pi_i - \widetilde{\pi}_i)x_i\Phi(s - x_i)|}{\alpha(1,\pi)\alpha(1,\widetilde{\pi})} \\
&\leq L\sum_{i=1}^{n} |\pi_i - \widetilde{\pi}_i|.
\end{aligned}$$

The last inequality follows from $x_i\Phi(s - x_i) \leq x_{\max} < \infty$, $\Phi(s - x_i) \leq 1$ for all i, and $\alpha(1, \pi) \geq \Phi(s - x_{\min}) \geq \Phi(C) > 0$. Together with (3.4), this proves the lemma. □

Remark 3.8. *Analogously, the static Glosten-Milgrom bid price is characterized by the condition that*

$$h(s, \pi) := E[X \mid X + \epsilon \leq s] = s. \tag{3.5}$$

By the conditions $1 > \Phi(0)$ *and*

$$-\Phi'(y) \leq \frac{K}{C}(1 - \Phi(y))$$

for all $y \in [-C, C]$ *and a constant* $K < 1$*, which appear in Theorem 2.3, it is guaranteed that analogs of Lemma 3.4, Theorem 3.6, and Lemma 3.7 hold for the bid price.*

4 Proof of Theorem 2.3

To prove the existence and uniqueness of a solution of a GMPS, we first characterize it as a fixed point of a functional F (see Definition 4.3) that is defined on the set of admissible pricing strategies (see Theorem 4.10). Then, we show a contraction property of F (see Lemma 4.13), which can be used to verify uniqueness. Finally, we conclude that F possesses a fixed point. Since F does not generally map into the set of admissible strategies, and the contraction property generally holds only if the arguments are admissible strategies, we cannot just use a Picard-iteration.

4.1 Glosten-Milgrom strategies as fixed points

The filtration of the market maker $\mathbb{F}^S$ does not satisfy the usual conditions, since it does not contain all null sets. We now define the completion $\widetilde{\mathbb{F}}^S$ of $\mathbb{F}^S$.

Definition 4.1. *For any $\mathcal{F} \otimes \mathcal{B}([0,\infty))$-measurable process $S = (\overline{S}, \underline{S})$ let the filtration $\widetilde{\mathbb{F}}^S$ be defined by*

$$\widetilde{\mathcal{F}}_t^S := \mathcal{F}_t^S \vee \mathcal{N},$$

where $\mathcal{N}$ are all P-null sets of $\mathcal{F}$.

$\widetilde{\mathbb{F}}^S$ is used in the proof, but it is not needed to state our main result, Theorem 2.3.

Lemma 4.2. *For any $\mathcal{F} \otimes \mathcal{B}([0,\infty))$-measurable process $S = (\overline{S}, \underline{S})$, there exists a unique (up to indistinguishability) $\widetilde{\mathbb{F}}^S$-adapted càdlàg process π^S with*

$$\pi_\tau^S = \left(P\left[X_\tau = x_i | \mathcal{F}_\tau^S\right]\right)_{i=1,\ldots,n} \qquad P\text{-a.s.} \tag{4.1}$$

for all finite stopping times τ.

Proof. Since $\widetilde{\mathbb{F}}^S$ satisfies the usual conditions, we can apply Theorems 2.7 and 2.9 of Bain and Crisan (2008) to $\widetilde{\mathbb{F}}^S$ and the process $\left(1_{\{X_t=x_i\}}\right)_{i=1,\ldots,n}$, which gives us a càdlàg optional projection π^S, i.e. an $\widetilde{\mathbb{F}}^S$-optional process satisfying

$$\pi_\tau^S = \left(P\left[X_\tau = x_i | \widetilde{\mathcal{F}}_\tau^S\right]\right)_{i=1,\ldots,n} \qquad P\text{-a.s.}$$

for all finite stopping times τ. Since $E[\,\cdot\,|\mathcal{F}_\tau]$ and $E[\,\cdot\,|\widetilde{\mathcal{F}}_\tau]$ only differ by a P-null set, (4.1) follows. □

Since π^S is càdlàg , π_{t-}^S exists, and we can now define the before mentioned functional.

Definition 4.3. *For admissible S we define $F(S) : \Omega \times [0,\infty) \to \mathbb{R}^2$ by*

$$F(S)_t := \left(\overline{F(S)}_t, \underline{F(S)}_t\right) := \left(g\left(\overline{S}_t, \pi_{t-}^S\right), h\left(\underline{S}_t, \pi_{t-}^S\right)\right),$$

where g is defined in Definition 3.5, h in (3.5), and π^S in (4.1).

As a continuous function of $\widetilde{\mathbb{F}}^S$-predictable processes, $F(S)$ is $\widetilde{\mathbb{F}}^S$-predictable. By definition of g and h and the fact that S is admissible, it follows that $\overline{F(S)}_t(\omega) \geq \underline{F(S)}_t(\omega)$ for all (ω, t). However, $F(S)$ is not necessarily predictable with respect to the filtration $\mathbb{F}^{F(S)}$ that is defined as in (2.5) after replacing $\overline{S}$ by $\overline{F(S)}$ and $\underline{S}$ by $\underline{F(S)}$ in (2.2) respectively (2.3). In general, $\mathbb{F}^S \neq \mathbb{F}^{F(S)}$ (also after completion of the filtrations). Thus, $F(S)$ need not be admissible. Economically, this means that the strategy $F(S)$ uses information that is only available if the market maker realizes the strategy S.

We now define a larger filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ by

$$\mathcal{F}_t := \sigma\left(X_s, N_s, \epsilon_i 1_{\{\tau_i \leq s\}}, s \leq t, i \in \mathbb{N}\right) \vee \mathcal{N},$$

which contains all information up to time t. The following lemma describes the intensity of the jump process N^B that counts actual buys (for a given pricing strategy S) as described in Section 2.

Lemma 4.4. *The $\mathbb{F}$-intensity of N^B (in the sense of Brémaud (1981) II, D7) is given by $\lambda\Phi(\overline{S} - X_-)$.*

Proof. Let C be a nonnegative $\mathbb{F}$-predictable process. Then

$$\begin{aligned} E\left[\int_0^\infty C_s dN_s^B\right] &= E\left[\sum_{i=1}^\infty C_{\tau_i} 1_{\{X_{\tau_i}+\epsilon_i \geq \overline{S}_{\tau_i}\}}\right] \\ &= \sum_{i=1}^\infty E\left[E\left[C_{\tau_i} 1_{\{X_{\tau_i}+\epsilon_i \geq \overline{S}_{\tau_i}\}} | \mathcal{F}_{\tau_i-}\right]\right] \\ &= \sum_{i=1}^\infty E\left[C_{\tau_i}\Phi(\overline{S}_{\tau_i} - X_{\tau_i-})\right] \\ &= E\left[\int_0^\infty C_s\Phi(\overline{S}_s - X_{s-}) dN_s\right] \\ &= E\left[\int_0^\infty C_s\lambda\Phi(\overline{S}_s - X_{s-}) ds\right], \end{aligned}$$

where we use $X_{\tau_i} = X_{\tau_i-}$ P-a.s. for the third equation. □

We now derive the filtering equation for π^S.

