Continuous time trading of a small investor in a limit order market

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Abstract

We provide a mathematical framework to model continuous time trading of a small investor in limit order markets. We show how elementary strategies can be extended in a suitable way to general continuous time strategies containing orders with infinitely many different limit prices. The general limit buy order strategies are predictable processes with values in the set of nonincreasing demand functions. It turns out that our strategy set of limit and market orders is closed, but limit orders can turn into market orders when passing to the limit, and any element can be approximated by a sequence of elementary strategies.

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1 Introduction

In today's electronic markets the predominant market structure is the limit order market (or continuous double auction) where traders can continuously place market and limit orders. A market order is executed immediately at the best currently available price whereas a limit order is stored in the book until it can be executed at its limit price. The limit orders can be chosen from a continuum of limit prices. By the enormous increase of trading speed and a reduction of immediate order execution costs, there appears a huge demand for sophisticated mathematical models of high-frequency trading that take the precise price formation mechanism into account and allow for the computation of optimal trading strategies. This article provides a mathematical background to model self-financing continuous time portfolio processes allowing for a continuum of limit prices of a "small" trader whose transactions have no impact on the order book dynamics. Under the assumption that the order sizes of the investor are small compared to the orders in the book, trading solely with market orders corresponds to models with proportional transaction costs. The small investor buys at the best-ask price and sells at the lower best-bid price. These models and their arbitrage theory are very well developed and we can apply some of these results. However, the modeling of limit order execution is more complex. The trader can submit limit orders at different prices and orders may be stored in the order book waiting for execution.

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The aim of this article is not to *explain* the evolution of the order book or the transaction price as e.g. in the models by Cont, Stoikov, and Talreja [4], Cvitanić and Kirilenko [5], Osterrieder [15], Luckock [13], Roşu [18]. We rather model the trading opportunities of *one* investor *given* the order book dynamics. In contrast to Alfonsi and Schied [1] and Predoiu, Shaikhet, and Shreve [16] among others who consider the price impact of market orders and the order book resilience, we assume that the trader under consideration is small. Models with both market and limit orders have already been considered in Guilbaud and Pham [9] and Kühn and Stroh [12] among others. But, in all these models only special limit order prices are permitted, especially the current best-bid price or one tick above it (for buy orders) and the best-ask price or one tick below it (for sell orders). As the best-bid and the best-ask may move continuously in time, it is interesting to investigate in a more general framework how these strategies can be approximated by strategies with piecewise constant limit prices (see Example 6.2). More importantly, in the model introduced in this article orders with arbitrary limit prices can be placed. A limit buy order with a higher limit price has a higher execution probability. In particular, the execution probability can be increased by placing orders in the inner of the bid-ask-spread. Thus, the model can be used as a framework to analyze the trade-off between the risks and the rewards connected with the placement of limit orders with different limit prices.

The article is organized as follows. In Subsection 1.1 we provide a motivation of the order execution mechanism behind our model. For the convenience of the reader we briefly introduce random measures in Section 2. They are needed to introduce the model formally. This and the statement of the main results, Theorem 3.13 and Theorem 3.17, are done in Section 3. In addition, in Subsection 3.4 we discuss the arbitrage theory of the model and Subsection 3.5 is about the special case of a finite price grid. The proofs of Theorem 3.13 and Theorem 3.17 can be found in Section 4 and Section 5. The article ends with two examples in Section 6 and a conclusion.

1.1 A motivation of the execution mechanism

The basic assumption is that the investor is small, only trades of other market participants, called exogenous orders in what follows, change the state of the order book, whereas the impact of the orders of the small investor is neglected. Being small also implies that there are no partial executions of his limit orders. A single limit order of the small investor with limit price L is either completely executed or not.

One building block of the model are the exogenous best-bid and best-ask price processes (not including the orders placed by the small trader). They are modeled by the càdlàg stochastic processes $\underline{S}$ and $\overline{S}$ with $\underline{S} < \overline{S}$. Market buy orders are immediately executed at $\overline{S}$ and market sell orders at $\underline{S}$. Let t be the point in time at which the exogenous order arrives. $\overline{S}$ and $\underline{S}$ are càdlàg. Thus $\underline{S}_t$ and $\overline{S}_t$ are interpreted as the prices immediately after the order execution or cancelation at time t and $\underline{S}_{t-}$ and $\overline{S}_{t-}$ are the prices immediately before this event. Let us discuss some typical "events" driven by the actions of the exogenous market participants and their effects on the small investor to get an idea of what our model should cover.

