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# Optimal portfolios of a small investor in a limit order market: a shadow price approach

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**Abstract** We study Merton's portfolio optimization problem in a limit order market. An investor trading in a limit order market has the choice between market orders that allow immediate transactions and limit orders that trade at more favorable prices but are executed only when another market participant places a corresponding market order. Assuming Poisson arrivals of market orders from other traders we use a shadow price approach, similar to Kallsen and Muhle-Karbe [9] for models with proportional transaction costs, to show that the optimal strategy consists of using market orders to keep the proportion of wealth invested in the risky asset within certain boundaries, similar to the result for proportional transaction costs, while within these boundaries limit orders are used to profit from the bid-ask spread. Although the given best-bid and best-ask price processes are geometric Brownian motions the resulting shadow price process possesses jumps.

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## 1 Introduction

A *portfolio problem* in mathematical finance is the optimization problem of an investor possessing a given initial endowment of assets who has to decide how many shares of each asset to hold at each time instant in order to maximize his expected utility from consumption (see [11]). To change the asset allocation of his portfolio or finance consumption, the investor can buy or sell assets at the market. Merton [15, 16] solved the portfolio problem for a continuous time frictionless market consisting of one risky asset and one riskless asset. When the price process of the risky asset is modeled as a geometric Brownian motion (GBM), Merton was able to show that the investor's optimal strategy consists of keeping the fraction of wealth invested in the risky asset constant. Due to the fluctuations of the GBM this leads to incessant trading.

The assumption that investors can purchase and sell arbitrary amounts of the risky asset at a fixed price per share is quite unrealistic in a less liquid market which possesses a significant

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bid-ask spread. In today's electronic markets the predominant market structure is the limit order market, where traders can continuously place market and limit orders, and change or delete them as long as they are not executed. When a trader wants to buy shares for example, he has a basic choice to make. He can either place a market buy order or he can submit a limit buy order, with the limit specifying the maximum price he would be willing to pay per share. If he uses a market order his order is executed immediately, but he is paying at least the best-ask price (the lowest limit of all unexecuted limit sell orders), and an even higher average price if the order size is large. By using a limit buy order with a limit lower than the current best-ask price he pays less, but he cannot be sure if and when the order is executed by an incoming sell order matching his limit.

We introduce a new model for continuous-time trading using both market and limit orders. This allows us to analyze e.g. the *trade-off between rebalancing the portfolio quickly and trading at favorable prices*. To obtain a mathematically tractable model we keep some idealized assumptions of the frictionless market model resp. the model with proportional transaction costs. E.g. we assume that the investor under consideration is small, i.e. the size of his orders is sufficiently small to be absorbed by the orders in the order book. The best-ask and the best-bid price processes solely result from the behavior of the other market participants and can thus be given *exogenously*. Furthermore, we assume that the investor's limit orders are small enough to be executed against any arising market order whose arrival times are also exogenously given and modeled as Poisson times. We also assume that limit orders can be submitted and taken out of the order book for free.

The model tries to close a gap between the market microstructure literature which lacks analytical tractability when it comes to dynamic trading and the literature on portfolio optimization under idealized assumptions with powerful closed-form and duality results.

In the economic literature on limit order markets (see e.g. the survey by Parlour and Seppi [17] for an overview) the incentive to trade quickly (and therefore submit market orders) is usually modeled exogenously by a preference for immediacy. This is e.g. the case in the multi-period equilibrium models of Foucault, Kadan, and Kandel [2] and Roşu [21], which model the limit order market as a stochastic sequential game. Even in research concerning the optimal behavior of a single agent, this exogenous motivation to trade is common. Consider e.g. Harris [5], which deals with optimal order submission strategies for certain stylized trading problems, e.g. for a *risk-neutral* trader who has to sell one share before some deadline. By contrast, in our model the trading decision is directly derived from the maximization of expected utility from a consumption stream (thus from "first principles"), i.e. the incentive to trade quickly is explained. Furthermore, in Harris [5] the order size is discarded and the focus is on the selection of the right limit price at each point in time. In our work the limit prices used by the small investor are effectively reduced to selling at the best-ask and buying at the best-bid, but in view of the agent's underlying portfolio problem, the size of these limit orders is a key question. *There is a trade-off between placing large limit orders to profit from the spread and taking too much risk by the resulting large positions (usually called* inventory risk *in the literature on market making)*.

In Section 2 we introduce the market model on a quite general level. In Section 3 we specify stochastic processes for which we study the problem of maximizing expected logarithmic utility from consumption over an infinite horizon. Namely, we let the best-bid and best-ask price processes be geometric Brownian motions and the spread be proportional to them. Market orders of the other traders arise according to two independent Poisson processes with constant rates. In Section 3 we also provide some intuition on how we obtain a promising candidate for an optimal strategy and connect it to the solution of a suitable free boundary problem. In Section 4 we prove the existence of a solution of this free boundary problem. The verification that the constructed solution is indeed optimal is done in Section 5.

The optimal strategy consists in placing the minimal amount of market orders which is necessary to keep the proportion of wealth invested in the risky asset within certain boundaries – similar to the result of Davis and Norman [1] for transaction costs – while within these boundaries limit orders are used to hit one of the boundaries when at a Poisson time trading is possible at a favorable price (i.e. the investor adjusts the sizes of his limit orders continuously in such a way that the proportion invested in the risky asset jumps to one of the boundaries whenever a limit order is executed by an incoming exogenous market order). By the latter the investor profits from the bid-ask spread. Thus, although the structure of the solution looks at first glance quite similar to the case with proportional transaction costs, a key incentive of the investor is now to capitalize on the spread by placing limit orders. Whereas the investor generally tries to avoid using market orders, he is always willing to trade using limit orders. In a sense, trading with limit orders corresponds to negative proportional transactions costs.

We derive the optimal trading strategy by showing the existence of a shadow price process of the asset – similar to the work of Kallsen and Muhle-Karbe [9] with proportional transaction costs. A shadow price process $\widetilde{S}$ for the risky asset has to satisfy the following two properties. Firstly, in a fictitious frictionless market without spread and with price process $\widetilde{S}$ *any* transaction feasible in the original market can be implemented at better or equal prices. Secondly, there is an *optimal* trading strategy in the fictitious market which can also be realized in the original market leading to the same trading gains.

The main difference of the shadow price process in our model compared to [9] is that it possesses jumps – namely at the Poisson arrival times of the exogenous market orders.

## 2 The model

### 2.1 Trading of a small investor in a limit order market

Let $(\Omega, \mathscr{F}, P, (\mathscr{F}_t)_{t\geq 0})$ be a filtered probability space satisfying the usual conditions. Regarding conventions and notation we mostly follow Jacod and Shiryaev [7]. For a process $X$ with left and right limits (also called làglàd) let $\Delta X_t := X_t - X_{t-}$ denote the jump at time $t$ and let $\Delta^+ X_t := X_{t+} - X_t$ denote the jump immediately after time $t$. If we write $X = Y$ for two stochastic processes $X$ and $Y$, we mean equality up to indistinguishability.

We model the best-bid price $\underline{S}$ and the best-ask price $\overline{S}$ as two *continuous*, adapted, exogenously given stochastic processes such that $\underline{S} \leq \overline{S}$. The continuity of $\underline{S}$ and $\overline{S}$ will play a key role in the reduction of the dimension of the strategy space. The arrivals of market sell orders and market buy orders by the other traders are modeled exogenously by counting processes $N^1$ and $N^2$ (as defined e.g. in [19], Section 1.3).

*In our model (formally introduced in Definition 1) the investor may submit market buy and sell orders which are immediately executed at price $\overline{S}$ and $\underline{S}$, resp. In addition, he may submit limit buy and sell orders. The limit buy price is restricted to $\underline{S}$ and these orders are executed at the jump times of $N^1$ at price $\underline{S}$. Accordingly, the limit sell price is restricted to $\overline{S}$ and the limit sell orders are executed at the jump times of $N^2$ at the price $\overline{S}$.*

This restriction is an immense reduction of the dimensionality of the problem, as we do not consider limit orders at arbitrary limit prices. It can be justified by the following considerations. A superior limit order strategy of the small investor is to place a limit buy order at a *"marginally" higher price than the current best-bid price $\underline{S}$* (of course this necessitates to update the limit price according to the movements of the best-bid price, which could in practice be approximately realized as long as the submission and deletion of orders is for free). Then, the limit buy order is executed as soon as the next limit sell order by the other traders arrives (i.e. at the next jump time of $N^1$). As $\underline{S}$ is continuous there is no reason to submit a limit buy order at a limit price strictly lower than the current best-bid price. Namely, such an order could not be executed before $\underline{S}$ hits the lower limit buy price of the order. As this appears at a *predictable* stopping time it is sufficient to place the order at this stopping time and take the current best-bid price as the limit price. On the other hand, a limit buy order with limit price in $(\underline{S}, \overline{S})$ is executed at the same time as a buy order with limit price $\underline{S}$ (resp. "marginally" higher than $\underline{S}$), but at a higher price than $\underline{S}$ (assuming that market sell orders of the other traders still arise according to $N^1$).

Thus, in our model it is implicitly assumed that the small investor does not influence the best-ask price or the best-bid price and his orders are small enough to be executed against any market order arising at $\Delta N^1 = 1$ and $\Delta N^2 = 1$. Furthermore, the market orders arising at $\Delta N^1 = 1$, $\Delta N^2 = 1$ (although being large in comparison to the size of the orders of the small investor) are sufficiently small to be absorbed by the orders in the book, i.e. a jump of $N^1$ or $N^2$ does not cause a movement of $\underline{S}$ and $\overline{S}$.

*With the considerations above we are in the quite fortunate situation that the quadruple* $(\underline{S}, \overline{S}, N^1, N^2)$ *is sufficient to model the trading opportunities of the small investor.* Thus, our mathematical model can be build on these processes alone (rather than on the dynamics of the whole order book).

A possible economic interpretation is that $\underline{S}$ and $\overline{S}$ move as nonaggressive traders update their limit prices with varying fundamentals whereas $N^1$ and $N^2$ model immediate supply and demand for the asset.

*Remark 1* Note that the investor in our model does not play the role of a market maker who, however, also wants to profit from the spread. The market maker can influence the spread and he is *forced* to trade with arising market orders.

**Definition 1** Let $M^B$, $M^S$, $L^B$ and $L^S$ be predictable processes. Furthermore, let $M^B$ and $M^S$ be non-decreasing with $M_0^B = M_0^S = 0$ and $L^B$ and $L^S$ non-negative. Let $c$ be an optional process. A quintuple $\mathfrak{S} = (M^B, M^S, L^B, L^S, c)$ is called a *strategy*. For $\eta^0, \eta^1 \in \mathbb{R}$ we define the *portfolio process* $(\varphi^0, \varphi^1)(\mathfrak{S}, \eta^0, \eta^1)$ *associated with strategy* $\mathfrak{S}$ *and initial portfolio* $(\eta^0, \eta^1)$ to be

$$\begin{aligned} \varphi_t^0 &:= \eta^0 - \int_0^t c_s ds - \int_0^t \overline{S}_s \, dM_s^B + \int_0^t \underline{S}_s \, dM_s^S \\ &\quad - \int_0^{t-} L_s^B \underline{S}_s \, dN_s^1 + \int_0^{t-} L_s^S \overline{S}_s \, dN_s^2 \\ \varphi_t^1 &:= \eta^1 + M_t^B - M_t^S + \int_0^{t-} L_s^B \, dN_s^1 - \int_0^{t-} L_s^S \, dN_s^2. \end{aligned} \tag{2.1}$$

$\varphi^0$ is the number of risk-free assets and $\varphi^1$ the number of risky assets. For simplicity, we assume there is a risk-free interest rate, which is equal to zero. The interpretation is that *aggregated* market buy or sell orders up to time $t$ are modeled with $M_t^B$ and $M_t^S$, whereas $L_t^B$ (resp. $L_t^S$) specifies the size of a limit buy order with limit price $\underline{S}$ (resp. the size of a limit sell order with limit price $\overline{S}$), i.e. the amount that is bought or sold favorably if an exogenous market sell or buy order arrives at time $t$. $L^B$ and $L^S$ can be arbitrary predictable processes which is justified under the condition that submission and deletion of orders which are not yet executed is for free. Finally, $c_t$ is interpreted as the rate of consumption at time $t$.

Note that integrating w.r.t. the processes $M^B$ and $M^S$ which are of finite variation and therefore have left and right limits is a trivial case of integrating w.r.t. optional semimartingales (as discussed e.g. in [3] and [12]). For a càdlàg process $Y$ we define the integral $\int (Y_-, Y) dM^B$ by

$$\int_0^t (Y_{s-}, Y_s) dM_s^B := \int_0^t Y_{s-} d(M_s^B)^r + \sum_{0 \le s < t} Y_s \Delta^+ M_s^B, \quad t \ge 0, \tag{2.2}$$

where $(M^B)_t^r := M_t^B - \sum_{0 \le s < t} \Delta^+ M_s^B$. The first term on the right-hand side of (2.2) is just a standard Lebesgue-Stieltjes integral. For a continuous integrand $Y$, as e.g. in (2.1), we set $\int Y dM^B := \int (Y, Y) dM^B$ (which is consistent with the integral w.r.t. càdlàg integrators).