Lemma 4.5. *The process* π^S *satisfies the following SDE*

$$
\begin{aligned}
d\pi_t^{S,i} =& \pi_{t-}^{S,i}\left(\frac{\Phi(\overline{S}_t - x_i)}{\sum_j \pi_{t-}^{S,j}\Phi(\overline{S}_t - x_j)} - 1\right) dN_t^B \\
&+ \pi_{t-}^{S,i}\left(\frac{\Psi(\underline{S}_t - x_i)}{\sum_j \pi_{t-}^{S,j}\Psi(\underline{S}_t - x_j)} - 1\right) dN_t^C \\
&- \Bigg(\lambda \pi_t^{S,i}\Bigg(\Psi(\underline{S}_t - x_i) + \Phi(\overline{S}_t - x_i) \\
&- \sum_j \pi_t^{S,j}\left(\Psi(\underline{S}_t - x_j) + \Phi(\overline{S}_t - x_j)\right)\Bigg) \\
&+ \sum_j \pi_t^{S,j} q(j,i)\Bigg) dt, \quad t \geq 0,
\end{aligned} \tag{4.2}
$$

up to indistinguishability, with initial condition $\pi_0^{S,i} = P[X_0 = x_i]$*, where* $\Psi(x) = P[\epsilon_1 \leq x]$*.*

Proof. Following Brémaud (1981) IV, T2, the filtering equation for π^S reads

$$
\pi_t^S = \pi_0^S + \int_0^t K_s^B dN_s^B + \int_0^t K_s^C dN_s^C + \int_0^t \left(-K_s^B \widehat{\lambda}^B - K_s^C \widehat{\lambda}^C + f_s\right) ds,
$$

where $\widehat{\lambda}^B$ and $\widehat{\lambda}^C$ are the $\widetilde{\mathbb{F}}^S$-intensities of N^B and N^C respectively, f is the $\widetilde{\mathbb{F}}^S$-compensator of X, which is given by $\sum_j \pi^{S,j} q(j,\cdot)$, $K_s^B = \Psi_s^B - \pi_{s-}^S$ and $K_s^C = \Psi_s^C - \pi_{s-}^S$, where Ψ_s^B is the unique (up to a $(P\otimes\lambda)$-null set) $\widetilde{\mathbb{F}}^S$-predictable process satisfying

$$
E\left[\int_0^t C_s 1_{\{X_s = x_i\}} \lambda_s^B ds\right] = E\left[\int_0^t C_s \Psi_s^{B,i} \widehat{\lambda}_s^B ds\right] \tag{4.3}
$$

for all $\widetilde{\mathbb{F}}^S$-predictable nonnegative bounded processes C, $i = 1, \dots, n$ and all $t \geq 0$, where $\lambda^B, \widehat{\lambda}^B$ are the $\mathbb{F}, \widetilde{\mathbb{F}}^S$-intensities of N^B respectively. A similar equation holds for Ψ_s^C.

From Lemma 4.4, we have that $\lambda^B = \lambda\Phi(\overline{S} - X_-)$ and

$$
\begin{aligned}
E\left[\lambda_s^B \mid \mathcal{F}_s^S\right] &= \sum_{i=1}^n E\left[1_{\{X_{s-} = x_i\}} \lambda\Phi(\overline{S}_s - x_i) \mid \mathcal{F}_s^S\right] \\
&= \sum_{i=1}^n \lambda\Phi(\overline{S}_s - x_i) E\left[1_{\{X_{s-} = x_i\}} \mid \mathcal{F}_s^S\right] \ P\text{-a.s..}
\end{aligned}
$$

Since, for fixed $s \in \mathbb{R}_+$, $X_s = X_{s-}$ P-a.s., $\lambda \sum_{i=1}^n \pi_{s-}^{S,i} \Phi(\overline{S}_s - x_i)$ is a version of $\widehat{\lambda}_s^B$. From this and as $\pi_-^{S,i}$ is the $\widetilde{\mathbb{F}}^S$-compensator of $1_{\{X=x_i\}}$, it follows that

$$\Psi_s^{B,i} = \frac{\pi_{s-}^{S,i} \Phi(\overline{S}_s - x_i)}{\sum_{j=1}^n \pi_{s-}^{S,j} \Phi(\overline{S}_s - x_j)}$$

solves (4.3), which gives us K_s^B and the analog K_s^C as stated in the lemma.

For the buy-side part of the dt-term, we get

$$-K_s^{B,i} \widehat{\lambda}_s^B = -\left(\frac{\pi_{s-}^{S,i} \Phi(\overline{S}_s - x_i)}{\sum_{j=1}^n \pi_{s-}^{S,j} \Phi(\overline{S}_s - x_j)} - \pi_s^{S,i} \right) \lambda \sum_{j=1}^n \pi_{s-}^{S,j} \Phi(\overline{S}_s - x_j),$$

which simplifies to

$$-\lambda \pi_{s-}^{S,i} \Phi(\overline{S}_s - x_i) + \lambda \pi_{s-}^{S,i} \sum_{j=1}^n \pi_{s-}^{S,j} \Phi(\overline{S}_s - x_j).$$

Together with f and similar results for the sell-side, we obtain the dt-term stated in the lemma which completes the proof. □

We are now in the position to prove the following lemma.

Lemma 4.6. *We have*

$$\overline{F(S)}_{B_i} = E[X_{B_i} | \mathcal{F}_{B_i}^S] \qquad \text{and} \qquad \underline{F(S)}_{C_i} = E[X_{C_i} | \mathcal{F}_{C_i}^S] \qquad P\text{-a.s.}$$

for all $i \in \mathbb{N}$.

Proof. Because of the filtering equation in Lemma 4.5 and the fact that N^B and N^C have no common jumps, it follows that

$$\pi_{B_i}^{S,k} = \frac{\pi_{B_i-}^{S,k} \Phi(\overline{S}_{B_i} - x_k)}{\sum_{j=1}^n \pi_{B_i-}^{S,j} \Phi(\overline{S}_{B_i} - x_j)}, \qquad \text{for } k = 1, \dots, n. \tag{4.4}$$

By definition of F and g and by (4.4), we have

$$\begin{aligned} \overline{F(S)}_{B_i} = g\left(\overline{S}_{B_i}, \pi_{B_i-}^S\right) &= \frac{\sum_{j=1}^n x_j \pi_{B_i-}^{S,j} \Phi(\overline{S}_{B_i} - x_j)}{\sum_{j=1}^n \pi_{B_i-}^{S,j} \Phi(\overline{S}_{B_i} - x_j)} \\ &= \sum_{j=1}^n x_j \pi_{B_i}^{S,j} = E[X_{B_i} | \mathcal{F}_{B_i}^S]. \end{aligned}$$

The same holds true at the times when a sell occurs. □

The importance of that assertion becomes clear if we put it together with the next lemma that provides an equivalent, and maybe even more intuitive, characterization of a GMPS than the criterion in Definition 2.2.

Lemma 4.7. *S is a GMPS iff it is admissible and*

$$\overline{S}_{B_i} = E[X_{B_i}|\mathcal{F}^S_{B_i}] \qquad \textit{and} \qquad \underline{S}_{C_i} = E[X_{C_i}|\mathcal{F}^S_{C_i}] \qquad P\text{-a.s.}$$

for all $i \in \mathbb{N}$.

This means that all trades in a GMPS are executed at a price that is the expectation of X, given the information available to market makers at that time. The interesting point about this characterization of GMPS is that a trade occuring at that very moment is included in the filtration but its occurrence and especially its direction are not predictable, in contrast to S.