(i) Market buy order arrives: The best-bid price is certainly unchanged, but the best-ask price may or may not jump upwards, depending on whether the market buy order eats into the book or not, i.e. $\underline{S}_t = \underline{S}_{t-}$ and $\overline{S}_t \geq \overline{S}_{t-}$. In addition, all limit sell orders with limit price smaller (or equal) some x with $x \in [\underline{S}_t, \overline{S}_t]$ are executed.

(ii) Market sell order arrives: The best-ask price is certainly unchanged, but the best-bid price may or may not jump downwards, depending on whether the market sell order eats into the book or not, i.e. $\overline{S}_t = \overline{S}_{t-}$ and $\underline{S}_t \leq \underline{S}_{t-}$. In addition, all limit buy orders with limit price higher (or equal) some x with $x \in [\underline{S}_t, \overline{S}_t]$ are executed.

(iii) Limit buy order with limit price L arrives:

(a) $L \leq \underline{S}_{t-}$. Nothing changes, i.e. $\underline{S}_t = \underline{S}_{t-}$ and $\overline{S}_t = \overline{S}_{t-}$.

(b) $\underline{S}_{t-} < L < \overline{S}_{t-}$. The best-bid price increases to L, while the best-ask price does not change, i.e. $\underline{S}_t = L$ and $\overline{S}_t = \overline{S}_{t-}$. In addition, all limit sell orders of the small trader with limit price smaller or equal L are executed. Note that $\overline{S}$ is the best-ask price without the small trader's orders.

(c) $L \geq \overline{S}_{t-}$. The same impact as in (i).

(iv) Limit sell order with limit price L arrives:

(a) $L \geq \overline{S}_{t-}$. Nothing changes, i.e. $\underline{S}_t = \underline{S}_{t-}$ and $\overline{S}_t = \overline{S}_{t-}$.

(b) $\underline{S}_{t-} < L < \overline{S}_{t-}$. The best-ask price decreases to L, while the best-bid price does not change, i.e. $\overline{S}_t = L$ and $\underline{S}_t = \underline{S}_{t-}$. In addition, all limit buy orders of the small trader with limit price higher or equal L are executed.

(c) $L \leq \underline{S}_{t-}$. The same impact as in (ii).

(v) Limit buy order is canceled: The best-ask price does not change, but depending on whether the canceled limit order is the only one at the best-bid price, the best-bid price may move downwards, i.e. $\overline{S}_t = \overline{S}_{t-}$ and $\underline{S}_t \leq \underline{S}_{t-}$.

(vi) Limit sell order is canceled: The best-bid price does not change, but depending on whether the canceled limit order is the only one at the best-ask price, the best-ask price may move upwards, i.e. $\underline{S}_t = \underline{S}_{t-}$ and $\overline{S}_t \geq \overline{S}_{t-}$.

It is important to note that the execution mechanism is not determined solely from the best-bid price and the best-ask price processes. Namely, a downward jump of the best-bid price from $\underline{S}_{t-}$ to $\underline{S}_t$ may or may not execute a limit buy order of the small investor with limit price $\underline{S}_t < L < \underline{S}_{t-}$. Essentially, this depends whether the downward jump is triggered by a large exogenous market sell order eating into the book (as in (ii)) or by a cancelation of a limit buy order in the book (as in (v)). Therefore we introduce two integer-valued random measures that model the execution of the limit orders of the small investor explicitly. They have to be in line with the processes $\underline{S}$ and $\overline{S}$, but they cannot be derived from them. This is in contrast to the models of Smid [20] where limit buy (sell) orders are only executed when the best-ask (bid) process hits the limit price. In the model considered in Osterrieder [15] the execution of limit orders is triggered by an exogenous transaction price process.

2 Notation

Throughout the article we fix a terminal time $T \in \mathbb{R}_+$ and a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ satisfying the usual conditions. Denote by $\mathcal{O}$ (resp. by $\mathcal{P}$) the optional σ-algebra (resp. the predictable σ-algebra) on $\Omega \times [0, T]$.

2.1 Random measures

Most of the following definitions are from Chapter XI in [10]. As they are the building blocks for our model we quote them here rather completely for the convenience of the reader.