In (2.1) the integrals w.r.t. $N^1$ and $N^2$ are only up to time $t-$, a limit order triggered by $\Delta N_t^i = 1$ is not yet included in $\varphi_t$. The integrals w.r.t. $M^B$ and $M^S$ are up to time $t$, but note that by (2.2) just the orders $\Delta M_t^B$ and $\Delta M_t^S$ (corresponding to trades at time $t-$) are already included in $\varphi_t$ at time $t$, whereas the orders $\Delta^+ M_t^B$ and $\Delta^+ M_t^S$ (corresponding to trades at time $t$) are only included in $\varphi_t$ right after time $t$. Hence, (2.1) goes conform to the usual interpretation of $\varphi_t$ as the holdings at time $t-$ (and the amount invested in the jump at time $t$) and for $\underline{S} = \overline{S}$ it coincides with the self-financing condition in frictionless markets (up to the restriction to finite variation strategies).

### 2.2 The Merton problem in a limit order market

Given initial endowment $(\eta^0, \eta^1)$ a strategy $\mathfrak{S}$ is called *admissible* if its associated portfolio process $(\varphi^0, \varphi^1)(\mathfrak{S}, \eta^0, \eta^1)$ satisfies

$$\varphi_t^0 + 1_{\{\varphi_t^1 \geq 0\}} \underline{S}_t \varphi_t^1 + 1_{\{\varphi_t^1 < 0\}} \overline{S}_t \varphi_t^1 \geq 0, \quad \forall t \geq 0. \tag{2.3}$$

Thus, a strategy is considered admissible if at any time a market order can be used to liquidate the position in the risky asset resulting in a non-negative amount held in the risk-free asset. Let $\mathscr{A}(\eta^0, \eta^1)$ denote the *set of admissible strategies* for initial endowment $(\eta^0, \eta^1)$.

Now the *value function* $V$ for the optimization problem of an investor with initial holdings $\eta^0$ in the risk-free asset and $\eta^1$ in the risky asset and logarithmic utility function who wants to maximize expected utility from consumption can be written as

$$V(\eta^0, \eta^1) := \sup_{\mathfrak{S} \in \mathscr{A}(\eta^0, \eta^1)} \mathscr{J}(\mathfrak{S}) := \sup_{\mathfrak{S} \in \mathscr{A}(\eta^0, \eta^1)} E\left(\int_0^\infty e^{-\delta t} \log(c_t) dt\right), \tag{2.4}$$

where $\delta > 0$ models the time preference. Note that due to the spread the optimization problem is not myopic.

### 2.3 Fictitious markets and shadow prices

To solve (2.4) we consider – similar to [9] – a fictitious frictionless market comprising of the same two assets as above. In this frictionless market the discounted price process of the risky asset is modeled as a real-valued semimartingale $\widetilde{S}$. Any amount of the risky asset can be bought or sold instantly at price $\widetilde{S}$.

Let $(\psi^0, \psi^1)$ be a two-dimensional predictable process, integrable w.r.t. to the two-dimensional semimartingale $(1, \widetilde{S})$, i.e. $(\psi^0, \psi^1) \in L((1, \widetilde{S}))$ in the notation of [7]. Suppose $c$ is an optional process. We call $\widetilde{\mathfrak{S}} = (\psi^0, \psi^1, c)$ a *self-financing* strategy with initial endowment $(\eta^0, \eta^1)$ if it satisfies

$$\psi_t^0 + \psi_t^1 \widetilde{S}_t = \eta^0 + \eta^1 \widetilde{S}_0 + \int_0^t \psi_s^1 d\widetilde{S}_s - \int_0^t c_s ds.$$

A self-financing strategy $\widetilde{\mathfrak{S}}$ is called *admissible* if

$$\psi_t^0 + \psi_t^1 \widetilde{S}_t \geq 0, \quad \forall t \geq 0.$$

Denote by $\widetilde{\mathscr{A}}(\eta^0, \eta^1)$ the set of all admissible strategies given initial endowment $(\eta^0, \eta^1)$. Again, we introduce a value function $\widetilde{V}$ by

$$\widetilde{V}(\eta^0, \eta^1) := \sup_{\widetilde{\mathfrak{S}} \in \widetilde{\mathscr{A}}(\eta^0, \eta^1)} \widetilde{\mathscr{J}}(\widetilde{\mathfrak{S}}) := \sup_{\widetilde{\mathfrak{S}} \in \widetilde{\mathscr{A}}(\eta^0, \eta^1)} E\left(\int_0^\infty e^{-\delta t} \log(c_t) dt\right).$$

Note that because the spread is zero, for another initial portfolio $(\zeta^0, \zeta^1)$ we have $V(\eta^0, \eta^1) = V(\zeta^0, \zeta^1)$ if $\eta^0 + \eta^1 \widetilde{S}_0 = \zeta^0 + \zeta^1 \widetilde{S}_0$. Nonetheless, to keep the notation for the frictionless market close to the notation for the limit order market we write $\widetilde{V}(\eta^0, \eta^1)$.

**Definition 2** We call the real-valued semimartingale $\widetilde{S}$ a *shadow price process* of the risky asset if it satisfies for all $t \geq 0$:

$$\underline{S}_t \leq \widetilde{S}_t \leq \overline{S}_t, \qquad \widetilde{S}_t = \begin{cases} \underline{S}_t & \text{if} \quad \Delta N_t^1 = 1 \\ \overline{S}_t & \text{if} \quad \Delta N_t^2 = 1 \end{cases} \tag{2.5}$$

and if there exists a strategy $\mathfrak{S} = (M^B, M^S, L^B, L^S, c) \in \mathscr{A}(\eta^0, \eta^1)$ in the limit order market such that for the associated portfolio process $(\varphi^0, \varphi^1)$ we have $\widetilde{\mathfrak{S}} = (\varphi^0, \varphi^1, c) \in \widetilde{\mathscr{A}}(\eta^0, \eta^1)$

and $\widetilde{\mathscr{J}}(\widetilde{\mathfrak{S}}) = \widetilde{V}(\eta^0, \eta^1)$ in the frictionless market with $\widetilde{S}$ as the discounted price process of the risky asset, i.e. the associated portfolio process of $\mathfrak{S}$ paired with the consumption rate $c$ of $\mathfrak{S}$ has to be an optimal strategy in the frictionless market.

The concept of a shadow price process consists of two parts. Firstly, trading in the frictionless market at prices given by the shadow price process should be at least as favorable as in the market with frictions. The investor can use a market order *at any time* to buy the risky asset at price $\overline{S}$. Hence, we have to require $\widetilde{S}_t \leq \overline{S}_t$ *for all* $t \geq 0$ to make sure that he never has to pay more than in the market with frictions. Analogously, to take care of the market sell orders, we demand $\underline{S} \leq \widetilde{S}_t$ for all $t \geq 0$. In a market with proportional transaction costs this would be sufficient, but in our limit order market the investor can also buy at $\underline{S}$ *whenever an exogenous market sell order arrives*. Thus, we have to require $\widetilde{S}_t \leq \underline{S}_t$ *whenever* $\Delta N_t^1 = 1$. Accordingly, to cover the opportunities to sell at $\overline{S}$ using limit sell orders, we need to demand $\widetilde{S}_t \geq \overline{S}_t$ whenever $\Delta N_t^2 = 1$. Combining these four requirements, we arrive at condition (2.5). Secondly, the maximal utility which can be achieved by trading at the shadow price must not be higher than by trading in the market with frictions. This is ensured by the second part of the definition. Note that for a shadow price to exist, $N^1$ and $N^2$ must not jump simultaneously at any time at which $\underline{S} < \overline{S}$ holds, otherwise (2.5) cannot be satisfied.

The following lemma is a reformulation of Lemma 2.2 in [9]. We quote it for convenience of the reader.

**Lemma 1** *(Kallsen and Muhle-Karbe [9]) Let $S$ be a real-valued semimartingale and let $\varphi \in L(S)$ be a finite variation process (not necessarily right-continuous). Then their product $\varphi S$ can be written as*

$$\begin{aligned}\varphi_t S_t &= \varphi_0 S_0 + \int_0^t \varphi_s dS_s + \int_0^t (S_{s-}, S_s) d\varphi_s \\ &= \varphi_0 S_0 + \int_0^t \varphi_s dS_s + \int_0^t S_{s-} d\varphi_s^r + \sum_{0 \leq s < t} S_s \Delta^+ \varphi_s.\end{aligned}$$

**Proposition 1** *If $\widetilde{S}$ is a shadow price process and $\mathfrak{S}$ is a strategy in the limit order market corresponding to an optimal strategy $\widetilde{\mathfrak{S}}$ in the frictionless market as in Definition 2, then $\mathfrak{S}$ is an optimal strategy in the limit order market, i.e. $\mathscr{J}(\mathfrak{S}) = V(\eta^0, \eta^1)$.*

*Proof Step 1.* We begin by showing $V(\eta^0, \eta^1) \leq \widetilde{V}(\eta^0, \eta^1)$. Let $\mathfrak{S} \in \mathscr{A}(\eta^0, \eta^1)$ with corresponding portfolio process $(\varphi^0, \varphi^1)$. Define

$$\begin{aligned}\psi_t^0 := \eta^0 &- \int_0^t c_s ds - \int_0^t (\widetilde{S}_{s-}, \widetilde{S}_s) dM_s^B + \int_0^t (\widetilde{S}_{s-}, \widetilde{S}_s) dM_s^S \\ &- \int_0^{t-} L_s^B \widetilde{S}_s dN_s^1 + \int_0^{t-} L_s^S \widetilde{S}_s dN_s^2\end{aligned}$$

and $\psi^1 := \varphi^1$. Applying Lemma 1 we get

$$\psi_t^1 \widetilde{S}_t = \eta^1 \widetilde{S}_0 + \int_0^t \psi_s^1 d\widetilde{S}_s + \int_0^t (\widetilde{S}_{s-}, \widetilde{S}_s) d\psi_s^1.$$

This equation is equivalent to

$$\psi_t^0 + \psi_t^1 \widetilde{S}_t - \eta^0 - \eta^1 \widetilde{S}_0 - \int_0^t \psi_s^1 d\widetilde{S}_s + \int_0^t c_s ds = \psi_t^0 + \int_0^t (\widetilde{S}_{s-}, \widetilde{S}_s) d\psi_s^1 - \eta^0 + \int_0^t c_s ds. \tag{2.6}$$

By definition of $\psi^0$ and $\psi^1$ and associativity of the integral the term on the right side is equal to 0. Hence (2.6) implies that $(\psi^0, \psi^1, c)$ is a self-financing strategy in the frictionless market. Furthermore, by (2.5) and (2.3) we get

$$\psi_t^0 + \psi_t^1 \widetilde{S}_t \geq \varphi_t^0 + \varphi_t^1 \widetilde{S}_t \geq \varphi_t^0 + 1_{\{\varphi_t^1 \geq 0\}} \varphi_t^1 \underline{S}_t + 1_{\{\varphi_t^1 < 0\}} \varphi_t^1 \overline{S}_t \geq 0.$$

Thus for every $\mathfrak{S} \in \mathscr{A}(\eta^0, \eta^1)$ we have an admissible strategy $\widetilde{\mathfrak{S}} = (\psi^0, \psi^1, c) \in \widetilde{\mathscr{A}}(\eta^0, \eta^1)$ with the same consumption rate.

*Step 2.* By the definition of a shadow price there is a strategy $\mathfrak{S} = (M^B, M^S, L^B, L^S, c)$ in the limit order market with associated portfolio process $(\varphi^0, \varphi^1)$ such that $\widetilde{\mathfrak{S}} = (\varphi^0, \varphi^1, c)$ is an optimal strategy in the frictionless market, i.e.

$$\mathscr{J}(\mathfrak{S}) = \widetilde{\mathscr{J}}(\widetilde{\mathfrak{S}}) = \widetilde{V}(\eta^0, \eta^1).$$

By Step 1 this implies $\mathscr{J}(\mathfrak{S}) = V(\eta^0, \eta^1)$, hence $\mathfrak{S}$ is optimal.

## 3 Heuristic derivation of a candidate for a shadow price process

The model of a small investor trading in a limit order market makes sense in the generality introduced above. Still, to get enough tractability to be able to construct a shadow price process we reduce the complexity by restricting ourselves to a more concrete case. From now on we model the spread as proportional to the best-bid price, which is modeled as a standard geometric Brownian motion with starting value $\underline{s}$, i.e.

$$d\underline{S}_t = \underline{S}_t(\mu dt + \sigma dW_t), \qquad \underline{S}_0 = \underline{s}, \tag{3.1}$$

with $\mu, \sigma \in \mathbb{R}_+ \setminus \{0\}$. The size of the spread is modeled with a constant $\lambda > 0$. Similarly to [9] define

$$\overline{C} := \log(1+\lambda) \qquad \text{and} \qquad \overline{S} := \underline{S}e^{\overline{C}} = \underline{S}(1+\lambda).$$

Let $\alpha_1, \alpha_2 \in \mathbb{R}_+$. The arrival of exogenous market orders is modeled as two independent time-homogenous Poisson processes $N^1$ and $N^2$ with rates $\alpha_1$ and $\alpha_2$. These memoryless and stationary arrival times, the time-independent coefficients in the dynamics of the best bid price, the proportional spread, and the infinite horizon of the optimization problem (2.4) will lead to a time-homogenous structure of the solution.