Remark 4.8. *In view of Lemma 4.6 and Lemma 4.7, we can give an intuitive description of F. $F(S)$ are the Glosten-Milgrom prices (i.e. zero expected profits) a market maker would have in mind if actually the prices S were quoted, and the market reacted with buys and sells to them (which leads to the filtration $\mathbb{F}^S$), see Figure 2.*

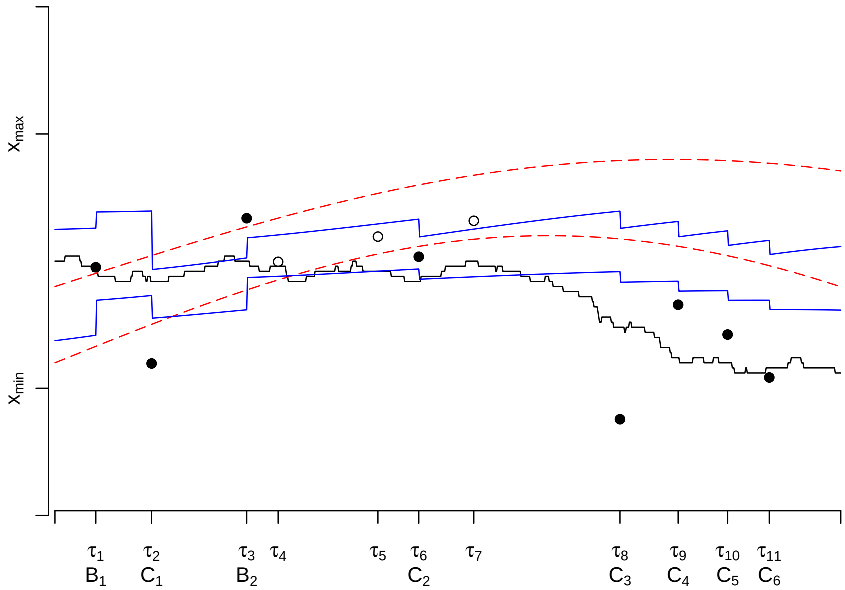

Figure 2: We add $F(S)$ to Figure 1, which are the fictitious Glosten-Milgrom prices (i.e. zero expected profits) of the market maker if prices S were actually quoted, and the market reacted with buys and sells to them.

Proof of Lemma 4.7. We only consider buys. Let S be a GMPS. For fixed $i \in \mathbb{N}$, we consider sets of the form

$$C := \{B_{i-1} \le t < B_i\} \cap A, \quad t \in \mathbb{R}_+, \ A \in \mathcal{F}^S_t. \tag{4.5}$$

For $n \in \mathbb{N}, n > t$, let $\kappa_n(\omega) := t$ if $\omega \notin C$ and $\kappa_n(\omega) := B_i(\omega) \wedge n$ if $\omega \in C$. Hence $t \leq \kappa_n$ for all n and both are bounded $\mathbb{F}^S$-stopping times. Thus, we have

$$\begin{aligned} 0 &= E\left[\sum_{B_j \leq \kappa_n} (\overline{S}_{B_j} - X_{B_j})\right] - E\left[\sum_{B_j \leq t} (\overline{S}_{B_j} - X_{B_j})\right] \\ &= E\left[\sum_{t < B_j \leq \kappa_n} (\overline{S}_{B_j} - X_{B_j}) 1_C\right] = E\left[(\overline{S}_{B_i} - X_{B_i}) 1_{C \cap \{B_i \leq n\}}\right] \end{aligned}$$

for all n and therefore

$$E\left[(\overline{S}_{B_i} - X_{B_i}) 1_C\right] = 0. \tag{4.6}$$

$\mathcal{F}^S_{B_i-}$ is generated by sets of the form $\{t < B_i\} \cap A$, where $A \in \mathcal{F}^S_t$ and $t \in \mathbb{R}_+$, which can be written as

$$\{t < B_i\} \cap A = \cup_{t \leq t_n \in \mathbb{Q}} \{B_{i-1} \leq t_n < B_i\} \cap A.$$

Hence, as $A \in \mathcal{F}_t \subset \mathcal{F}_{t_n}$ for $t_n \geq t$ and as $\{B_{i-1} \leq t_n\} \in \mathcal{F}^S_{t_n}$, the sets from (4.5) generate $\mathcal{F}^S_{B_i-}$. Since in addition $\overline{S}$ is predictable, it follows from (4.6) that

$$\overline{S}_{B_i} = E\left[X_{B_i} | \mathcal{F}^S_{B_i-}\right] \qquad P\text{-a.s.}. \tag{4.7}$$

Let us show that $\mathcal{F}^S_{B_i-} = \mathcal{F}^S_{B_i}$. We consider the marked point process $(T_n, Z_n)_{n \in \mathbb{N}}$ with

$$T_n := \inf\left\{t \geq 0 \mid \sum_{i \geq 1} 1_{\{B_i \leq t\}} + \sum_{i \geq 1} 1_{\{C_i \leq t\}} \geq n\right\},$$

which are the times of trades (buys and sells), and $Z_n := 1$ if $T_n = B_i$ for some i and $Z_n := -1$ if $T_n = C_i$ (here we use that buys and sells never happen simultaneously).

We have

$$\mathcal{F}^S_{B_i} = \{A \mid A = \cup_{n \in \mathbb{N}} A_n \cap \{B_i = T_n\} \text{ for } A_n \in \mathcal{F}^S_{T_n} \text{ for all } n\} \tag{4.8}$$

and

$$\mathcal{F}^S_{B_i-} = \{A \mid A = \cup_{n \in \mathbb{N}} A_n \cap \{B_i = T_n\} \text{ for } A_n \in \mathcal{F}^S_{T_n-} \text{ for all } n\}. \tag{4.9}$$

Indeed, the first equation is obvious. For the inclusion "$\subset$" of the second it suffices to show that sets of the form $A \cap \{t < B_i\}$, $A \in \mathcal{F}^S_t$ are contained in the set on the RHS. This can be done by choosing $A_n = A \cap \{t < T_n\}$. For the inclusion "$\supset$" it again suffices to consider sets of the form $A_n = \widetilde{A}_n \cap \{t < T_n\}$, $\widetilde{A}_n \in \mathcal{F}^S_t$. It then remains to show that $\{B_i = T_n\} \in \mathcal{F}^S_{B_i-}$, which follows from

$$\{T_n < B_i\} = \cup_{q \in \mathbb{Q}} \{T_n < q\} \cap \{q < B_i\} \in \mathcal{F}^S_{B_i-}$$

and

$$\{B_i = T_n\} = \{T_{n-1} < B_i\} \cap \{T_n < B_i\}^c \in \mathcal{F}^S_{B_i-}.$$

Now, by Theorem III, T2 in Brémaud (1981) applied to the marked point process $(T_n, Z_n)_{n \in \mathbb{N}}$, any $A_n \in \mathcal{F}^S_{T_n}$ can be written as

$$A_n = (M_1 \cap \{Z_n = 1\}) \cup (M_2 \cap \{Z_n = -1\})$$

for some $M_1, M_2 \in \mathcal{F}^S_{T_n-}$. Since $\{B_i = T_n\} \subset \{Z_n = 1\}$, we have

$$A_n \cap \{B_i = T_n\} = M_1 \cap \{B_i = T_n\},$$

and $\mathcal{F}^S_{B_i-} = \mathcal{F}^S_{B_i}$ follows from (4.8) and (4.9). Together with (4.7), one direction of the lemma is proven.