Definition 2.1. *Define*

$$\begin{aligned} (\widetilde{\Omega}, \widetilde{\mathcal{F}}) &:= (\Omega \times [0,T] \times \mathbb{R}_+, \mathcal{F} \otimes \mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}_+)), \\ \widetilde{\mathcal{O}} &:= \mathcal{O} \otimes \mathcal{B}(\mathbb{R}_+), \qquad \widetilde{\mathcal{P}} := \mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+). \end{aligned}$$

We call $\widetilde{\mathcal{O}}$ the optional σ-field *in $\widetilde{\Omega}$ and $\widetilde{\mathcal{P}}$ the* predictable σ-field *in $\widetilde{\Omega}$.*

Definition 2.2. *An extended real function μ defined on $\Omega \times \mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}_+)$ is called a* random measure *if*

(i) $\mu(\omega, \cdot)$ is a σ-finite measure on $\mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}_+)$ for all $\omega \in \Omega$ and

(ii) $\mu(\cdot, \hat{B})$ is a random variable on $(\Omega, \mathcal{F})$ for all $\hat{B} \in \mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}_+)$.

Definition 2.3. *For any $\widetilde{B} \in \widetilde{\mathcal{F}}$ define*

$$M_\mu(\widetilde{B}) := E\left[\int_{[0,T]\times\mathbb{R}_+} 1_{\widetilde{B}}(\omega, t, x)\mu(\omega, dt, dx)\right].$$

For $A \in \mathcal{F} \otimes \mathcal{B}([0,T])$ define $\widehat{M_\mu}(A) := M_\mu(A \times \mathbb{R}_+)$.

Note that M_μ is a measure on $\widetilde{\mathcal{F}}$ and is called the *measure generated by μ*. μ is said to be *integrable* if M_μ is a finite measure, i.e. $M_\mu(\widetilde{\Omega}) < \infty$. μ is said to be *optionally* (resp. *predictably*) *σ-integrable*, if the restriction of M_μ on $\widetilde{\mathcal{O}}$ (resp. $\widetilde{\mathcal{P}}$) is a σ-finite measure.

Definition 2.4. *A $\widetilde{\mathcal{F}}/\mathcal{B}(\overline{\mathbb{R}})$-measurable function H that satisfies*

$$\int_{[0,t]\times\mathbb{R}_+} |H(\omega, s, x)|\mu(\omega, ds, dx) < \infty, \qquad \forall t \in [0,T]$$

and all $\omega \in M$ for a set $M \in \mathcal{F}$ with $P(M) = 1$ is called μ-integrable.

For a *μ-integrable* H the integral $\int_{[0,\cdot]\times\mathbb{R}_+} H(\omega, s, x)\mu(\omega, ds, dx)$ is well-defined up to evanescence.

A random measure μ is said to be *optional* (resp. *predictable*), if for any $\widetilde{\mathcal{O}}$-measurable function H (resp. $\widetilde{\mathcal{P}}$-measurable function H) s.t. $\int_{[0,\cdot]\times\mathbb{R}_+} H(\cdot, s, x)\mu(ds, dx)$ exists, $\int_{[0,\cdot]\times\mathbb{R}_+} H(\cdot, s, x)\mu(ds, dx)$ is an optional (resp. predictable) process.

Definition 2.5. *A random measure μ is called an* integer-valued random measure *if*

(i) μ takes only values in $\mathbb{N}_0 \cup \{\infty\}$,

(ii) $\mu(\omega, \{t\} \times \mathbb{R}_+) \leq 1$ for all $\omega \in \Omega$, $t \geq 0$, and

(iii) μ is optional and optionally σ-integrable.

3 A model of a small investor trading in a limit order book

3.1 Description of the model

Let $\underline{S}$ and $\overline{S}$ be two adapted càdlàg processes s.t. $0 \leq \underline{S}_t(\omega) < \overline{S}_t(\omega)$ for all $(\omega, t) \in \Omega \times [0, T]$. Denote $\Delta\underline{S}_t := \underline{S}_t - \underline{S}_{t-}$ and $\Delta\overline{S}_t := \overline{S}_t - \overline{S}_{t-}$. One may interpret $\underline{S}$ as the best-bid price and $\overline{S}$ as the best-ask price without the orders of the small investor. Let μ, ν be two integer-valued random measures. The random measure μ models when and up to which price the limit buy orders of the small trader are executed. The random measure ν models when and up to which price the limit sell orders of the small trader are executed. Throughout the article let the following assumption hold.