For $\alpha_1 = \alpha_2 = 0$ the model reduces to the model with proportional transaction costs as e.g. in [1], [9] or [22]. For $\lambda = 0$ and by allowing to trade *only* at the jump times of the Poisson process we would arrive at an illiquidity model introduced by Rogers and Zane [20] which is widely investigated in the literature, see e.g. Matsumoto [13] who studies optimal portfolios w.r.t. terminal wealth in this model. Pham and Tankov [18] recently introduced a related illiquidity model under which the price of the risky asset cannot even be observed apart from the Poisson times at which trading is possible.

We will show (under certain restrictions to the parameters $\mu, \sigma, \lambda, \alpha_1, \alpha_2$, see Proposition 2) that it is optimal to control the portfolio as follows. There exist $\pi_{\min}, \pi_{\max} \in \mathbb{R}_+$ with $0 < \pi_{\min} < \pi_{\max}$ such that the proportion of wealth invested in the risky asset (measured in terms of the best bid price) is kept in the interval $[\pi_{\min}, \pi_{\max}]$ by using **market orders**, i.e.

$$\pi_{\min} \leq \frac{\varphi_t^1 \underline{S}_t}{\varphi_t^0 + \varphi_t^1 \underline{S}_t} \leq \pi_{\max}, \quad \forall t > 0 \tag{3.2}$$

(Note that, as $\underline{S}$ and $\overline{S}$ only differ in a constant factor, the structure of the solution would remain unaffected if wealth was measured in terms of the best-ask price instead of the best-bid price – only the numbers $\pi_{\min}$ and $\pi_{\max}$ would change). To keep the proportion within this interval, as is the case with proportional transaction costs, $M^B$ and $M^S$ will have local time at the boundary. In the inner they are constant. Furthermore, at all times two **limit orders** are kept in the order book such that

$$\frac{\varphi_t^1 \underline{S}_t}{\varphi_t^0 + \varphi_t^1 \underline{S}_t} = \pi_{\max}, \quad \text{after limit buy order is executed with limit } \underline{S}_t \tag{3.3}$$

$$\frac{\varphi_t^1 \underline{S}_t}{\varphi_t^0 + \varphi_t^1 \underline{S}_t} = \pi_{\min}, \quad \text{after limit sell order is executed with limit } \overline{S}_t. \tag{3.4}$$

To follow this strategy both limit *prices* and limit order *sizes* have to be permanently adjusted. The former to stay at $\underline{S}$ and $\overline{S}$, resp. The latter as after a successful execution of a limit order the *proportion of wealth* invested in the risky asset and not the number of risk assets is time-homogenous. Finally, optimal consumption is proportional to wealth measured w.r.t. the shadow price.

In this section we provide some intuition on how to use the *guessed* properties of the optimal strategy described in (3.2), (3.3), and (3.4) to find a promising candidate for a shadow price process by combining some properties a shadow price process should satisfy. Later, in Section 5, we construct a semimartingale that satisfies these properties by using solutions of a suitable free boundary problem and a related Skorohod problem. This semimartingale is then *verified* to be indeed a shadow price process of the risky asset.

The definition of a shadow price process suggests that if for example market order sales become worthwhile, $\widetilde{S}$ approaches $\underline{S}$ as in [9]. Moreover, by (2.5) if an exogenous market buy order arises (i.e. the asset can be sold expensively), the shadow price has to jump to $\overline{S}$. Consider a $[0,\overline{C}]$-valued Markov process which satisfies

$$dC_t = \widetilde{\mu}(C_{t-})dt + \widetilde{\sigma}(C_{t-})dW_t - C_{t-}dN_t^1 + (\overline{C} - C_{t-})dN_t^2,$$

where the real functions $\widetilde{\mu}$ and $\widetilde{\sigma}$ are not yet specified, but are assumed to be sufficiently nice for a solution $C$ of the stochastic differential equation to exist. As an ansatz for the shadow price $\widetilde{S}$ we use $\widetilde{S} := \underline{S}\exp(C)$. $C$ is similar to the process in [9] apart from its jumps. From Itô's formula we get

$$\begin{aligned} d\widetilde{S}_t =& \widetilde{S}_{t-}\left[\left(\mu + \frac{\widetilde{\sigma}(C_{t-})^2}{2} + \sigma\widetilde{\sigma}(C_{t-}) + \widetilde{\mu}(C_{t-})\right)dt + (\sigma + \widetilde{\sigma}(C_{t-}))\,dW_t \right.\\ &+ \left.\left(e^{-C_{t-}\Delta N_t^1 + (\overline{C}-C_{t-})\Delta N_t^2} - 1\right)\right]. \end{aligned}$$

For $\widetilde{S}$ to be a shadow price process, we have to be able to find a strategy which is optimal in the frictionless market with price process $\widetilde{S}$, but can also be carried out in the limit order market at the same prices. Fortunately, optimal behavior in the frictionless market is well understood for logarithmic utility. The plan is to choose the dynamics of $\widetilde{S}$ in such a way, that the portfolio process of the suspected optimal strategy described in (3.2), (3.3), and (3.4) is an optimal strategy in the frictionless market. To do this, we can use a theorem by Goll and Kallsen [4] (Theorem 3.1) which gives a sufficient condition for a strategy in a frictionless markets to be optimal. It says that if the triple $(\widetilde{b},\widetilde{c},\widetilde{F})$ is the differential semimartingale characteristics of the special semimartingale $\widetilde{S}$ (w.r.t. to the predictable increasing process $I(\omega,t) := t$ and "truncation function" $h(x) = x$, see e.g. [7] (Proposition II.2.9)) and if the equation

$$\widetilde{b}_t - \widetilde{c}_t H_t + \int \left(\frac{x}{1+H_t x} - x\right)\widetilde{F}_t(dx) = 0$$

was fulfilled $(P\otimes I)$-a.e on $\Omega\times[0,\infty)$ by $H := \varphi^1/\widetilde{V}_-$, then $H$ would be optimal. Using that $N^1$ and $N^2$ are independent and thus

$$\Delta N^1 \Delta N^2 = 0 \quad \text{and} \quad e^{-C_-\Delta N^1 + (\overline{C}-C_-)\Delta N^2} - 1 = e^{-C_-\Delta N^1} - 1 + e^{(\overline{C}-C_-)\Delta N^2} - 1$$

up to evanescence, the characteristic triple of $\widetilde{S}$ becomes

$$\begin{aligned} \widetilde{b}_t &= \widetilde{S}_{t-}\left(\mu + \frac{\widetilde{\sigma}(C_{t-})^2}{2} + \sigma\widetilde{\sigma}(C_{t-}) + \widetilde{\mu}(C_{t-})\right) + \int x\widetilde{F}_t(dx) \\ \widetilde{c}_t &= \left(\widetilde{S}_{t-}(\sigma + \widetilde{\sigma}(C_{t-}))\right)^2 \\ \widetilde{F}_t(\omega,dx) &= \alpha_1\delta_{x_1(\omega,t)}(dx) + \alpha_2\delta_{x_2(\omega,t)}(dx), \end{aligned}$$

with

$$x_1(\omega,t) := \widetilde{S}_{t-}(\omega)(e^{-C_{t-}(\omega)}-1), \quad x_2(\omega,t) := \widetilde{S}_{t-}(\omega)(e^{\overline{C}-C_{t-}(\omega)}-1).$$

Denote by $\widetilde{\pi}_t := H_t\widetilde{S}_{t-}$ the optimal fraction invested in the risky asset, measured in terms of the shadow price. Even though we cannot write down $\widetilde{\pi}_t$ explicitly, we know that a $\widetilde{\pi}$ is optimal, if it satisfies

$$\begin{aligned} F(C_{t-},\widetilde{\pi}_t) &:= \mu + \frac{\widetilde{\sigma}(C_{t-})^2}{2} + \sigma\widetilde{\sigma}(C_{t-}) + \widetilde{\mu}(C_{t-}) - \widetilde{\pi}_t(\sigma+\widetilde{\sigma}(C_{t-}))^2 \\ &+ \alpha_1(e^{-C_{t-}}-1)\left(\frac{1}{1+\widetilde{\pi}_t(e^{-C_{t-}}-1)}\right) \\ &+ \alpha_2(e^{\overline{C}-C_{t-}}-1)\left(\frac{1}{1+\widetilde{\pi}_t(e^{\overline{C}-C_{t-}}-1)}\right) \\ &= 0. \end{aligned} \tag{3.5}$$

Consider the stopping time

$$\tau := \inf\left\{t>0 : C_t \in \{0,\overline{C}\}\right\}.$$

As long as $\underline{S} < \widetilde{S} < \overline{S}$, it should be optimal in the frictionless market to keep the number of shares in the risky asset constant, i.e. there is no trading. Thus, on $]\!]0,\tau[\![$ we should have that

$$d\log(\varphi_t^0) = \frac{-c_t}{\varphi_t^0}dt = \frac{-\delta\widetilde{V}_t}{\widetilde{V}_t - \widetilde{\pi}_t\widetilde{V}_t}dt = \frac{-\delta}{1-\widetilde{\pi}_t}dt,$$

where $(\varphi^0,\varphi^1)$ are the optimal amounts of securities. The second equality holds as optimal consumption is given by $c = \delta\widetilde{V}$ (again by Theorem 3.1 in [4]). Using the same approach to simplify the calculations as in [9] we introduce

$$\beta := \log\left(\frac{\widetilde{\pi}}{1-\widetilde{\pi}}\right) = \log\left(\frac{\varphi^1\widetilde{S}}{\varphi^0}\right).$$

On $]\!]0,\tau[\![$ we have $C = C_-$, hence the dynamics of $\beta_t$ on $]\!]0,\tau[\![$ can be written as

$$\begin{aligned} d\beta_t &= d\log(\varphi_t^1) + d\log(\widetilde{S}_t) - d\log(\varphi_t^0) \\ &= \left(\mu - \frac{\sigma^2}{2} + \widetilde{\mu}(C_t) + \frac{\delta}{1-\widetilde{\pi}(C_t)}\right)dt + (\sigma+\widetilde{\sigma}(C_t))dW_t. \end{aligned} \tag{3.6}$$

Furthermore, $\widetilde{\pi}$ is a function of $C_-$ implicitly given by optimality equation (3.5). On $]\!]0,\tau[\![$ we can even write $\beta = f(C)$ for some function $f$ which, however, depends on the functions $\widetilde{\mu}$ and $\widetilde{\sigma}$ that are not yet specified. Assume that $f \in C^2$. By Itô's formula we get

$$d\beta_t = \left(f'(C_t)\widetilde{\mu}(C_t) + f''(C_t)\frac{\widetilde{\sigma}(C_t)^2}{2}\right)dt + f'(C_t)\widetilde{\sigma}(C_t)dW_t. \tag{3.7}$$

By comparing the factors of (3.6) and (3.7) we can write down $\widetilde{\mu}$ and $\widetilde{\sigma}$ as functions of $f,\mu$ and $\sigma$:

$$\begin{aligned} \widetilde{\sigma} &= \frac{\sigma}{f'-1} \\ \widetilde{\mu} &= \left(\mu - \frac{\sigma^2}{2} + \frac{\delta(1+e^{-f})}{e^{-f}} - \frac{\sigma^2}{2}\frac{f''}{(f'-1)^2}\right)\frac{1}{f'-1}. \end{aligned}$$

Note that to get rid of $\widetilde{\pi}_t$ we have used that from $f(C) = \beta = \log\left(\frac{\widetilde{\pi}}{1-\widetilde{\pi}}\right)$ follows $\widetilde{\pi} = \frac{1}{1+e^{-f(C)}}$.