Now let $\overline{S}_{B_i} = E[X_{B_i} | \mathcal{F}^S_{B_i}]$ for all $i \in \mathbb{N}$ and τ be a bounded $\mathbb{F}^S$-stopping time. We obtain that

$$\begin{aligned} E\left[\sum_{B_i \le \tau} (\overline{S}_{B_i} - X_{B_i})\right] &= \sum_{i=1}^{\infty} E\left[1_{\{B_i \le \tau\}} (\overline{S}_{B_i} - X_{B_i})\right] \\ &= \sum_{i=1}^{\infty} E\left[E\left[1_{\{B_i \le \tau\}} (\overline{S}_{B_i} - X_{B_i}) | \mathcal{F}^S_{B_i}\right]\right] \\ &= \sum_{i=1}^{\infty} E\left[1_{\{B_i \le \tau\}} (\overline{S}_{B_i} - E\left[X_{B_i} | \mathcal{F}^S_{B_i}\right])\right] \\ &= 0. \end{aligned}$$

□

Definition 4.9. *We say that an admissible strategy S is a fixed point of F iff $S = F(S)$ $(P \otimes \lambda)$-a.e. (where λ denotes the Lebesgue measure on $\mathbb{R}_+$).*

Theorem 4.10. *An admissible strategy S is a solution of the GMPS-problem iff S is a fixed point of F.*

Proof. For a predictable set B, we have

$$(P \otimes \lambda)(B) = \sum_{j=1}^{\infty} P[(\omega, \tau_j(\omega)) \in B],$$

i.e. predictable sets coincide $(P \otimes \lambda)$-a.e. iff they coincide at all Poisson times P-a.s..

Let S be a fixed point of F. With Lemma 4.6 we obtain

$$\overline{S}_{B_i} = \overline{F(S)}_{B_i} = E[X_{B_i}|\mathcal{F}^S_{B_i}] \text{ and } \underline{S}_{C_i} = \underline{F(S)}_{C_i} = E[X_{C_i}|\mathcal{F}^S_{C_i}] \qquad P\text{-a.s.}$$

for all $i \in \mathbb{N}$. Lemma 4.7 now yields that S is a GMPS.

Now let S be a GMPS and $j \in \mathbb{N}$ be fixed. We consider the set

$$A := \left\{ \overline{S}_{\tau_j} \neq \overline{F(S)}_{\tau_j} \right\},$$

which is in $\mathcal{F}_{\tau_j -}$ since S and $F(S)$ are $\mathbb{F}^S$-predictable and hence $\mathbb{F}$-predictable. Because S is a GMPS and ϵ_j is independent of $\mathcal{F}_{\tau_j -}$, we have by Lemma 4.7

$$\begin{aligned} 0 &= P\left[\overline{S}_{B_i} \neq \overline{F(S)}_{B_i} \text{ for some } i \in \mathbb{N} \right] \\ &\geq P[\{\tau_j = B_i \text{ for some } i \in \mathbb{N}\} \cap A] = P[\{X_{\tau_j} + \epsilon_j \geq \overline{S}_{\tau_j}\} \cap A] \\ &\geq P[\{\epsilon_j \geq x_{\max} - x_{\min}\} \cap A] = P[A]P[\epsilon_j \geq x_{\max} - x_{\min}]. \end{aligned}$$

Since by Lemma 3.4 $P[\epsilon_j \geq x_{\max} - x_{\min}] > 0$, it follows that $P[A] = 0$, i.e.

$$\overline{S}_{\tau_j} = \overline{F(S)}_{\tau_j} \qquad P\text{-a.s.}. \qquad \square$$

4.2 Uniqueness

First the uniqueness of the solution is shown by proving that F is a contraction in the sense of Lemma 4.13. Let S and T be admissible pricing strategies.

Definition 4.11. *For given pricing strategies S, T let*

$$A^1_s := \{X_{\tau_i} + \epsilon_i \notin [\min\{\overline{S}_{\tau_i}, \overline{T}_{\tau_i}\}, \max\{\overline{S}_{\tau_i}, \overline{T}_{\tau_i}\}) \textit{ for all } \tau_i \leq s\}.$$

A^1_s is the event that until s no buy occurred only under one of the two pricing strategies S and T.

Lemma 4.12. *We have that*

$$P\left[(A^1_s)^c\right] \leq \lambda M E\left[\int_0^s |\overline{S}_u - \overline{T}_u| du\right], \quad \textit{where } M := \max\{\Phi'(x) | x \in [-C, C]\}.$$

Proof. Let Y be the process that counts the number of buys that only occur under one of the two pricing strategies, i.e.

$$Y_t := \sum_{i \in \mathbb{N}} 1_{\left\{\tau_i \leq t, X_{\tau_i} + \epsilon_i \in [\min\{\overline{S}_{\tau_i}, \overline{T}_{\tau_i}\}, \max\{\overline{S}_{\tau_i}, \overline{T}_{\tau_i}\})\right\}}.$$

We now show (essentially with the methods of the proof of Lemma 4.4) that the $\mathbb{F}$-intensity of Y is given by

$$\lambda_t^Y := \lambda(\Phi(\min\{\overline{S}_t, \overline{T}_t\} - X_{t-}) - \Phi(\max\{\overline{S}_t, \overline{T}_t\} - X_{t-})) \leq \lambda M |\overline{S}_t - \overline{T}_t|.$$

Let C be a nonnegative $\mathbb{F}$-predictable process. As $\overline{S}$ and $\overline{T}$ are $\mathbb{F}$-predictable and $P[X_{\tau_i} = X_{\tau_i -}] = 1$, we obtain

$$\begin{aligned} E\left[\int_0^\infty C_s dY_s\right] &= E\left[\sum_{i=1}^\infty C_{\tau_i} 1_{\{\min\{\overline{S}_{\tau_i}, \overline{T}_{\tau_i}\} \leq X_{\tau_i} + \epsilon_i < \max\{\overline{S}_{\tau_i}, \overline{T}_{\tau_i}\}\}}\right] \\ &= \sum_{i=1}^\infty E\left[E\left[C_{\tau_i} 1_{\{\min\{\overline{S}_{\tau_i}, \overline{T}_{\tau_i}\} \leq X_{\tau_i} + \epsilon_i < \max\{\overline{S}_{\tau_i}, \overline{T}_{\tau_i}\}\}} | \mathcal{F}_{\tau_i -}\right]\right] \\ &= \sum_{i=1}^\infty E\left[C_{\tau_i} \lambda_{\tau_i}^Y\right] = E\left[\int_0^\infty C_s \lambda_s^Y ds\right]. \end{aligned}$$

We define $\tau_Y := \inf\{t \geq 0 \mid Y_t = 1\}$ and get

$$\begin{aligned} P[(A_s^1)^c] = P[Y_s \neq 0] = E\left[\int_0^s 1_{\{\tau_Y \geq u\}} dY_u\right] &= E\left[\int_0^s 1_{\{\tau_Y \geq u\}} \lambda_u^Y du\right] \\ \leq E\left[\int_0^s \lambda_u^Y du\right] \leq \lambda M E\left[\int_0^s |\overline{S}_u - \overline{T}_u| du\right]. \end{aligned}$$ □