Assumption 3.1. *(i) For all $\omega \in \Omega$ it holds that*

$$\mu(\omega, \{(t, x) \in [0, T] \times \mathbb{R}_+ \mid x < \underline{S}_t(\omega) \text{ or } x > \overline{S}_t(\omega)\}) = 0$$
$$\nu(\omega, \{(t, x) \in [0, T] \times \mathbb{R}_+ \mid x < \underline{S}_t(\omega) \text{ or } x > \overline{S}_t(\omega)\}) = 0.$$

(ii) For all $(\omega, t) \in \Omega \times [0, T]$ it holds that

$$\Delta\overline{S}_t(\omega) < 0 \quad \Rightarrow \quad \exists x \in [\underline{S}_t(\omega), \overline{S}_t(\omega)] \text{ with } \mu(\omega, \{t\} \times \{x\}) = 1,$$
$$\Delta\underline{S}_t(\omega) > 0 \quad \Rightarrow \quad \exists x \in [\underline{S}_t(\omega), \overline{S}_t(\omega)] \text{ with } \nu(\omega, \{t\} \times \{x\}) = 1.$$

(iii) For all $\omega \in \Omega$ we have that

$$\mu(\omega, \{(t, x) \in [0, T] \times \mathbb{R}_+ \mid x < \overline{S}_t(\omega)\}) \quad < \quad \infty$$
$$\nu(\omega, \{(t, x) \in [0, T] \times \mathbb{R}_+ \mid x > \underline{S}_t(\omega)\}) \quad < \quad \infty.$$

(iv) For all $(\omega, t) \in \Omega \times [0, T]$ we have that

$$\mu(\omega, \{t\} \times \{\overline{S}_t(\omega)\}) = 1 \quad \Rightarrow \quad \Delta\overline{S}_t(\omega) < 0$$
$$\nu(\omega, \{t\} \times \{\underline{S}_t(\omega)\}) = 1 \quad \Rightarrow \quad \Delta\underline{S}_t(\omega) > 0.$$

(v) There does not exist a pair $(\omega, t) \in \Omega \times [0, T]$ with

$$\mu(\omega, \{t\} \times [0, \overline{S}_t(\omega))) = 1 \qquad \text{and} \qquad \nu(\omega, \{t\} \times (\underline{S}_t(\omega), \infty)) = 1.$$

(vi) For all $\omega \in \Omega$ we have that

$$\mu(\omega, \{0, T\} \times \mathbb{R}_+) = \nu(\omega, \{0, T\} \times \mathbb{R}_+) = 0.$$

Remark 3.2. *For any càdlàg processes $\underline{S}$ and $\overline{S}$ with $\underline{S} < \overline{S}$ there exist random measures μ and ν satisfying Assumption 3.1. Thus, the assumptions are no restriction on the best-bid and the best-ask price process.*

Let us discuss Assumption 3.1. (i) and (ii) are justified by the considerations in Subsection 1.1. As $\underline{S}_t$ stands for the highest remaining exogenous limit buy order in the book, clearly no limit buy order of the small investor with a limit price strictly below $\underline{S}_t$ can be executed at time t. Similarly it would not make sense that a limit buy order of the small investor with a limit price strictly higher than $\overline{S}_t$ persists, because $\overline{S}_t$ denotes the lowest limit price of outstanding exogenous limit sell orders. The first part of Assumption 3.1(ii) means that a downward jump of the best-ask entails that at least all limit buy orders of the small investor with limit prices larger or equal the best-ask after the jump are executed.

Assumption 3.1(iii) says that there are only finitely many executions of limit orders of the small investor up to time T leading to a better trade than using market orders. This assumption is made as in reasonable models with continual execution of limit orders at favorable prices the small investor could make riskless gains by placing simultaneously a limit buy order close to $\underline{S}$ and a limit sell order close to $\overline{S}$. Note however that there can be countably many executions of limit buy orders by (small) downward jumps of $\overline{S}$ (this is the reason why we do not restrict to finite random measures). These executions do not lead to arbitrage as the buyer has to pay at least the new best-ask price which is the price he has to pay when using a market order (cf. also (iv)). Condition (v) is needed to exclude simultaneous limit buy and sell order executions at similar prices which could cancel each other out and thus they would possibly not enter in the portfolio process. Assumption 3.1 (vi) is made w.l.o.g. and only to keep the notation simpler. It is in the same vein as Assumption 2.2. in [3] and could be discarded by starting the model at time -1, finishing it at $T+1$, and demanding that on $[-1,0) \cup (T, T+1]$ nothing happens (compare Remark 4.2 in [3]).