Now that we have expressions for $\widetilde{\mu}$ and $\widetilde{\sigma}$ we can insert them into the optimality equation (3.5) to get an ODE similar to the one in [9]. The ODE in our case is

$$\begin{aligned}
&\mu+\frac{1}{2}\left(\frac{\sigma}{f'(x)-1}\right)^2+\frac{\sigma^2}{f'(x)-1} \\
&+\left(\mu-\frac{\sigma^2}{2}+\frac{\delta(1+e^{-f(x)})}{e^{-f(x)}}-\frac{\sigma^2}{2}\frac{f''(x)}{(f'(x)-1)^2}\right)\frac{1}{f'(x)-1} \\
&-\frac{(\sigma+\frac{\sigma}{f'(x)-1})^2}{1+e^{-f(x)}}+\alpha_1(e^{-x}-1)\left(\frac{1}{1+\frac{e^{-x}-1}{1+e^{-f(x)}}}\right)+\alpha_2(e^{\overline{C}-x}-1)\left(\frac{1}{1+\frac{e^{\overline{C}-x}-1}{1+e^{-f(x)}}}\right) \\
&=0.
\end{aligned} \tag{3.8}$$

Remember that apart from a possible bulk trade at time 0 in our suspected optimal strategy the aggregated market buy and sell orders are local times. This implies that the fraction invested in the risky asset also has a local time component, and hence the same is true for $\beta$. Thus a smooth function $f$ with $\beta=f(C)$ has to possess an exploding first derivative as in $C$ no local time appears (the ansatz that $C$ resp. $\widetilde{S}$ has no local time makes sense, as it is well known that a local time component in the discounted price process would imply arbitrage, see e.g. Appendix B in [10] or [8] for an introduction to the problematics). To avoid an explosion, we turn the problem around by considering $C$ as a function of $\beta$, i.e. $C=g(\beta):=f^{-1}(\beta)$. Defining

$$B(y,z):=\alpha_1(e^{-z}-1)\left(\frac{1}{1+\frac{e^{-z}-1}{1+e^{-y}}}\right)+\alpha_2(e^{\overline{C}-z}-1)\left(\frac{1}{1+\frac{e^{\overline{C}-z}-1}{1+e^{-y}}}\right), \tag{3.9}$$

we can invert ODE (3.8) and get

$$\begin{aligned}
g''(y)=&-\frac{2}{\sigma^2}B(y,g(y))-\frac{2\mu}{\sigma^2}+\frac{2}{1+e^{-y}} \\
&+\left(\frac{6}{\sigma^2}B(y,g(y))+\frac{4\mu}{\sigma^2}-\frac{2}{1+e^{-y}}-1-\frac{2\delta}{\sigma^2}(1+e^{y})\right)g'(y) \\
&+\left(-\frac{6}{\sigma^2}B(y,g(y))-\frac{2\mu}{\sigma^2}+1+\frac{4\delta}{\sigma^2}(1+e^{y})\right)(g'(y))^2 \\
&+\left(\frac{2}{\sigma^2}B(y,g(y))-\frac{2\delta}{\sigma^2}(1+e^{y})\right)(g'(y))^3.
\end{aligned} \tag{3.10}$$

Note that this equation without the term $B$ is the same ODE as in [9]. We need to take care that the local time in $\beta$ does not show up in $C$ but since local time only occurs at $\underline{\beta}$ and $\overline{\beta}$ by choosing the right boundary conditions for $g'$ this can be avoided easily. Namely, $g'$ has to vanish at the boundaries. Similar to [9] we arrive at the boundary conditions

$$g(\underline{\beta})=\overline{C},\qquad g(\overline{\beta})=0,\qquad g'(\underline{\beta})=g'(\overline{\beta})=0, \tag{3.11}$$

where $\underline{\beta}$ and $\overline{\beta}$ have to be chosen. Indeed, an application of Itô's formula shows that these boundary conditions for $g'$ imply that $C$ does not have a local time component.

## 4 Existence of a solution to the free boundary problem

**Proposition 2** *Let $\alpha_1<\mu\frac{1+\lambda}{\lambda}$, $\alpha_2<(\sigma^2-\mu)\frac{1+\lambda}{\lambda}$, and $\delta>\alpha_2\lambda$. Then the free boundary problem (3.10)/(3.11) admits a solution $(g,\underline{\beta},\overline{\beta})$ such that $g:[\underline{\beta},\overline{\beta}]\to[0,\overline{C}]$ and $g$ is strictly decreasing.*

The first two parameter restrictions can be interpreted economically quite well, whereas the last restriction is a technical condition, which is sufficient for the existence of a shadow price.

As $\alpha_1, \alpha_2 \geq 0$ the first two parameter restrictions imply that

$$0 < \mu < \sigma^2. \tag{4.1}$$

In the case with proportional transaction costs, (4.1) guarantees that $0 < \pi_{\min} < \pi_{\max} < 1$, i.e. the optimal strategy entails neither leveraging nor shorting of the risky asset. This is not the case when the opportunity to trade at more favorable prices using limit orders exists. Namely, short selling the stock by a limit order and liquidating this position again after the successful execution of a limit buy order leads to some additional expected return whose rate is for small $\lambda$ roughly $\alpha_1\lambda$ (note that $\alpha_1$ is the rate of the arrival times which allow to buy the stock cheaply back, the expected return is earned as long as the investor holds a short position). Thus $\alpha_1 < \mu\frac{1+\lambda}{\lambda}$ guarantees that short selling is not worthwhile. Analogously long positions that are build up with limit buy orders yield an additional expected return with approximative rate $\alpha_2\lambda$. Thus $\alpha_2 < (\sigma^2 - \mu)\frac{1+\lambda}{\lambda}$ becomes necessary to exclude leveraging. Summing up, the first two conditions are necessary to avoid leveraging and short selling.

*Proof* Define for $y, z \in \mathbb{R}$

$$\widetilde{B}(y,z) := \begin{cases} B(y,z) & \text{if } z \in [0,\overline{C}], \\ \alpha_2\left(e^{\overline{C}} - 1\right)\left(1 + \frac{e^{\overline{C}}-1}{1+e^{-y}}\right)^{-1} & \text{if } z < 0, \\ \alpha_1\left(e^{-\overline{C}} - 1\right)\left(1 + \frac{e^{-\overline{C}}-1}{1+e^{-y}}\right)^{-1} & \text{if } z > \overline{C}. \end{cases}$$

Note that $\widetilde{B}(y,z)$ is decreasing in $y$ and $z$. Furthermore, for all $y, z \in \mathbb{R}$ we have

$$-\frac{\alpha_1\lambda}{1+\lambda} < \widetilde{B}(y,z) < \alpha_2\lambda.$$

Instead of dealing with the original free boundary problem (3.10)/(3.11), we now replace (3.10) with

$$\begin{aligned} g''(y) &= -\frac{2}{\sigma^2}\widetilde{B}(y,g(y)) - \frac{2\mu}{\sigma^2} + \frac{2}{1+e^{-y}} \\ &+ \left(\frac{6}{\sigma^2}\widetilde{B}(y,g(y)) + \frac{4\mu}{\sigma^2} - \frac{2}{1+e^{-y}} - 1 - \frac{2\delta}{\sigma^2}(1+e^y)\right)g'(y) \\ &+ \left(-\frac{6}{\sigma^2}\widetilde{B}(y,g(y)) - \frac{2\mu}{\sigma^2} + 1 + \frac{4\delta}{\sigma^2}(1+e^y)\right)(g'(y))^2 \\ &+ \left(\frac{2}{\sigma^2}\widetilde{B}(y,g(y)) - \frac{2\delta}{\sigma^2}(1+e^y)\right)(g'(y))^3, \end{aligned} \tag{4.2}$$

whereas the boundary condition (3.11) stays the same. We will see that the change from $B$ to $\widetilde{B}$ guarantees that functions satisfying the ODE do not explode, because the impact of $g(y)$ on $g''(y)$ remains bounded, even when $g(y)$ leaves $[0,\overline{C}]$. Note that if we show the existence of a solution $g : [\underline{\beta}, \overline{\beta}] \to [0,\overline{C}]$ to this modified free boundary problem, we have also shown the existence of a solution to the original free boundary problem, since $B(y,z) = \widetilde{B}(y,z)$ on $\mathbb{R} \times [0,\overline{C}]$. Denote by $y_0$ the unique root of the function

$$H(y) := \frac{-\alpha_1}{\sigma^2}\left(e^{-\overline{C}} - 1\right)\left(1 + \frac{e^{-\overline{C}}-1}{1+e^{-y}}\right)^{-1} - \frac{\mu}{\sigma^2} + \frac{1}{1+e^{-y}}.$$

Such an $y_0$ exists. Indeed, we have assumed $\alpha_1 < \mu\frac{1+\lambda}{\lambda}$, which implies $\frac{\alpha_1\lambda - \mu - \mu\lambda}{\sigma^2 + \sigma^2\lambda} < 0$. Thus, as $\overline{C} = \log(1+\lambda)$, it follows $\lim_{y\to-\infty} H(y) < 0$. $e^{-\overline{C}} - 1 < 0$ and $\mu < \sigma^2$ imply $\lim_{y\to\infty} H(y) > 0$.

Since $H$ is continuous, the intermediate value theorem implies the existence of a $y_0$, which is unique since $H$ is strictly increasing.

For any $\Delta > 0$ let $\underline{\beta}_\Delta := y_0 - \Delta$. For any choice of $\Delta > 0$ the initial value problem given by (4.2) with initial conditions $g(\underline{\beta}_\Delta) = \overline{C}$ and $g'(\underline{\beta}_\Delta) = 0$ admits a unique local solution $g_\Delta$. Because $\delta - \alpha_2\lambda > 0$, we can define a real number $\underline{M} < 0$ by

$$\underline{M} := \min\left\{-\sqrt[3]{\frac{3(\alpha_2\lambda+\mu)}{\delta-\alpha_2\lambda}}, -\sqrt{\frac{3(3\alpha_2\lambda+2\mu)}{\delta-\alpha_2\lambda}}, -\frac{3\alpha_2\lambda+\mu}{\delta-\alpha_2\lambda}\right\}.$$

For $g'_\Delta(y) < \underline{M}$ we have $g''_\Delta(y) > 0$. Similarly, define a real number $\overline{M} > 0$ by

$$\overline{M} := \max\left\{\sqrt[3]{\frac{3\left(\frac{\alpha_1\lambda}{1+\lambda}+\sigma^2\right)}{\delta-\alpha_2\lambda}}, \sqrt{\frac{3(3\alpha_2\lambda+2\mu)}{\delta-\alpha_2\lambda}}, \frac{3\left(\frac{6\alpha_1\lambda}{1+\lambda}+\sigma^2+4\delta\right)}{2(\delta-\alpha_2\lambda)}\right\}.$$

For $g'_\Delta(y) > \overline{M}$ we have $g''_\Delta(y) < 0$. Hence, $g'_\Delta(y) \in [\underline{M}, \overline{M}]$ for all $y \geq \underline{\beta}_\Delta$ and the maximal interval of existence for $g_\Delta$ is $\mathbb{R}$. Note that $\underline{M}, \overline{M}$ do not depend on the choice of $\Delta$.

By $\alpha_2 < (\sigma^2 - \mu)\frac{1+\lambda}{\lambda}$, there exist $y^\star \in \mathbb{R}$ and $\varepsilon > 0$ such that

$$-\frac{2}{\sigma^2}\widetilde{B}(y,z) - \frac{2\mu}{\sigma^2} + \frac{2}{1+e^{-y}} > \varepsilon$$

for all $y \geq y^\star, z \in \mathbb{R}$ (this can be proved analogously to the existence of $y_0$). Combining this with (4.2) shows that there even exists an $\overline{y}_\Delta$ such that $g''_\Delta(y) > \varepsilon$ for $g'_\Delta(y) \leq 0$ and $y \geq \overline{y}_\Delta$. Thus, $g'_\Delta$ has at least another root larger than $\underline{\beta}_\Delta$, i.e.

$$\overline{\beta}_\Delta := \min\{y > \underline{\beta}_\Delta : g'_\Delta(y) = 0\} < \infty.$$

Hence, by definition $g_\Delta$ is decreasing on $[\underline{\beta}_\Delta, \overline{\beta}_\Delta]$. The remainder of the proof consists in showing that $g_\Delta(\overline{\beta}_\Delta) \to \overline{C}$ for $\Delta \to 0$, $g_\Delta(\overline{\beta}_\Delta) \to -\infty$ for $\Delta \to \infty$ and that $\Delta \mapsto g_\Delta(\overline{\beta}_\Delta)$ is a continuous mapping. Then, by the intermediate value theorem, there exists a $\Delta$ such that $g_\Delta$ is a solution to the free boundary problem (4.2)/(3.11).