The same holds true for sells. Hence, for

$$A_s^2 := \{X_{\tau_i} + \epsilon_i \notin (\min\{\underline{S}_{\tau_i}, \underline{T}_{\tau_i}\}, \max\{\underline{S}_{\tau_i}, \underline{T}_{\tau_i}\}] \text{ for all } \tau_i \leq s\}$$

we obtain a similar estimate, and for $A_s := A_s^1 \cap A_s^2$, which is the event that the same buys and sells occur under the two pricing strategies S and T, we have

$$P[A_s^c] \leq 2\lambda M E\left[\int_0^s \|S_u - T_u\| du\right], \qquad \text{where} \tag{4.10}$$

$$\|S_u - T_u\| := \max\left\{|\overline{S}_u - \overline{T}_u|, |\underline{S}_u - \underline{T}_u|\right\}.$$

Lemma 4.13. *There is a constant $K_1 < \infty$ such that*

$$E\left[\int_0^t \|F(S)_s - F(T)_s\| ds\right] \leq (K + tK_1) E\left[\int_0^t \|S_s - T_s\| ds\right]$$

for all $t \geq 0$ and for K from Theorem 2.3.

Proof. First we estimate the difference of the conditional distributions of the true value resulting from different pricing strategies. We obtain

(4.11)

$$
\begin{aligned}
E\left[\left|\pi_s^{S,i}-\pi_s^{T,i}\right|\right] &= E\left[\left|P\left[X_s=x_i|\mathcal{F}_s^S\right]-P\left[X_s=x_i|\mathcal{F}_s^T\right]\right|\left(1_{A_s^c}+1_{A_s}\right)\right] \\
&\le E\left[1_{A_s^c}\right]+E\left[\left|E\left[1_{\{X_s=x_i\}}|\mathcal{F}_s^S\right]-E\left[1_{\{X_s=x_i\}}|\mathcal{F}_s^T\right]\right|1_{A_s\}}\right] \\
&= P\left[A_s^c\right]+E[\left|E\left[1_{\{X_s=x_i\}}\left(1_{A_s^c}+1_{A_s}\right)\right|\mathcal{F}_s^S\right] \\
&\qquad -E\left[1_{\{X_s=x_i\}}\left(1_{A_s^c}+1_{A_s}\right)|\mathcal{F}_s^T\right]|1_{A_s}] \\
&\le 3P\left[A_s^c\right]+E[|E\left[1_{\{X_s=x_i\}}1_{A_s}|\mathcal{F}_s^S\right] \\
&\qquad -E\left[1_{\{X_s=x_i\}}1_{A_s}|\mathcal{F}_s^T\right]|1_{A_s}] \\
&= 3P\left[A_s^c\right], \qquad i=1,\dots,n.
\end{aligned}
$$

The last equation holds true since $\mathcal{F}_s^S \cap A_s = \mathcal{F}^T \cap A_s$. The equality of the trace σ-algebras holds due to

$$
\{B_i^S \le u\} \cap A_s = \{B_i^T \le u\} \cap A_s \text{ and } \{C_i^S \le u\} \cap A_s = \{C_i^T \le u\} \cap A_s
$$

for all $i \in \mathbb{N}$ and $u \le s$. Putting (4.10) and (4.11) together, we obtain

$$
\begin{aligned}
E\left[\int_0^t \sum_{i=1}^n |\pi_s^{S,i}-\pi_s^{T,i}|ds\right] &= \int_0^t \sum_{i=1}^n E\left[|\pi_s^{S,i}-\pi_s^{T,i}|\right]ds \\
&\le \int_0^t \sum_{i=1}^n 3P\left[A_s^c\right]ds \\
&\le 6n\lambda M \int_0^t E\left[\int_0^s \|S_u-T_u\|\,du\right]ds \\
&\le 6n\lambda M t E\left[\int_0^t \|S_s-T_s\|\,ds\right].
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
E\left[\int_0^t \left|\overline{F(S)}_s-\overline{F(T)}_s\right|ds\right] &= E\left[\int_0^t \left|g\left(\overline{S}_s,\pi_{s-}^S\right)-g\left(\overline{T}_s,\pi_{s-}^T\right)\right|ds\right] \\
&\le E\left[\int_0^t K|\overline{S}_s-\overline{T}_s|+L\sum_{i=1}^n|\pi_s^{S,i}-\pi_s^{T,i}|ds\right] \\
&\le E\left[\int_0^t K|\overline{S}_s-\overline{T}_s|+6Ln\lambda Mt\,\|S_s-T_s\|\,ds\right],
\end{aligned}
$$

where the first inequality is due to Lemma 3.7 for $K<1$ defined in Theorem 2.3 and $L=2x_{\max}/\Phi(C)^2$. A similar estimate can be obtained for

$$
E\left[\int_0^t \left|\underline{F(S)}_s-\underline{F(T)}_s\right|ds\right].
$$

We then get the desired result with $K_1 = 12Ln\lambda M$. $\square$

Proof of uniqueness in Theorem 2.3. Let S, T be two solutions of the GMPS problem. By Theorem 4.10, every solution is a fixed point of F. Applying Lemma 4.13 with some $t > 0$ such that $K + tK_1 < 1$, we obtain that S and T coincide $P \otimes \lambda|_{[0,t]}$-a.e.. Note that K and K_1 only depend on $x_{\min}, x_{\max}$, the distribution of the ϵ_i, and λ, but they are independent of the probabilities $P[X_0 = x_i]$.

But, if $S = T$ $P \otimes \lambda|_{[0,t]}$-a.e. so are the P-completions of $\mathcal{F}_t^S$ and $\mathcal{F}_t^T$. By proceeding iteratively, it follows that S and T are equal on $[0, \infty)$ as all arguments from above hold true also for a non-trivial $\mathcal{F}_0$. $\square$

4.3 Existence

To show existence, we proceed as follows. We define an n-dimensional process ϕ as a pathwise solution of a stochastic integral equation from which we expect that it has to be satisfied by the conditional distribution of the true value under the filtration of a GMPS. We then define prices as the static solutions for every (ω, t), inserting the conditional distribution of the true value, and construct the corresponding market maker's filtration. Then, we show that, under the constructed filtration, ϕ is adapted and solves the filtering equation of the conditional distribution of the true value. This, along with the results in Subsection 4.1, shows that we have indeed constructed a GMPS.

Definition 4.14. *Let $\phi \in [0,1]^n$ such that $\sum_{i=1}^n \phi_i = 1$. With $G(\phi)$ and $H(\phi)$ we denote the unique solutions s of*

$$g(s, \phi) = s \text{ and } h(s, \phi) = s$$

respectively, where g and h are defined in Definition 3.5 and (3.5) respectively. The existence and uniqueness of that solution is guaranteed by Theorem 3.6.

In the following, we still use the notation $\Phi(x) = P[\epsilon_1 \geq x]$ and further denote the distribution function of the ϵ_i by $\Psi(x) = P[\epsilon_1 \leq x]$.