Now we define the set of general continuous time strategies and the self-financing condition for the small trader.

Definition 3.3. *Denote by $\mathcal{L}^B$ the set of all $\widetilde{\mathcal{P}}/\mathcal{B}(\overline{\mathbb{R}}_+)$-measurable functions $L^B : \widetilde{\Omega} \to \overline{\mathbb{R}}_+$, which satisfy*

(i) $x \mapsto L^B(\omega, t, x)$ is nonincreasing, for all $(\omega, t) \in \Omega \times [0, T]$,

(ii) $L^B(\omega, t, x) = 0$ for all $(\omega, t) \in \Omega \times [0, T]$ and $x \geq \overline{S}_{t-}(\omega)$,

(iii) L^B is μ-integrable.

Similarly, let $\mathcal{L}^S$ be the set of all $\widetilde{\mathcal{P}}/\mathcal{B}(\overline{\mathbb{R}}_+)$-measurable functions $L^S : \widetilde{\Omega} \to \overline{\mathbb{R}}_+$, which satisfy

(iv) $x \mapsto L^S(\omega, t, x)$ is nondecreasing, for all $(\omega, t) \in \Omega \times [0, T]$,

(v) $L^S(\omega, t, x) = 0$ for all $(\omega, t) \in \Omega \times [0, T]$ and $x \leq \underline{S}_{t-}(\omega)$,

(vi) L^S is ν-integrable.

$L^B(\omega, t, x)$ is the sum of outstanding limit buy orders of the small investor with limit price x or higher, which could possibly be executed at time t. The orders are placed (resp. updated) with the information $\mathcal{F}_{t-}$, i.e. in general without the knowledge of the order flow at time t. This reflects the fact that a limit order has to be in the book in advance before it can be executed by a market order. Condition (i) is self-explanatory. A limit buy order of the small trader placed at $\overline{S}_-$ or above would be executed immediately at $\overline{S}$, hence such an order would in effect be a market order. Thus condition (ii) separates limit from market orders and is no restriction, see Subsection 3.2 for the relation to real-world strategies.

Definition 3.4. *Let M^B, M^S be two real-valued predictable nondecreasing processes with $M_0^B = M_0^S = 0$ and let $L^B \in \mathcal{L}^B$ and let $L^S \in \mathcal{L}^S$. We call a quadruple $\mathfrak{S} = (M^B, M^S, L^B, L^S)$ a* trading strategy.

M_t^B (resp. M_t^S) is interpreted as the *accumulated* purchases (resp. sells) by market orders up to time t.

At several places in the article we have to define integrals w.r.t. processes of finite variation which are neither left- nor right-continuous. Let X be a process of finite variation. It follows that X is làglàd, i.e. it possesses left and right limits, but it can have double jumps. Let $\Delta X_t := X_t - X_{t-}$ denote the jump at time t and let $\Delta^+ X_t := X_{t+} - X_t$ denote the jump immediately after time t. For a càdlàg integrand Y we define the integral $(Y_-, Y) \bullet X$ by

$$(Y_-, Y) \bullet X_t := (Y_- \bullet X^r)_t + \sum_{0 \le s < t} Y_s \Delta^+ X_s, \quad t \ge 0, \tag{3.1}$$

where $X_t^r := X_t - \sum_{0 \le s < t} \Delta^+ X_s$. The first term on the right-hand side of (3.1) is just a standard Lebesgue-Stieltjes integral. As the notation indicates, the left jumps of X are weighted with Y_- and the right jumps with Y. If Y is continuous we use the shorter notation $Y \bullet X$ for the integral defined in (3.1). Note that the notations are consistent with the common integral w.r.t. càdlàg integrators.