*Step 1.* We prove that $g_\Delta(\overline{\beta}_\Delta) \to \overline{C}$ for $\Delta \to 0$. The boundedness of $(\Delta, y) \mapsto g'_\Delta(y)$ together with (4.2) implies that $|g''_\Delta(y)|$ is bounded by a constant $M''$ on $[y_0-1, y_0+1]$. For $\Delta < 1$ and $y \in [y_0-1, y_0+1]$ we get $|g'_\Delta(y)| \leq (y - y_0 + \Delta)M''$. Hence, by (4.2), $g_\Delta(y) \to \overline{C}$ for $\Delta \to 0$ and $y \to y_0$, the continuity of $\widetilde{B}$, and the definition of $y_0$ we have that

$$\sup_{y\in[y_0-\Delta, y_0+\widetilde{\Delta}]} |g''_\Delta(y)| \to 0 \qquad \text{for } \Delta, \widetilde{\Delta} \downarrow 0. \tag{4.3}$$

Firstly, by (4.3) the last three summands in (4.2) are of order $o(y - y_0 + \Delta)$ for $(\Delta, y) \to (0, y_0)$. Let us rewrite the first summand of (4.2) as

$$\begin{aligned} &-\frac{2}{\sigma^2}\widetilde{B}(y, g_\Delta(y)) - \frac{2\mu}{\sigma^2} + \frac{2}{1+e^{-y}} \\ &= \left(-\frac{2}{\sigma^2}\widetilde{B}(y, g_\Delta(y)) + \frac{2}{\sigma^2}\widetilde{B}(y, \overline{C})\right) + \left(-\frac{2}{\sigma^2}\widetilde{B}(y, \overline{C}) - \frac{2\mu}{\sigma^2} + \frac{2}{1+e^{-y}}\right). \end{aligned} \tag{4.4}$$

Secondly, because of $g'_\Delta(y_0 - \Delta) = 0$, a first order Taylor expansion of the first summand in (4.4) at $y_0 - \Delta$ shows that

$$\begin{aligned} &-\frac{2}{\sigma^2}\widetilde{B}(y, g_\Delta(y)) + \frac{2}{\sigma^2}\widetilde{B}(y, \overline{C}) \\ &= \frac{1}{2}\left(g''_\Delta(\xi_\Delta)\partial_2\widetilde{B}(y, g_\Delta(\xi_\Delta)) + (g'_\Delta(\xi_\Delta))^2\partial_{22}\widetilde{B}(y, g_\Delta(\xi_\Delta))\right)(y - y_0 + \Delta)^2, \end{aligned}$$

for $\xi_\Delta \in [y_0-\Delta, y]$, i.e. this term is also of order $o(y-y_0+\Delta)$ for $(\Delta,y)\to(0,y_0)$. Thirdly, a first order Taylor expansion of the second summand in (4.4) at $y_0$ shows that the term can be written as $a(y-y_0)+o(y-y_0)$, where $a := -\frac{2}{\sigma^2}\partial_1\widetilde{B}(y_0,\overline{C})) + \frac{2e^{-y_0}}{(1+e^{-y_0})^2} > 0$. Combining the three points above it follows that

$$g''_\Delta(y) = a(y-y_0)+o(y-y_0)+o(y-y_0+\Delta), \quad \text{for } (\Delta,y)\to(0,y_0).$$

Thus, for any constant $K>0$ we can choose $\Delta$ small enough that $g''_\Delta(y) > \frac{a}{2}\Delta$ on $y\in[y_0+\Delta, y_0+(K+1)\Delta]$. Hence,

$$\overline{\beta}_\Delta - \underline{\beta}_\Delta < 2\Delta + \frac{4\Delta\sup_{y\in[y_0-\Delta,y_0+\Delta]}|g''_\Delta(y)|}{a\Delta} \to 0, \qquad \text{for } \Delta\to 0.$$

Since $(y,\Delta)\mapsto g'_\Delta$ is bounded it follows that $g_\Delta(\overline{\beta}_\Delta)\to\overline{C}$ for $\Delta\to 0$.

*Step 2.* We prove that $g_\Delta(\overline{\beta}_\Delta)\to-\infty$ for $\Delta\to\infty$. Remember that the definition of $y_0$ and the strict monotonicity of $H$ imply $H(y^\star)<0$ for any $y^\star<y_0$. Let

$$\widetilde{M}(y^\star) := \max\left\{ \frac{\frac{1}{3}H(y^\star)}{\frac{6}{\sigma^2}\frac{\alpha_1\lambda}{1+\lambda}+3+\frac{2\delta}{\sigma^2}(1+e^{y^\star})}, -\sqrt{\frac{-\frac{1}{3}H(y^\star)}{\frac{6}{\sigma^2}\frac{\alpha_1\lambda}{1+\lambda}+1+\frac{4\delta}{\sigma^2}(1+e^{y^\star})}}, -\sqrt[3]{\frac{-\frac{1}{3}H(y^\star)}{\frac{2}{\sigma^2}\frac{\alpha_1\lambda}{1+\lambda}+\frac{2\delta}{\sigma^2}(1+e^{y^\star})}} \right\} < 0.$$

For $y\le y^\star$ and $0\ge g'_\Delta(y) > \widetilde{M}(y^\star)$ we have that $g''_\Delta(y) < H(y^\star) < 0$. By $g''_\Delta(\underline{\beta}_\Delta)<0$, this yields $g'_\Delta(y)<0$ for $y\le y^\star$ and also $g'_\Delta(y)\le\widetilde{M}(y^\star)$ for $y\in[y_0-\Delta+\frac{\widetilde{M}(y^\star)}{H(y^\star)}, y^\star]$. Therefore, $g_\Delta(\overline{\beta}_\Delta)\to-\infty$ as $\Delta\to\infty$.

*Step 3.* We prove that $\Delta\mapsto g_\Delta(\overline{\beta}_\Delta)$ is continuous. By Theorem 2.1 in [6] and because for every choice of $\Delta\in(0,\infty)$ the maximal interval of existence of $g_\Delta$ is $\mathbb{R}$, it follows that the general solution $(g,g')(\Delta,y) := \big(g_\Delta(y), g'_\Delta(y)\big) : (0,\infty)\times\mathbb{R}\to\mathbb{R}^2$ is continuous. Thus, $\big(g_\Delta, g'_\Delta\big)$ converges to $\Big(g_{\Delta_0}, g'_{\Delta_0}\Big)$ uniformly on compacts as $\Delta\to\Delta_0$.

Therefore, it is sufficient to show that $\Delta\to\Delta_0$ implies $\overline{\beta}_\Delta\to\overline{\beta}_{\Delta_0}$. Fix $\Delta_0\in(0,\infty)$. To verify that $\liminf_{\Delta\to\Delta_0}\overline{\beta}_\Delta \ge \overline{\beta}_{\Delta_0}$ note that by Step 2 we have $g'_\Delta(y)<0$ for all $\Delta>0$, $y<y_0$ (as $y^\star$ was chosen arbitrary). In addition, given an $\varepsilon>0$, $g'_{\Delta_0}$ is strictly separated from $[0,\infty)$ on $[y_0, \overline{\beta}_{\Delta_0}-\varepsilon]$. By the uniform convergence on compacts of $g'_\Delta$ to $g'_{\Delta_0}$, it follows that $\liminf_{\Delta\to\Delta_0}\overline{\beta}_\Delta\ge\overline{\beta}_{\Delta_0}$.

By the continuity of $g''_{\Delta_0}$ we have $g''_{\Delta_0}(\overline{\beta}_{\Delta_0})\ge 0$. In the case that $g''_{\Delta_0}(\overline{\beta}_{\Delta_0})>0$, a first order Taylor expansion of $g'_{\Delta_0}$ at $\overline{\beta}_{\Delta_0}$ shows that $g'_{\Delta_0}(y)>0$ immediately after $\overline{\beta}_{\Delta_0}$. Otherwise, i.e. if $g''_{\Delta_0}(\overline{\beta}_{\Delta_0})=0$, the same fact follows from a second order Taylor expansion of $g'_{\Delta_0}$ at $\overline{\beta}_{\Delta_0}$, because for $g'_{\Delta_0}(\overline{\beta}_{\Delta_0}) = g''_{\Delta_0}(\overline{\beta}_{\Delta_0}) = 0$ we have $g'''_{\Delta_0}(\overline{\beta}_{\Delta_0}) = -\frac{2}{\sigma^2}\partial_1\widetilde{B}(\overline{\beta}_{\Delta_0}, g_{\Delta_0}(\overline{\beta}_{\Delta_0})) + \frac{2\exp(-\overline{\beta}_{\Delta_0})}{(1+\exp(-\overline{\beta}_{\Delta_0}))^2} > 0$. Here the definition of $\widetilde{B}$ requires us to assume $g_{\Delta_0}(\overline{\beta}_{\Delta_0})\ne 0$ to ensure the differentiability of $g''_{\Delta_0}$ at $\overline{\beta}_{\Delta_0}$, but this is not problematic, because otherwise $(g_{\Delta_0}, \underline{\beta}_{\Delta_0}, \overline{\beta}_{\Delta_0})$ would already be a solution to the free boundary problem. Thus, there exists an $\varepsilon_0>0$ such that $g'_{\Delta_0}(\overline{\beta}_{\Delta_0}+\varepsilon)>0$ for any $\varepsilon\in(0,\varepsilon_0)$. This implies that $\limsup_{\Delta\to\Delta_0}\overline{\beta}_\Delta\le\overline{\beta}_{\Delta_0}$ and altogether continuity.

## 5 Proof of the existence of a shadow price

Throughout the section we assume that the assumptions of Proposition 2 are satisfied so that the free boundary problem specified in (3.10) and (3.11) has a solution $(g, \underline{\beta}, \overline{\beta})$ with $g : [\underline{\beta}, \overline{\beta}] \to [0, \overline{C}]$ strictly decreasing.

**Lemma 2** *Let $\beta_0 \in [\underline{\beta}, \overline{\beta}]$ and*

$$a(y) := \left(\mu - \frac{\sigma^2}{2} + \delta(1+e^y) + \frac{\sigma^2 g''(y)}{2(1-g'(y))^2}\right)\frac{1}{1-g'(y)}, \qquad b(y) := \frac{\sigma}{1-g'(y)}$$

*for $y \in [\underline{\beta}, \overline{\beta}]$. Then there exists a unique solution $(\beta, \Psi)$ to the following stochastic variational inequality*

*(i)* *$\beta$ is càdlàg and takes values in $[\underline{\beta}, \overline{\beta}]$. $\Psi$ is continuous and of finite variation with starting value $\Psi_0 = 0$,*

*(ii)*

$$\begin{aligned}\beta_t &= \beta_0 + \int_0^t a(\beta_{s-})ds + \int_0^t b(\beta_{s-})dW_s \\ &\quad + \sum_{s \le t}\left((\overline{\beta} - \beta_{s-})\Delta N_s^1 + (\underline{\beta} - \beta_{s-})\Delta N_s^2\right) + \Psi_t,\end{aligned} \tag{5.1}$$

*(iii)* *for every progressively measurable process $z$ which has càdlàg paths and takes values in $[\underline{\beta}, \overline{\beta}]$, we have*

$$\int_0^t (\beta_s - z_s)\, d\Psi_s \le 0, \qquad \forall t \le 0. \tag{5.2}$$

*Proof* We want to apply Theorem 1 in [14], which guarantees existence and uniqueness of reflected diffusion processes with jumps in convex domains under certain conditions. Thus we only need to verify that the conditions of the theorem are fulfilled in our setting.

Firstly, $(\underline{\beta}, \overline{\beta})$ is trivially bounded and convex. Secondly, the jump term in (5.1) ensures that all jumps from $[\underline{\beta}, \overline{\beta}]$ are inside $[\underline{\beta}, \overline{\beta}]$. All that is left is to check a Lipschitz-type condition. Note that if $g$ is a solution to ODE (3.10) on $[\underline{\beta}, \overline{\beta}]$ the functions $g$, $g'$ and $g''$ are continuous and therefore bounded on the compact set $[\underline{\beta}, \overline{\beta}]$. Furthermore, as we know that $g' \le 0$ on $[\underline{\beta}, \overline{\beta}]$, the derivative $b'$ of $b$ is bounded on $[\underline{\beta}, \overline{\beta}]$. In addition, this also implies that $B$ defined in (3.9) is bounded on $[\underline{\beta}, \overline{\beta}]$ as well, and the same is true for $\partial_1 B$ and $\partial_2 B$. Thus also $g'''$ is bounded on $[\underline{\beta}, \overline{\beta}]$ (using that the solution $g$ of the free boundary problem (3.10)/(3.11) can be extended to some neighborhood of $\underline{\beta}$ and $\overline{\beta}$, resp.) This implies that even the derivative $a'$ of $a$ is bounded on $[\underline{\beta}, \overline{\beta}]$.

*Remark 2* Since $\Psi$ is of finite variation there exist two non-decreasing processes $\overline{\Psi}$ and $\underline{\Psi}$ such that $\Psi = \overline{\Psi} - \underline{\Psi}$ and $\mathrm{Var}(\Psi) = \overline{\Psi} + \underline{\Psi}$. Furthermore, (5.2) implies that $\overline{\Psi}$ increases only on $\{\beta = \underline{\beta}\}$ (resp. on $\{\beta_- = \underline{\beta}\}$)and $\underline{\Psi}$ increases only on $\{\beta = \overline{\beta}\}$ (resp. on $\{\beta_- = \overline{\beta}\}$).