Proof of existence in Theorem 2.3. Step 1: For $\phi : \Omega \times [0, \infty) \to [0,1]^n$ we consider the SDE

(4.12)

$$\phi_t^i = \phi_0^i + \sum_{\tau_k \le t} \phi_{\tau_k-}^i \left(\frac{\Phi(G(\phi_{\tau_k-}) - x_i)}{\sum_j \phi_{\tau_k-}^j \Phi(G(\phi_{\tau_k-}) - x_j)} - 1 \right) 1_{\{X_{\tau_k} + \epsilon_k \ge G(\phi_{\tau_k-})\}}$$

$$+ \sum_{\tau_k \le t} \phi_{\tau_k-}^i \left(\frac{\Psi(H(\phi_{\tau_k-}) - x_i)}{\sum_j \phi_{\tau_k-}^j \Psi(H(\phi_{\tau_k-}) - x_j)} - 1 \right) 1_{\{X_{\tau_k} + \epsilon_k \le H(\phi_{\tau_k-})\}}$$

$$- \int_0^t \left(\lambda \phi_s^i \left(\Psi(H(\phi_s) - x_i) + \Phi(G(\phi_s) - x_i) \right.\right.$$

$$\left. - \sum_j \phi_s^j \left(\Psi(H(\phi_s) - x_j) + \Phi(G(\phi_s) - x_j) \right) \right)$$

$$\left. - \sum_j \phi_s^j q(j,i) \right) \, ds$$

with initial condition $\phi_0^i = P[X_0 = x_i]$ for all $i = 1, \dots, n$. In a first step, we consider this SDE only pathwise and show that it has a unique solution with càdlàg paths (we do not yet have a filtration). We start by showing that G (and H) are Lipschitz continuous. Let $s, \widetilde{s}$ be such that $G(\phi) = s$, i.e. $g(s, \phi) = s$, and $G(\widetilde{\phi}) = \widetilde{s}$. By Lemma 3.7, we have

$$|s - \widetilde{s}| = \left| g(s,\phi) - g(\widetilde{s}, \widetilde{\phi}) \right| \le K|s - \widetilde{s}| + L \sum_{i=1}^n |\phi_i - \widetilde{\phi}_i|,$$

where $K < 1$ and $L < \infty$. By rearranging, we get

$$|G(\phi) - G(\widetilde{\phi})| = |s - \widetilde{s}| \le \frac{L}{1-K} \sum_{i=1}^n |\phi_i - \widetilde{\phi}_i|.$$

Further, the functions Φ and Ψ are differentiable, and the derivatives are bounded by $K/C < \infty$ on the compact set $[-C, C]$. In addition, Φ and Ψ are bounded by 1. Integration by parts yields that the ds-term in (4.12), considered as a function of ϕ, can be modified to a function $f(\phi)$ that is Lipschitz continuous and f coincides with the original function for all $\phi \in \mathbb{R}^n$ with $\phi^i \ge 0$ and $\sum_{i=1}^n \phi^i = 1$. The system of ordinary differential equations consisting only of the modified ds-terms then has a unique solution and, by the construction of the ODEs, the solution stays in the set of probabilities. Thus, it also solves the differential equations with the original ds-terms, i.e.

$$
\begin{aligned}
d\psi_t^i = - \Bigg(\lambda \psi_t^i \Bigg(& \Psi(H(\psi_t) - x_i) + \Phi(G(\psi_t) - x_i) \\
& - \sum_j \psi_t^j \left(\Psi(H(\psi_t) - x_j) + \Phi(G(\psi_t) - x_j) \right) \Bigg) \\
& - \sum_j \psi_t^j q(j,i) \Bigg) \, dt.
\end{aligned}
$$

With this solution, we can construct a candidate for the original problem up to τ_1, i.e. $\phi_t := \psi_t$ for all $t < \tau_1$, and

$$
\begin{aligned}
\phi_{\tau_1}^i = & \psi_{\tau_1-}^i + \psi_{\tau_1-}^i \left(\frac{\Phi(G(\psi_{\tau_1-}) - x_i)}{\sum_j \psi_{\tau_1-}^j \Phi(G(\psi_{\tau_1-}) - x_j)} - 1 \right) 1_{\{X_{\tau_1} + \epsilon_1 \geq G(\psi_{\tau_1-})\}} \\
& + \psi_{\tau_1-}^i \left(\frac{\Psi(H(\psi_{\tau_1-}) - x_i)}{\sum_j \psi_{\tau_1-}^j \Psi(H(\psi_{\tau_1-}) - x_j)} - 1 \right) 1_{\{X_{\tau_1} + \epsilon_1 \leq H(\psi_{\tau_1-})\}}.
\end{aligned}
$$

We also obtain a solution $\widetilde{\psi}$ of the ordinary differential equation above for every state of $\phi_{\tau_1}^i$ as initial condition. Considered as a parameter-dependent differential equation, the solution is continuous in the initial condition. We then define a solution of the original problem on (τ_1, τ_2) by

$$
\phi_t = \widetilde{\psi}_{t-\tau_1}
$$

and so on. By proceeding iteratively, we obtain a process that satisfies (4.12) up to all τ_i. Then, one may define $\phi_t^i(\omega) = 1/n$ for $t \in \mathbb{R}_+$ with $t \geq \sup_{i \in \mathbb{N}} \tau_i(\omega)$. As τ_i are Poisson times, this definition affects only a P-null set of course, but the construction ensures measurability (see Step 2) without needing the usual conditions and without the additional assumption that $\sup_{i \in \mathbb{N}} \tau_i(\omega) = \infty$ for all $\omega \in \Omega$. The process $\phi : \Omega \times \mathbb{R}_+ \mapsto \mathbb{R}^n$ has càdlàg paths at least on $[0, \sup_{i \in \mathbb{N}} \tau_i(\omega))$.

Step 2: We now set $S_t := (G(\phi_{t-}), H(\phi_{t-}))$ on $(0, \sup_{i \in \mathbb{N}} \tau_i(\omega))$ and, say, $S := (x_{\max}, x_{\min})$ elsewhere. With S, the processes N^B, N^C (with jump times B_k and C_k respectively) and the filtration $\mathbb{F}^S$ are defined according to (2.4) and (2.5) respectively. It follows that the jumps in (4.12) take place only at actual buys and sells with prices S. Therefore, and by the construction of ϕ (using the continuity in the initial condition), for every $t \in \mathbb{R}$, ϕ_t can be written as a measurable function of $B_k 1_{\{B_k \leq t\}}$ and $C_k 1_{\{C_k \leq t\}}$, $k \in \mathbb{N}$. Thus, ϕ_t is $\mathcal{F}_t^S$-measurable, i.e. ϕ is $\mathbb{F}^S$-adapted. It follows that the process S which is left-continuous on $(0, \sup_{i \in \mathbb{N}} \tau_i(\omega))$ is $\mathbb{F}^S$-predictable and hence admissible in the sense of Definition 2.2. By using this pathwise construction, we obtain pricing strategies that are $\mathbb{F}^S$-predictable and not only predictable with respect to the completed filtration $\widetilde{\mathbb{F}}^S$ which satisfies the usual conditions. By (2.4), we can write (4.12) as

$$
\begin{aligned}
d\phi_t^i =& \phi_{t-}^i \left(\frac{\Phi(\overline{S}_t - x_i)}{\sum_j \phi_{t-}^j \Phi(\overline{S}_t - x_j)} - 1 \right) dN_t^B \\
&+ \phi_{t-}^i \left(\frac{\Psi(\underline{S}_t - x_i)}{\sum_j \phi_{t-}^j \Psi(\underline{S}_t - x_j)} - 1 \right) dN_t^C \\
&- \Bigg(\lambda \phi_t^i \Bigg(\Psi(\underline{S}_t - x_i) + \Phi(\overline{S}_t - x_i) \\
&- \sum_j \phi_t^j \left(\Psi(\underline{S}_t - x_j) + \Phi(\overline{S}_t - x_j) \right) \Bigg) \\
&- \sum_j \phi_t^j q(j,i) \Bigg) dt.
\end{aligned} \tag{4.13}
$$

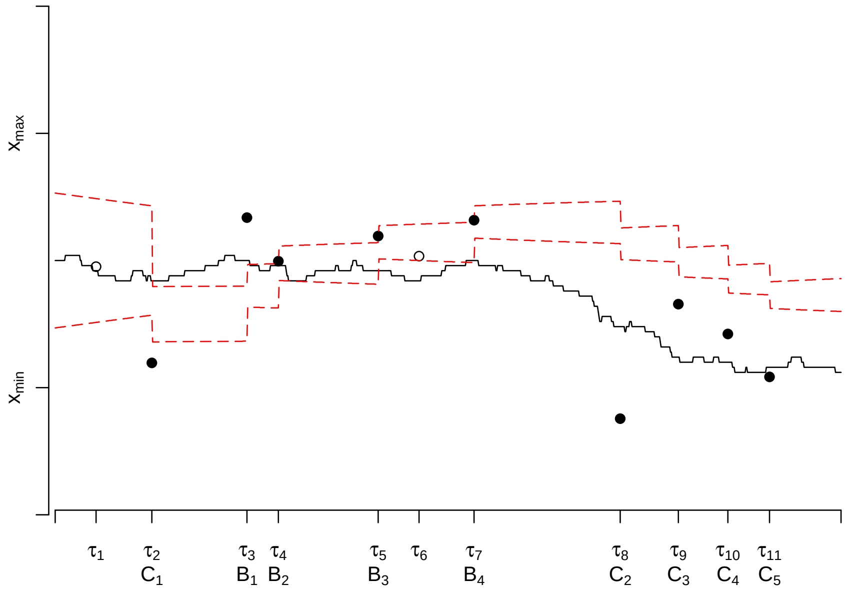

Figure 3: Glosten-Milgrom prices for the same scenario ω as in Figure 1.

Step 3: The filtering equation for $\pi_t^{S,i} = P\left[X_t = x_i | \mathcal{F}_\tau^S\right]$ is given in (4.2). In this equation, S, which depends on ϕ, is fixed. In terms of π^S, (4.2) has a unique solution, and ϕ is obviously a solution of this equation (uniqueness follows as the dt-term considered as a function of π^S is Lipschitz continuous, thus the arguments are similar, but simpler, as for (4.12)). Thus, as ϕ and π^S are both càdlàg, they are indistinguishable. For the ask price we then get

$$\overline{F(S)}_t = g\left(\overline{S}_t, \pi^S_{t-}\right) = g\left(G(\phi_{t-}), \phi_{t-}\right) = G(\phi_{t-}) = \overline{S}_t,$$

also up to indistinguishability, and thus $(P \otimes \lambda)$-a.e.. As the same holds for the bid price, Theorem 4.10 shows that S is a GMPS and existence is proven. □

5 Discussion

The focus of this article is the dynamic model, i.e. the purpose of Section 3 dealing with the corresponding static model is mainly to build some intuition and to prepare the proofs in Section 4. The conditions in Section 3 are chosen accordingly. But, it is worth noting that, to obtain unique prices, in the dynamic model stronger assumptions are needed than in the static one. To see this (and to justify the conditions in Section 4), we give a toy example of a model possessing unique static Glosten-Milgrom prices (for all probabilities of the finitely many states of the true value) in which the pair of initial prices can be extended to two different pairs of Glosten-Milgrom price *processes.*

Example 5.1. *[Perfect insiders and pure noise traders] Let* $P(\epsilon_j = 0) = P(\epsilon_j = \infty) = P(\epsilon_j = -\infty) = 1/3$. *This means that a new customer is either a perfect insider, a noise trader who wants to buy the asset (at all costs), or a noise trader who wants to sell, each with probability* $1/3$. *The initial distribution of the true value* X *is given by the states and probabilities*

$$(x_1, x_2, x_3, x_4) = (-5, -1, 1, 5) \quad \textit{and} \quad (\pi_1, \pi_2, \pi_3, \pi_4) = (1/4, 1/4, 1/4, 1/4).$$

Furthermore, we assume that the Markov process X *jumps with a small rate from* x_2 *to* x_1 *and from* x_3 *to* x_4 *and otherwise remains constant; more precisely we assume*

$$0 < q := q(2,1) = q(3,4) < \frac{\lambda}{6}, \tag{5.1}$$

where λ *is the rate of the customer arrival process (see (2.1) for the definition of* $q(i,j)$*).*

Straightforward calculations show that $\overline{S}_0 = 1$ and $\underline{S}_0 = -1$ are the unique static Glosten-Milgrom prices, i.e. expected profits vanish. It can also be shown that unique static Glosten-Milgrom prices exist for all bounded X if ϵ is chosen as above (in this case, the LHS of (3.1) is increasing and continuous in s).

Let us show that there are two pairs of Glosten-Milgrom price processes with the same initial value $(\overline{S}_0, \underline{S}_0) = (1, -1)$ but with different values afterwards. For this purpose, in the remainder of this section, we investigate how Glosten-Milgrom strategies may look after time zero *under the condition that no trade has yet occurred*. For the following considerations it is crucial that, by construction, the initial prices $\overline{S}_0 = 1$ and $\underline{S}_0 = -1$ coincide with some states of $X + \epsilon$ (putting $\epsilon = 0$, i.e. the insider is indifferent with positive probability). Consequently, for marginally different prices the learning effect by trades can differ considerably. If the ask price is slightly below 1 and $X = 1$, an insider buys the asset, but he abstains if the price is slightly above 1. Remember that for Lemma 4.12 we need ϵ_j to be continuously distributed on $[-(x_{\max}-x_{\min}), x_{\max}-x_{\min}]$, i.e. there cannot be a *perfect* insider. The lemma then says that, for two admissible strategy pairs, the probability that a trade takes place under only one of the two strategies is bounded by a constant times the L^1-norm (with respect to the product measure $P \otimes \lambda$) of the distance between the different ask prices plus the distance between the different bid prices. This excludes the situation described here.

We examine whether the four scenarios that the ask/bid price is increasing/decreasing after time zero are consistent with the zero expected profit condition. The scenario that *both* the ask price decreases below 1 and the bid price increases above -1 is impossible. Namely, in that case the insider would trade in all 4 states of X. Thus, from the observation that no trade occurs, the market maker cannot learn anything about X. Because, in addition, the unconditional probabilities of the extreme states $x_1 = -5$ and $x_4 = 5$ increase, the spread cannot decrease, and the scenario above cannot be a Glosten-Milgrom pair. The second symmetric scenario that both the ask price increases above 1 and the bid price decreases below -1 also turns out to be impossible. Here, the probability that a customer is both an insider *and* trades is 1/6 (an insider only trades if $X \in \{-5, 5\}$). Under the observation that no trade has occurred, the middle states -1 and 1 become more likely, and, due to Condition (5.1), this effect overcompensates for the jump rates of X. This is incompatible with a higher spread.