**Lemma 3** *For $\beta_0 \in [\underline{\beta}, \overline{\beta}]$ let $a(\cdot), b(\cdot)$ and the process $\beta$ be from Lemma 2. Then $C := g(\beta)$ is a $[0, \overline{C}]$-valued semimartingale with*

$$\begin{aligned}dC_t &= \left(g'(\beta_{t-})a(\beta_{t-}) + \frac{1}{2}g''(\beta_{t-})b(\beta_{t-})^2\right)dt + g'(\beta_{t-})b(\beta_{t-})dW_t \\ &\quad - g(\beta_{t-})dN_t^1 + (\overline{C} - g(\beta_{t-}))dN_t^2\end{aligned}$$

*and $\widetilde{S} := \underline{S}e^C$ satisfies*

$$\begin{aligned}d\widetilde{S}_t &= \widetilde{S}_{t-}\left(g'(\beta_{t-})a(\beta_{t-}) + \frac{1}{2}g''(\beta_{t-})b(\beta_{t-})^2 + \frac{1}{2}\left(g'(\beta_{t-})b(\beta_{t-})\right)^2 + \mu + \sigma g'(\beta_{t-})b(\beta_{t-})\right)dt \\ &\quad + \widetilde{S}_{t-}\left(g'(\beta_{t-})b(\beta_{t-}) + \sigma\right)dW_t \\ &\quad + \widetilde{S}_{t-}\left(\exp\{-g(\beta_{t-})\Delta N_t^1 + (\overline{C} - g(\beta_{t-}))\Delta N_t^2\} - 1\right).\end{aligned}$$

*Proof* Since $g'(\underline{\beta}) = g'(\overline{\beta}) = 0$ the result follows by Itô's lemma, the integration by parts formula and Remark 2.

**Lemma 4** *$\widetilde{S}$ is a special semimartingale. The differential semimartingale characteristics of $\widetilde{S}$ w.r.t $I$ and "truncation function" $h(x) = x$ are*

$$\begin{aligned}
\widetilde{b}_t &= \widetilde{S}_{t-}\left(-B(\beta_{t-}, g(\beta_{t-})) + \frac{1}{1+e^{-\beta_{t-}}}\left(\frac{\sigma}{1-g'(\beta_{t-})}\right)^2\right) + \int x \widetilde{F}_t(dx)\\
\widetilde{c}_t &= \widetilde{S}_{t-}^2\left(\frac{\sigma}{1-g'(\beta_{t-})}\right)^2\\
\widetilde{F}_t(\omega, dx) &= \alpha_1 \delta_{x_1(\omega,t)}(dx) + \alpha_2 \delta_{x_2(\omega,t)}(dx),
\end{aligned}$$

*with*

$$x_1(\omega,t) := \widetilde{S}_{t-}(\omega)(e^{-C_{t-}(\omega)} - 1), \quad x_2(\omega,t) := \widetilde{S}_{t-}(\omega)(e^{\overline{C}-C_{t-}(\omega)} - 1).$$

*Proof* With the definition of $a(\cdot)$ and $b(\cdot)$ in Lemma 2 and ODE (3.10) we get

$$\begin{aligned}
g'(\beta_{t-})a(\beta_{t-}) = &-\frac{\sigma^2}{2}\frac{g'(\beta_{t-})}{(1-g'(\beta_{t-}))^2} + g'(\beta_{t-})\delta(1+e^{\beta_{t-}}) - g'(\beta_{t-})B(\beta_{t-}, g(\beta_{t-}))\\
&+ \frac{\sigma^2}{1+e^{-\beta_{t-}}}\frac{g'(\beta_{t-})}{(1-g'(\beta_{t-}))^2},\\
\frac{1}{2}g''(\beta_{t-})b(\beta_{t-})^2 = &-B(\beta_{t-}, g(\beta_{t-}))(1-g'(\beta_{t-})) - \mu + \frac{\sigma^2}{1+e^{-\beta_{t-}}}\frac{1}{1-g'(\beta_{t-})}\\
&-\frac{\sigma^2}{2}\frac{g'(\beta_{t-})}{1-g'(\beta_{t-})} - g'(\beta_{t-})\delta(1+e^{\beta_{t-}}),\\
\frac{1}{2}\left(g'(\beta_{t-})b(\beta_{t-})\right)^2 = &\ \frac{\sigma^2}{2}\left(\frac{g'(\beta_{t-})}{1-g'(\beta_{t-})}\right)^2,\\
\sigma g'(\beta_{t-})b(\beta_{t-})) = &\ \sigma^2\frac{g'(\beta_{t-})}{1-g'(\beta_{t-})}.
\end{aligned}$$

The result now follows from Lemma 3.

**Proposition 3** *Given initial endowment $(\eta^0, \eta^1)$, let $\beta_0$ be defined by*

$$\beta_0 := \begin{cases} \underline{\beta} & \text{if} \quad \frac{\eta^1 \overline{s}}{\eta^0 + \eta^1 \overline{s}} < \frac{1}{1+e^{-\underline{\beta}}}, \qquad (\overline{s} := \overline{S}_0)\\ \overline{\beta} & \text{if} \quad \frac{\eta^1 \underline{s}}{\eta^0 + \eta^1 \underline{s}} > \frac{1}{1+e^{-\overline{\beta}}}, \end{cases}$$

*or else, let $\beta_0$ be the solution of*

$$\frac{\eta^1 e^{g(y)}\underline{s}}{\eta^0 + \eta^1 e^{g(y)}\underline{s}} = \frac{1}{1+e^{-y}}.$$

*Given the reflected jump-diffusion $\beta$ starting in $\beta_0$ as is Lemma 2 and the resulting $\widetilde{S}$ of Lemma 3 let*

$$\begin{aligned}
\widetilde{V}_t &:= (\eta^0 + \eta^1 \widetilde{S}_0)\mathscr{E}\left(\int_0^{\cdot} \frac{1}{(1+e^{-\beta_{s-}})\widetilde{S}_{s-}} d\widetilde{S}_s - \int_0^{\cdot} \delta ds\right)_t, \qquad t \geq 0,\\
c_t &:= \delta \widetilde{V}_t, \qquad t \geq 0,\\
\varphi_t^1 &:= \frac{1}{(1+e^{-\beta_{t-}})\widetilde{S}_{t-}}\widetilde{V}_{t-}, \qquad \varphi_t^0 := \widetilde{V}_{t-} - \varphi_t^1 \widetilde{S}_{t-}, \qquad t > 0,
\end{aligned}$$

*and let $\varphi_0^0 := \eta^0$ and $\varphi_0^1 := \eta^1$. Then $\widetilde{V}_t = \eta^0 + \eta^1 \widetilde{S}_0 + \int_0^t \varphi_s^1 d\widetilde{S}_s - \int_0^t c_s ds$ and $(\varphi^0, \varphi^1, c)$ is an optimal strategy for initial endowment $(\eta^0, \eta^1)$ in the frictionless market with price process $\widetilde{S}$.*

*Proof* Given the semimartingale characteristics in Lemma 4 we need to check that $H_t := \frac{1}{(1+e^{-\beta_{t-}})\widetilde{S}_{t-}}$ solves the optimality equation of Goll and Kallsen ([4], Theorem 3.1), i.e. that $(P \otimes I)$-a.e.

$$\widetilde{b}_t - \widetilde{c}_t H_t + \int \left( \frac{x}{1+H_t x} - x \right) \widetilde{F}_t(dx) = 0$$

holds. Of course the choice of $H_0$ is irrelevant for optimality.

Moreover, note that for $t > 0$ the term $-\widetilde{S}_{t-} B(\beta_{t-}, g(\beta_{t-})) + \int x \widetilde{F}_t(dx)$ in $\widetilde{b}_t$ and the integral term in the optimality equation cancel each other. The key to seeing this is

$$\begin{aligned}
\int \left( \frac{x}{1+H_t x} \right) \widetilde{F}_t(dx) &= \int \left( \frac{x}{1+H_t x} \right) \alpha_1 \delta_{x_1}(dx) + \int \left( \frac{x}{1+H_t x} \right) \alpha_2 \delta_{x_2}(dx) \\
&= \frac{\alpha_1 \widetilde{S}_{t-} \left( e^{-g(\beta_{t-})} - 1 \right)}{1 + \frac{\widetilde{S}_{t-} \left( e^{-g(\beta_{t-})} - 1 \right)}{\left(1+e^{-\beta_{t-}}\right)\widetilde{S}_{t-}}} + \frac{\alpha_2 \widetilde{S}_{t-} \left( e^{\overline{C} - g(\beta_{t-})} - 1 \right)}{1 + \frac{\widetilde{S}_{t-} \left( e^{\overline{C} - g(\beta_{t-})} - 1 \right)}{\left(1+e^{-\beta_{t-}}\right)\widetilde{S}_{t-}}} \\
&= \alpha_1 \widetilde{S}_{t-} \left( e^{-g(\beta_{t-})} - 1 \right) \left( \frac{1}{1 + \frac{\left(e^{-g(\beta_{t-})} - 1\right)}{1+e^{-\beta_{t-}}}} \right) \\
&\quad + \alpha_2 \widetilde{S}_{t-} \left( e^{\overline{C} - g(\beta_{t-})} - 1 \right) \left( \frac{1}{1 + \frac{\left(e^{\overline{C} - g(\beta_{t-})} - 1\right)}{1+e^{-\beta_{t-}}}} \right) \\
&= \widetilde{S}_{t-} B(\beta_{t-}, g(\beta_{t-})),
\end{aligned}$$

where the second equality follows from the definition of $x_1$ and $x_2$ (in Lemma 4) and the definition of $H$. Thus the specified strategy is optimal in the frictionless market.

**Lemma 5** *There exist two deterministic functions* $F^1 : [\underline{\beta}, \overline{\beta}] \to [0, \infty)$ *and* $F^2 : [\underline{\beta}, \overline{\beta}] \to (-\infty, 0]$ *such that for* $t > 0$

$$\varphi_t^1 - \varphi_0^1 = \int_0^t \frac{\varphi_s^1 e^{-\beta_{s-}}}{1+e^{-\beta_{s-}}} d\Psi_s + \sum_{0<s<t} \varphi_s^1 (e^{F^1(\beta_-)} - 1) \Delta N_s^1 + \sum_{0<s<t} \varphi_s^1 (e^{F^2(\beta_-)} - 1) \Delta N_s^2. \quad (5.3)$$

*Remark 3* As we will see in the proof of Theorem 1, Lemma 5 can be interpreted in the following way. The first summand on the right-hand side of (5.3) tells us that market orders are only used when the proportion invested in the risky asset is at the boundary. The last two summands imply that the sizes of the limit orders divided by the current holdings in the stock are deterministic functions of the current fraction of wealth invested in the stock (in terms of the shadow price).

*Proof* By Proposition 3 $\varphi^1$ is càglàd. Therefore, it is sufficient to show that (5.3) holds for the right-continuous versions of the processes on both sides of the equation.

After taking the logarithm of $\varphi_+^1$ we can write its dynamics as

$$d \log \varphi_{t+}^1 = d \log \widetilde{V}_t - d \log \widetilde{S}_t - d \log(1 + e^{-\beta_t}).$$

By Itô's formula and Proposition 3 we have that

$$
\begin{aligned}
d\log\widetilde{V}_t = \; & \frac{1}{(1+e^{-\beta_{t-}})\widetilde{S}_{t-}} d\widetilde{S}_t - \delta dt - \frac{1}{2}\left(\frac{1}{(1+e^{-\beta_{t-}})\widetilde{S}_{t-}}\right)^2 d[\widetilde{S},\widetilde{S}]^c_t \\
& + \log\left(1+\frac{1}{(1+e^{-\beta_{t-}})\widetilde{S}_{t-}}\Delta\widetilde{S}_t\right) - \frac{1}{(1+e^{-\beta_{t-}})\widetilde{S}_{t-}}\Delta\widetilde{S}_t \\
= \; & \bigg[\frac{1}{1+e^{-\beta_{t-}}}\Big(g'(\beta_{t-})a(\beta_{t-}) + \frac{1}{2}g''(\beta_{t-})b(\beta_{t-})^2 + \frac{1}{2}(g'(\beta_{t-})b(\beta_{t-}))^2 \\
& + \mu + \sigma g'(\beta_{t-})b(\beta_{t-})\Big) - \delta - \frac{1}{2}\frac{(g'(\beta_{t-})b(\beta_{t-})+\sigma)^2}{(1+e^{-\beta_{t-}})^2}\bigg] dt \\
& + \frac{g'(\beta_{t-})b(\beta_{t-})+\sigma}{1+e^{-\beta_{t-}}} dW_t \\
& + \log\left(1+\frac{\exp\{-g(\beta_{t-})\Delta N^1_t + (\overline{C}-g(\beta_{t-}))\Delta N^2_t\}-1}{1+e^{-\beta_{t-}}}\right).
\end{aligned}
\tag{5.4}
$$

Because $\widetilde{S}$ is defined as $\underline{S}\exp(C)$ we get

$$
\begin{aligned}
-d\log\widetilde{S}_t = \; & \left(\frac{\sigma^2}{2}-\mu\right)dt - \sigma dW_t - dC_t \\
= \; & \left(\frac{\sigma^2}{2}-\mu - g'(\beta_{t-})a(\beta_{t-}) - \frac{1}{2}g''(\beta_{t-})b(\beta_{t-})^2\right)dt \\
& - \left(g'(\beta_{t-})b(\beta_{t-})+\sigma\right)dW_t \\
& + g(\beta_{t-})\Delta N^1_t - \left(\overline{C}-g(\beta_{t-})\right)\Delta N^2_t.
\end{aligned}
$$

Using the properties of $\beta$ from Lemma 2, another application of Itô's formula yields

$$
\begin{aligned}
-d\log(1+e^{-\beta_t}) = \; & \frac{e^{-\beta_{t-}}}{1+e^{-\beta_{t-}}}d\beta_t - \frac{1}{2}\frac{e^{-\beta_{t-}}}{(1+e^{-\beta_{t-}})^2}d[\beta,\beta]^C_t \\
& - \left(\log(1+e^{-\beta_t}) - \log(1+e^{-\beta_{t-}})\right) - \frac{e^{-\beta_{t-}}}{1+e^{-\beta_{t-}}}\Delta\beta_t \\
= \; & \frac{e^{-\beta_{t-}}}{1+e^{-\beta_{t-}}}\left(a(\beta_{t-}) - \frac{1}{2}\frac{e^{-\beta_{t-}}}{(1+e^{-\beta_{t-}})^2}b(\beta_{t-})^2\right)dt \\
& + \frac{e^{-\beta_{t-}}}{1+e^{-\beta_{t-}}}b(\beta_{t-})dW_t \\
& + \frac{e^{-\beta_{t-}}}{1+e^{-\beta_{t-}}}\left(d\overline{\Psi}_t - d\underline{\Psi}_t\right) \\
& - \left(\log(1+e^{-\overline{\beta}}) - \log(1+e^{-\beta_{t-}})\right)\Delta N^1_t \\
& - \left(\log(1+e^{-\underline{\beta}}) - \log(1+e^{-\beta_{t-}})\right)\Delta N^2_t.
\end{aligned}
$$

Plugging in ODE (3.10) for $g''$ and summing up we see that all $dt$-terms and all $dW$-terms of the process $\log\varphi^1_+$ cancel out. Define

$$
\begin{aligned}
F^1(x) &:= \log\left(1+\frac{\exp\{-g(x)\}-1}{1+e^{-x}}\right) + g(x) - \log\left(\frac{1+e^{-\overline{\beta}}}{1+e^{-x}}\right), \\
F^2(x) &:= \log\left(1+\frac{\exp\{(\overline{C}-g(x))\}-1}{1+e^{-x}}\right) - \left(\overline{C}-g(x)\right) - \log\left(\frac{1+e^{-\underline{\beta}}}{1+e^{-x}}\right).
\end{aligned}
\tag{5.5}
$$

Itô's formula applied to the semimartingale $\log(\varphi^1_+)$ and the $C^2$-function $x \mapsto \exp(x)$ shows that (5.3) holds for the right-continuous versions. To finish the proof note that $F^1(x) \geq 0$ for all $x \in [\underline{\beta}, \overline{\beta}]$ follows from $g \geq 0$. $F^2(x) \leq 0$ for all $x \in [\underline{\beta}, \overline{\beta}]$ follows analogously, now making use of $\overline{C} - g \geq 0$.

**Theorem 1** *$\widetilde{S}$ is a shadow price process. An optimal strategy $\mathfrak{S}$ in the limit order market is given by*

$$M^B_t = 1_{\{t>0\}} \left( \frac{\eta^0 + \eta^1 \underline{s}(1+\lambda)}{\left(1+\exp(-\underline{\beta})\right) \underline{s}(1+\lambda)} - \eta^1 \right)^+ + \int_0^t 1_{\{\beta_- = \underline{\beta}\}} \frac{\varphi^1 e^{-\underline{\beta}}}{1+e^{-\underline{\beta}}} d\Psi,$$

$$M^S_t = 1_{\{t>0\}} \left( \frac{\eta^0 + \eta^1 \underline{s}}{\left(1+\exp(-\overline{\beta})\right) \underline{s}} - \eta^1 \right)^- - \int_0^t 1_{\{\beta_- = \overline{\beta}\}} \frac{\varphi^1 e^{-\overline{\beta}}}{1+e^{-\overline{\beta}}} d\Psi,$$

$$L^B_t = \varphi^1_t (e^{F^1(\beta_{t-})} - 1), \quad L^S_t = -\varphi^1_t (e^{F^2(\beta_{t-})} - 1),$$

*and $c_t = \delta \widetilde{V}_t$, where $F^1$, $F^2$ are defined in (5.5) and $\underline{s} = \underline{S}_0$. The strategy yields finite expected utility.*

*Remark 4* Theorem 1 can be interpreted as follows. $M^B$ is the minimal amount of risky assets the investor has to buy by market orders to prevent that the fraction of wealth invested in the risky asset leaves the acceptable interval at the lower boundary (the first summand of $M^B$ put the fraction on the lower boundary if it starts below the interval at time zero). Analogously, $M^S$ is the minimal amount of risky assets the investor has to sell by market orders to prevent that the fraction of wealth invested in the risky asset leaves the interval at the upper boundary. Mathematically these minimal trades correspond to the local time of the two dimensional wealth process at the boundaries of the cone illustrated in Fig. 4.

The choice of $L^B$ (resp. $L^S$) ensures that after a successful execution of the limit buy order (resp. the limit sell order) the fraction of wealth invested in the risky asset jumps on the upper boundary (resp. the lower boundary) of the interval. As $L^B > 0$ and $L^S > 0$ apart from the time at which the wealth process is at the boundary (which has Lebesgue measure zero) the investor is always willing to trade both with limit buy and with limit sell orders. However, the order sizes depend on how far away the wealth process is from the boundaries and they have to be adjusted continuously with the movements of the process $(\beta_t)_{t \geq 0}$.

*Remark 5* An important detail in model (2.1) is that a limit order has to be in the book already at $\Delta N^i = 1$ to be executed against the arising market order. This market mechanism is reflected in the condition that the limit order sizes $L^B$ and $L^S$ have to be predictable. By contrast, in the frictionless market with price process $\widetilde{S}$ the buying decision at a time $\tau$ at which $\widetilde{S}_\tau = \underline{S}_\tau$, may depend on all new information available at time $\tau$ (Note that by the standard convention in frictionless market models a simple purchase at time $\tau$ only affects the simple trading strategy on $(\tau, \infty)$, i.e. the value of the strategy at $\tau$ itself is not affected. Thus the latter is no contradiction to the fact that the strategy in the frictionless market with price process $\widetilde{S}$ is predictable as well. See also the discussion after Definition 1). However, as the jumps of $\widetilde{S}$ always land on one of the two continuous processes $\underline{S}$ or $\overline{S}$, and limit orders are submitted contingent that they can be executed, it turns out that this subtle distinction does not matter.

*Proof of Theorem 1* By construction of $\widetilde{S}$ (2.5) is clearly satisfied. All we have to do is to construct an admissible strategy $\mathfrak{S} = (M^B, M^S, L^B, L^S, c)$ in the limit order market such that the associated portfolio process of $\mathfrak{S}$ as defined in (2.1) is equal to the optimal strategy in the frictionless market $(\varphi^0, \varphi^1, c)$ from Proposition 3.

By Lemma 5 $\varphi^1$ is of finite variation, hence we can write it as the difference of two increasing càglàd processes $Z^1$ and $Z^2$, i.e. $\varphi^1 = \eta^1 + Z^1 - Z^2$. Since the sum $\sum_{s<t} \Delta^+ Z^i_s$ clearly

converges, we can define the continuous component $(Z^i)^c_t := Z^i_t - \sum_{s<t} \Delta^+ Z^i_s$ of $Z^i$ for $i \in \{1,2\}$. Note that $(Z^i)^c$ indeed has continuous paths since $Z^i$ has càglàd paths.

Now let $M^B_t := \Delta^+ Z^1_0 1_{\{t>0\}} + (Z^1)^c_t$ and $M^S_t := \Delta^+ Z^2_0 1_{\{t>0\}} + (Z^2)^c_t$. Clearly, $M^B$ and $M^S$ are non-decreasing predictable processes. Again by Lemma 5 and by Remark 2 we have

$$\int_0^{\cdot} 1_{\{\widetilde{S}\neq\overline{S}\}} dM^B = \int_0^{\cdot} 1_{\{\widetilde{S}\neq\underline{S}\}} dM^S = 0. \tag{5.6}$$

Thus, we have $\int_0^{\cdot} \overline{S} dM^B = \int_0^{\cdot} \widetilde{S} dM^B$ and $\int_0^{\cdot} \underline{S} dM^S = \int_0^{\cdot} \widetilde{S} dM^S$. Furthermore, let $L^B_t := \varphi^1_t(e^{F^1(\beta_{t-})} - 1)$ and $L^S_t := -\varphi^1_t(e^{F^2(\beta_{t-})} - 1)$. $L^B$ and $L^S$ are predictable and by Lemma 5 we have $\Delta^+ Z^1_t = L^B_t \Delta N^1_t$ and $\Delta^+ Z^2_t = L^S_t \Delta N^2_t$ for $t > 0$. Therefore, this construction of $\mathfrak{S}$ satisfies

$$\varphi^1_t = \eta^1 + M^B_t - M^S_t + \int_0^{t-} L^B dN^1 - \int_0^{t-} L^S dN^2, \qquad \forall t \geq 0.$$

Define

$$\psi^0_t := \eta^0 - \int_0^t c_s ds - \int_0^t \overline{S}_s dM^B_s + \int_0^t \underline{S}_s dM^S_s$$
$$- \int_0^{t-} L^B_s \underline{S}_s dN^1_s + \int_0^{t-} L^S_s \overline{S}_s dN^2_s,$$

where $c$ is from Proposition 3. By (5.6), $\underline{S} = \widetilde{S}$ on $\Delta N^1 = 1$ resp. $\overline{S} = \widetilde{S}$ on $\Delta N^2 = 1$ and Lemma 1, we have that $(\psi^0, \varphi^1, c)$ is self-financing in the frictionless market. Thus, $\psi^0 = \varphi^0$ implying that $(\varphi^0, \varphi^1)$ is indeed the associated portfolio process of $\mathfrak{S}$. From their definitions in Proposition 3 it can be seen that $\varphi^1 > 0$ and $\varphi^0 > 0$. Thus $(\varphi^0, \varphi^1, c)$ is clearly admissible.

The last term in (5.4) consists of $dt$-, $dW_t$-, $dN^1_t$-, and $dN^2_t$-integrals with bounded integrands. Together with the Poisson-distribution of $N^1_t$ and $N^2_t$, the fact that $c_t$ is proportional to $\widetilde{V}_t$, and $\delta > 0$, this yields that the discounted logarithmic utility from consumption is integrable.

In Theorem 1 the optimal strategy in the limit order market is expressed in terms of the shadow price process resp. the wealth process based on the shadow price. In the following proposition we want to the characterize $M^B, M^S, L^B$, and $L^S$ by the fraction of wealth invested in the risky asset *based on the best-bid price* $\underline{S}$. This verifies our guess (3.2)-(3.4). The optimal consumption rate is still expressed in terms of the wealth process based on the shadow price. We consider a reflected SDE – similar to that in Lemma 2.

**Proposition 4** *Let $\beta' := \log\left((\varphi^1_+ \underline{S})/\varphi^0_+\right)$, where $(\varphi^0, \varphi^1)$ is the optimal strategy from Proposition 3. Define $\beta'_{\min} := \underline{\beta} - \log(1+\lambda)$ and $\beta'_{\max} := \overline{\beta}$. Assume that $\beta'_0 \in [\beta'_{\min}, \beta'_{\max}]$. Let*

$$c(y) := \mu - \frac{\sigma^2}{2} + \delta(1 + \exp(h(y))), \quad y \in [\beta'_{\min}, \beta'_{\max}], \tag{5.7}$$

*where $h : [\beta'_{\min}, \beta'_{\max}] \to [\underline{\beta}, \overline{\beta}]$ is the inverse of $\mathrm{Id} - g$ (the inverse exists as $g' \leq 0$). Let $\Psi$ be the local time from Lemma 2. Then, given $\beta'_0$, $(\beta', \Psi)$ is the unique solution to the following stochastic variational inequality*

*(i) $\beta'$ is càdlàg and takes values in $[\beta'_{\min}, \beta'_{\max}]$. $\Psi$ is continuous and of finite variation with starting value $\Psi_0 = 0$,*

*(ii)*

$$\beta'_t = \beta'_0 + \int_0^t c(\beta'_{s-}) ds + \sigma W_t + \sum_{s \leq t} \left( (\beta'_{\max} - \beta'_{s-}) \Delta N^1_s + (\beta'_{\min} - \beta'_{s-}) \Delta N^2_s \right) + \Psi_t,$$

*(iii) for every progressively measurable process $z$ which has càdlàg paths and takes values in $[\beta'_{\min}, \beta'_{\max}]$, we have*

$$\int_0^t \left(\beta'_s - z_s\right) d\Psi_s \leq 0, \qquad \forall t \geq 0.$$

*Remark 6* The function $h$ in (5.7) converts the process $\beta'$ which is based on the valuation of portfolio positions by $(1,\underline{S})$ into the process $\beta$ which is based on $(1,\widetilde{S})$. This conversion is needed as the optimal consumption rate is proportional to the wealth based on the shadow price.