On the other hand, it turns out that the two asymmetric scenarios are possible. Our first ansatz is based on an ask price strictly above 1 and a bid price strictly above -1, immediately after time zero (cf. the blue curves in Figure 4): the conditional probabilities of $X = x_i$, given that no trade has yet occurred, are given by the unique solution of the ordinary differential equation

$$
\begin{aligned}
\frac{d\pi_t^1}{dt} &= -\frac{\lambda}{3}\pi_t^3\pi_t^1 + q\pi_t^2, & \frac{d\pi_t^2}{dt} &= -\frac{\lambda}{3}\pi_t^3\pi_t^2 - q\pi_t^2, \\
\frac{d\pi_t^3}{dt} &= \frac{\lambda}{3}\pi_t^3(1-\pi_t^3) - q\pi_t^3, & \frac{d\pi_t^4}{dt} &= -\frac{\lambda}{3}\pi_t^3\pi_t^4 + q\pi_t^3
\end{aligned}
\tag{5.2}
$$

with $\pi_0^i = 1/4$. (5.2) can be obtained as the dt-terms of the filtering equation in Lemma 4.5, assuming that the insider buys iff $X = 5$ and sells iff $X = -5$ or $X = -1$. Given the conditional distribution of X, we obtain a unique ask price satisfying the zero profit condition by

$$
\gamma(t) = G(\pi_t) = \frac{\sum_{i=1}^4 \pi_t^i x_i + \pi_t^4 x_4}{1 + \pi_t^4}
\tag{5.3}
$$

and a unique bid price by

$$
\delta(t) = H(\pi_t) = \frac{\sum_{i=1}^4 \pi_t^i x_i + \pi_t^1 x_1 + \pi_t^2 x_2}{1 + \pi_t^1 + \pi_t^2},
\tag{5.4}
$$

up to the first trading time. It remains to check that the ask price is indeed above 1 and the bid price above -1 for t sufficiently small. By

$$\frac{d\left(\sum_{i=1}^{4}\pi_t^i x_i + \pi_t^4 x_4\right)}{dt} = \left(-\frac{\lambda}{3}\pi_t^3\pi_t^1 + q\pi_t^2\right)(-5) + \left(-\frac{\lambda}{3}\pi_t^3\pi_t^2 - q\pi_t^2\right)(-1) + \left(\frac{\lambda}{3}\pi_t^3(1-\pi_t^3) - q\pi_t^3\right) + 2\left(-\frac{\lambda}{3}\pi_t^3\pi_t^4 + q\pi_t^3\right)5,$$

differentiating (5.3) with respect to t implies

$$\begin{aligned}\gamma'(0)\frac{5}{4} + \gamma(0)\left(-\frac{\lambda}{3}\frac{1}{16} + \frac{q}{4}\right) =& \frac{\lambda}{3}\frac{5}{16} - \frac{5q}{4} + \frac{\lambda}{3}\frac{1}{16} + \frac{q}{4} \\ &+ \frac{\lambda}{3}\frac{3}{16} - \frac{q}{4} - \frac{\lambda}{3}\frac{10}{16} + \frac{10q}{4} \\ =& \frac{5q}{4} - \frac{\lambda}{3}\frac{1}{16}.\end{aligned}$$

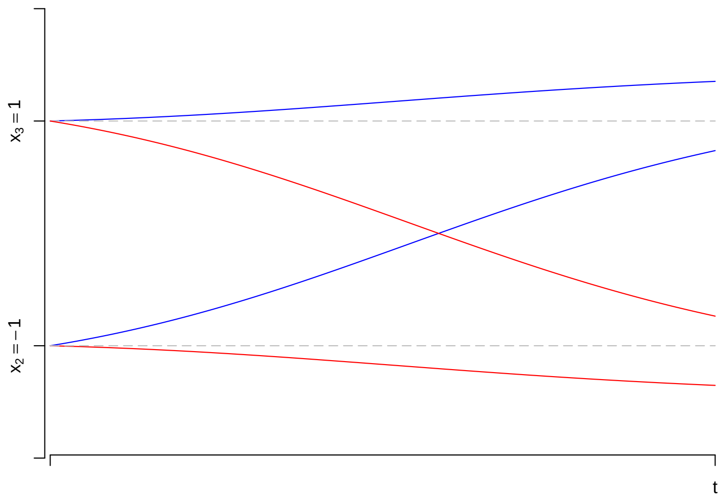

Figure 4: The blue curves show the pair $(\overline{S}_t, \underline{S}_t)$ of Glosten-Milgrom prices up to the first trade that we construct in this section. The red curves are $(-\underline{S}_t, -\overline{S}_t)$, which is another Glosten-Milgrom pricing strategy. This shows that in Example 5.1 only the initial prices are unique.

By $\gamma(0) = 1$, we arrive at $\gamma'(0) = 4q/5 > 0$. Thus, the ask price lies above 1 for t sufficiently small. This can be interpreted as follows. As $X = 1$ is the only state inside the spread, it becomes more likely at the cost of the other three states. As this has no impact on the profitability of the quote at price 1, the effect that the unconditional probabilities of the extreme states increase dominates, and thus the ask price increases.

For the numerator of the bid price we obtain

$$\begin{aligned}\frac{d\left(\sum_{i=1}^4 \pi_t^i x_i + \pi_t^1 x_1 + \pi_t^2 x_2\right)}{dt} =& 2\left(-\frac{\lambda}{3}\pi_t^3\pi_t^1 + q\pi_t^2\right)(-5) \\ &+ 2\left(-\frac{\lambda}{3}\pi_t^3\pi_t^2 - q\pi_t^2\right)(-1) \\ &+ \left(\frac{\lambda}{3}\pi_t^3(1-\pi_t^3) - q\pi_t^3\right) \\ &+ \left(-\frac{\lambda}{3}\pi_t^3\pi_t^4 + q\pi_t^3\right)5\end{aligned}$$

and thus

$$\begin{aligned}\delta'(0)\frac{6}{4} + \delta(0)\left(-\frac{\lambda}{3}\frac{1}{16} + \frac{q}{4} - \frac{\lambda}{3}\frac{1}{16} - \frac{q}{4}\right) =& \frac{\lambda}{3}\frac{10}{16} - \frac{10q}{4} + \frac{\lambda}{3}\frac{2}{16} + \frac{2q}{4} \\ &+ \frac{\lambda}{3}\frac{3}{16} - \frac{q}{4} - \frac{\lambda}{3}\frac{5}{16} + \frac{5q}{4} \\ =& \frac{\lambda}{3}\frac{10}{16} - q.\end{aligned}$$

We arrive at $\delta'(0) = \lambda/9 - 2q/3$, which is positive by (5.1). For the bid price, it matters that $X = 1$, the only state inside the spread, becomes more likely under the condition that no trade has yet occurred, because this lets the expected profits increase by sells of noise traders. As q is small relative to λ, the bid price goes up. Putting everything together, the solution of (5.2) does indeed describe the conditional probabilities of X under the filtration of the market maker. Thus, by $(\overline{S}_t, \underline{S}_t) = (\gamma(t), \delta(t))$, we have constructed a candidate for the Glosten-Milgrom pair for t small enough.

But, as Example 5.1 is symmetric around zero, with $(\overline{S}_t, \underline{S}_t)$, the pair $(-\underline{S}_t, -\overline{S}_t)$ is also a Glosten-Milgrom strategy up to the first trade (see the red curves in Figure 4). This indicates that prices are unique only at time zero.

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