*Proof of Proposition 4* At first note that by construction of the shadow price process

$$\left\{\beta_- = \underline{\beta}\right\} = \{\beta'_- = \beta'_{\min}\} \quad \text{and} \quad \left\{\beta_- = \overline{\beta}\right\} = \{\beta'_- = \beta'_{\max}\}.$$

Thus, $(P\otimes I)\left(\beta'_- \in \{\beta'_{\min},\beta'_{\max}\}\right) = 0$ (i.e. $dt$-terms and $dW_t$-terms acting solely on this set vanish). In addition, $(P\otimes N^i)\left(\beta'_- \in \{\beta'_{\min},\beta'_{\max}\}\right) = (P\otimes I)\left(\beta'_- \in \{\beta'_{\min},\beta'_{\max}\}\right) = 0$ for $i = 1,2$. By $\beta' = \log(\varphi^1)+\log(\underline{S})-\log(\varphi^0)$, this implies that

$$\int_0^t 1_{\{\beta'_- \in \{\beta'_{\min},\beta'_{\max}\}\}}\, d\beta' = \int_0^t 1_{\{\beta'_- \in \{\beta'_{\min},\beta'_{\max}\}\}}\, d\beta = \int_0^t 1_{\{\beta'_- \in \{\beta'_{\min},\beta'_{\max}\}\}}\, d\Psi,$$

where the latter equation follows by Lemma 2. As we have $\beta = \overline{\beta}$, $\underline{S} = \widetilde{S}$ on $\Delta N^1 = 1$ and $\beta = \underline{\beta}$, $\underline{S} = \frac{\widetilde{S}}{1+\lambda}$ on $\Delta N^2 = 1$, it follows from the definition of $\beta'$, $\beta'_{\min}$, and $\beta'_{\max}$ that

$$\beta' = \beta'_{\max} \quad \text{on} \quad \Delta N^1 = 1 \quad \text{and} \quad \beta' = \beta'_{\min} \quad \text{on} \quad \Delta N^2 = 1. \tag{5.8}$$

By (5.8) and Itô's formula we obtain

$$\begin{aligned}\int_0^t 1_{\{\beta'_{\min}<\beta'_-<\beta'_{\max}\}}\, d\beta' = &\int_0^t 1_{\{\beta'_{\min}<\beta'_-<\beta'_{\max}\}} a(\beta'_-) dI + \sigma \int_0^t 1_{\{\beta'_{\min}<\beta'_-<\beta'_{\max}\}}\, dW \\ &+ \sum_{s\le t}\left((\beta'_{\max}-\beta'_{s-})\Delta N^1_s + (\beta'_{\min}-\beta'_{s-})\Delta N^2_s\right).\end{aligned}$$

As $\beta'$ stays by construction in $[\beta'_{\min},\beta'_{\max}]$ we have that $(\beta',\Psi)$ is the solution of (i)-(iii).

## 6 An illustration of the optimal strategy

Let us fix parameters for the model such that the assumptions of Proposition 2 are satisfied:

$$\mu = 0.05, \qquad \sigma = 0.4, \qquad \lambda = 0.01, \qquad \alpha_1 = 1, \qquad \alpha_2 = 1, \qquad \delta = 0.1.$$

With these parameters specified, the free boundary problem consisting of (4.2) and (3.11) can be solved numerically. The approach used is based on the idea behind the proof of Proposition 2. It can be roughly described as follows. First a value $x$ for $\underline{\beta}$ is assumed, then a computer program for numerical calculations is used to solve the initial value problem consisting of (4.2) and the initial conditions $g(x) = \log(1+\lambda)$ and $g'(x) = 0$. Then the smallest $y > x$ with $g'(y) = 0$ is determined. Now if $g(y) < 0$ we choose a larger $x$ in the next iteration, if $g(y) > 0$ we choose a smaller $x$, and if $g(x) = 0$ the algorithm stops and we have found our boundary $\{\underline{\beta},\overline{\beta}\} = \{x,y\}$.

When the boundary $\{\underline{\beta},\overline{\beta}\}$ is now known, we can calculate the boundary for the fraction of wealth invested in the risky asset (here measured in the shadow price) by

$$\pi_{\min} = \frac{\exp(\underline{\beta})}{1+\exp(\underline{\beta})}, \qquad \pi_{\max} = \frac{\exp(\overline{\beta})}{1+\exp(\overline{\beta})}.$$

For our example this yields $\pi_{\min} = 0.206$ and $\pi_{\max} = 0.412$. In addition, in Table 1 we have calculated $\pi_{\min}$ and $\pi_{\max}$ for various values of $\alpha$ to illustrate the effects of a change in the arrival rate of exogenous market orders. We see that for small $\alpha$ $\pi_{\min}$ and $\pi_{\max}$ are close to the boundaries in the proportional transaction costs model.

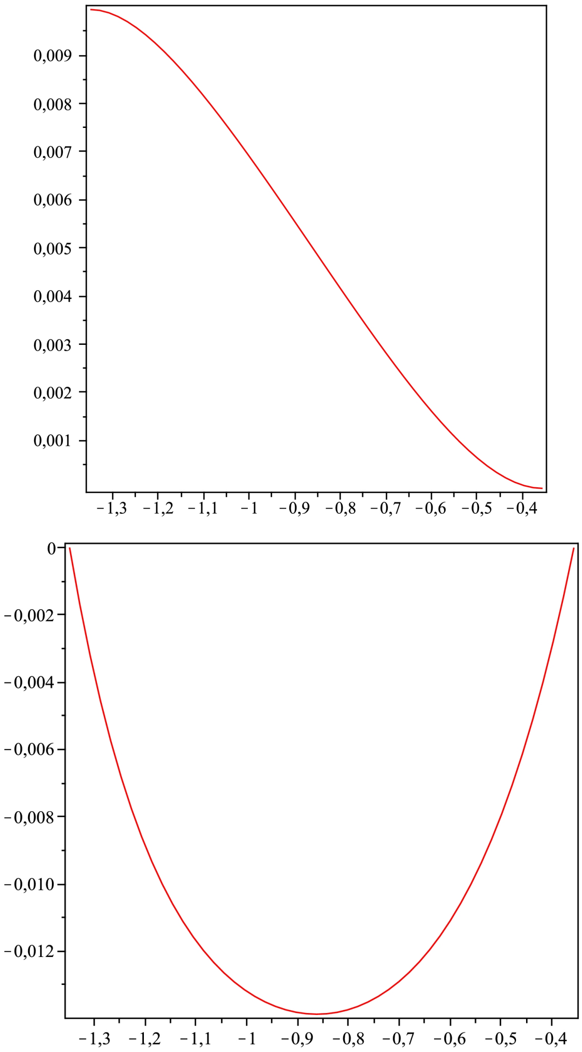


**Fig. 1** The function $C = g(\beta)$ and its derivative $g'(\beta)$

| $\alpha$ | $\pi_{\min}$ | $\pi_{\max}$ |
|---|---|---|
| 0 | 0.231 | 0.368 |
| 0.01 | 0.231 | 0.368 |
| 0.1 | 0.229 | 0.371 |
| 0.5 | 0.221 | 0.388 |
| 1 | 0.206 | 0.412 |
| 2 | 0.163 | 0.467 |
| 3 | 0.112 | 0.525 |
| 4 | 0.058 | 0.583 |

**Table 1** Optimal boundaries for different $\alpha$

The numerical solution to the free boundary problem can furthermore be used to simulate paths of various quantities. Fig. 2, Fig. 3, and Fig. 4 are the result of this procedure for the parameters given above and illustrate the structure of the solution.

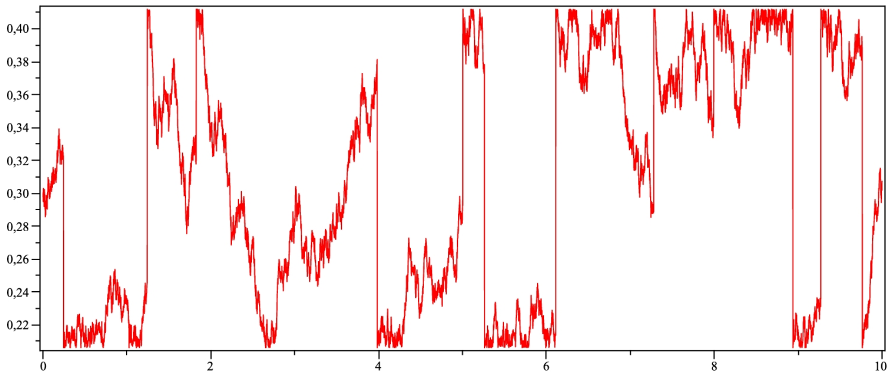


**Fig. 2** Optimal fraction $\widetilde{\pi}$ invested in stock (with local time at the boundaries)

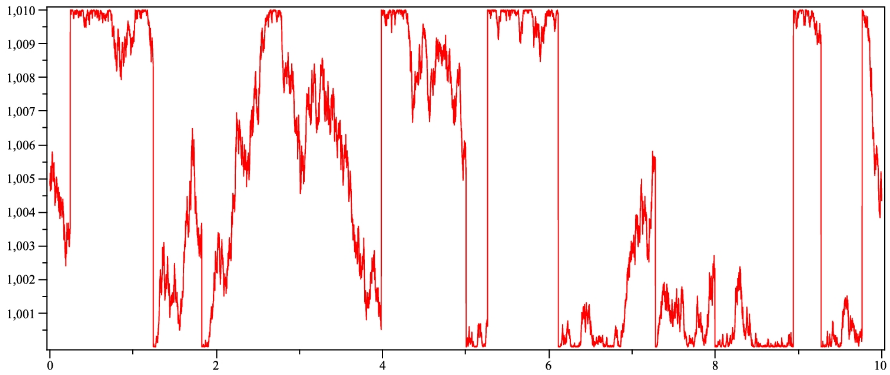


**Fig. 3** Shadow factor $\exp(C)$ (without local time)

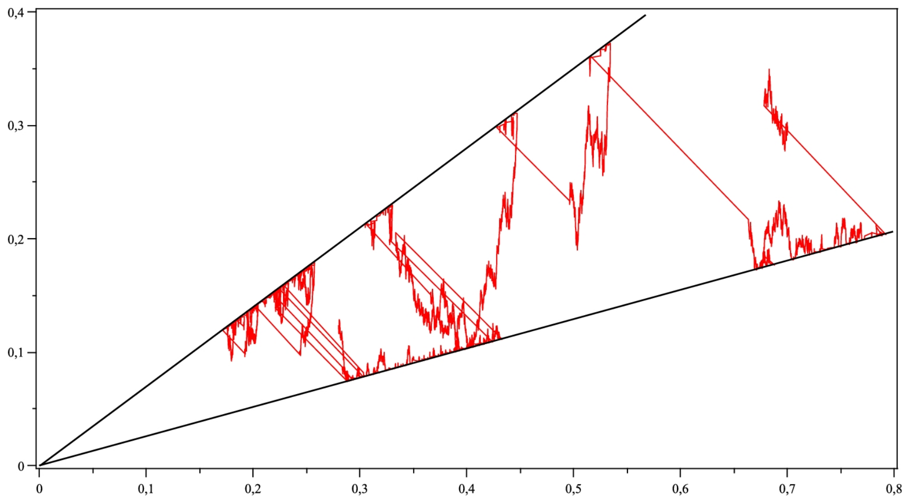


**Fig. 4** Wealth in bond $\varphi^0$, liquidation wealth in stock $\varphi^1 \underline{S}$

## 7 Conclusion

We introduced a simple, analytically tractable model for continuous-time trading in limit order markets. Although our mathematical results heavily rely on the quite idealized assumptions of the model, especially on the assumption that the considered investor is "small", i.e. his trades do not affect the dynamics of the order book, we think that in more complex situations the structure of the optimal strategy is still economically meaningful.

The investor tries to profit from the bid-ask spread by permanently holding both limit buy and limit sell orders in the book. After a successful execution of the limit buy order at the lower bid-price he holds a large stock position in his portfolio which is quite speculative. But, ideally he is able to liquidate the position quite shortly afterwards by the execution of the limit sell order at the higher ask-price. To limit the *inventory risk* he takes by this strategy the fraction of wealth he invests in the risky stock is always kept in a bounded interval (using market orders whenever the fraction is at the boundary of the interval). Thus the model carries the flavor of a market model with negative transaction costs, but which is arbitrage-free as favorable trades can only be realized at Poisson times.

Consider for example the case that the investor's limit orders are *not* small compared to the incoming market orders from other traders. Then, his wealth process does not always jump on the boundary of the cone (cf. Fig. 4), as incoming market orders may not be large enough to cover the full order size of his limit orders. But, still it seems to be worthwhile for the investor to place, say, limit buy orders as long as the fraction of wealth invested in stocks does not surpass a certain threshold. Under this scenario the threshold might be approached by several successive partial executions of these limit buy orders.

Furthermore, if the investor's *market* orders were not small enough to be filled by the orders placed at the best-bid resp. the best-ask price, such a large market order will eat into the book and is therefore executed against various limit orders with *different* limit prices at a *single* point in time. Hence, a shadow price can obviously not exist.

In this spirit we see the paper also as an impetus to solve more complicated portfolio optimization problems in continuous-time limit order markets (most probably in less explicit form).

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