Definition 3.5. *For a given initial endowment $(\eta^0, \eta^1) \in \mathbb{R}^2$ we define the* portfolio process *$(\varphi^0(\mathfrak{S}), \varphi^1(\mathfrak{S}))$ associated with the trading strategy $\mathfrak{S}$ by*

$$\begin{aligned}
\varphi_t^0(\mathfrak{S}) &:= \eta^0 - \int_0^t (\overline{S}_{s-}, \overline{S}_s) dM_s^B + \int_0^t (\underline{S}_{s-}, \underline{S}_s) dM_s^S \\
&+ \int_{[0,t) \times \mathbb{R}_+} \int_x^\infty y L^B(s, dy) \mu(ds, dx) + \int_{[0,t) \times \mathbb{R}_+} \int_0^x y L^S(s, dy) \nu(ds, dx) \\
\varphi_t^1(\mathfrak{S}) &:= \eta^1 + M_t^B - M_t^S + \int_{[0,t) \times \mathbb{R}_+} L_s^B d\mu_s - \int_{[0,t) \times \mathbb{R}_+} L_s^S d\nu_s
\end{aligned}$$

(cf. (3.1)). For $L \in \{L^B, L^S\}$, the integral $\int y L(s, dy)$ is defined by

$$\int_a^b y L(s, dy) := \int_{(a,b)} y L^c(s, dy) + \sum_{a < y \le b} y \Delta^- L(s, y) + \sum_{a \le y < b} y \Delta^+ L(s, y),$$

where L^c denotes the continuous part and $\Delta^- L(s, y)$ resp. $\Delta^+ L(s, y)$ the jumps of the function $y \mapsto L(s, y)$.

Definition 3.5 can be regarded as the self-financing condition of the model. $\Delta^- M_t^B$ are the purchases at time $t-$ paying the price $\overline{S}_{t-}$ per share whereas $\Delta^+ M_t^B$ are purchases at time t paying $\overline{S}_t$ (this is reflected in the notation $\int_0^t (\overline{S}_{s-}, \overline{S}_s) dM_s^B$). The need for double jumps of M^B and M^S in the time variable has already been discussed in the literature on transaction costs, see e.g. page 581 of [3]. Example 3.19 shows why we need double jumps of L^B and L^S in the price variable.

Definition 3.6. *For any $a > 0$ a trading strategy $\mathfrak{S}$ is called* admissible *with threshold a if its associated portfolio process $(\varphi^0(\mathfrak{S}), \varphi^1(\mathfrak{S}))$ satisfies*

$$\varphi^0(\mathfrak{S}) + a + \underline{S} \left(\varphi^1(\mathfrak{S}) + a \right) 1_{\{\varphi^1(\mathfrak{S}) + a \ge 0\}} + \overline{S} \left(\varphi^1(\mathfrak{S}) + a \right) 1_{\{\varphi^1(\mathfrak{S}) + a < 0\}} \ge 0. \tag{3.2}$$

This can be interpreted that given strategy $\mathfrak{S}$, if at all times we have a additional units of cash and a additional units of the stock in our portfolio, then we always would be able to close our position in the stock using market orders without going into debt. Note that an admissible portfolio process with threshold a as defined above is also admissible with threshold a in the sense of [3]. We will make use of this later on, when we prove the closedness result.

Remark 3.7. *The integer-valued random measures μ and ν can be written as*

$$\mu(dt, dx) = \sum_{i=1}^{\infty} \delta_{(\tau_i, Y_i)}(dt, dx),$$

where δ_x denotes the Dirac measure on x, $(\tau_i)_{i\in\mathbb{N}}$ is a sequence of stopping times with disjoint graphs, and Y_i are $\mathcal{F}_{\tau_i}$-measurable random variables, and

$$\nu(dt, dx) = \sum_{i=1}^{\infty} \delta_{(\sigma_i, Z_i)}(dt, dx),$$

where $(\sigma_i)_{i\in\mathbb{N}}$ is a sequence of stopping times with disjoint graphs and Z_i are $\mathcal{F}_{\sigma_i}$-measurable random variables (this is a consequence of Theorem 11.13 in [10]).

Example 3.8. *We want to show how reasonable examples for the execution measures μ and ν can be constructed s.t. Assumption 3.1 is satisfied. Let $\underline{S} < \overline{S}$ be arbitrary stochastic processes with càdlàg paths. Define*

$$\begin{aligned} \mu \quad := & \sum_{0<t\le T,\ \Delta\overline{S}_t<0,\ \Delta\underline{S}_t>-\varepsilon} \delta_{(t,\overline{S}_t)} + \sum_{0<t\le T,\ \Delta\underline{S}_t\le-\varepsilon,\ \Delta\overline{S}_t\le 0} \delta_{(t,\underline{S}_t)} + \sum_{0<t\le T,\ \Delta N^1_t=1} \delta_{(t,\underline{S}_t)} \\ & + \sum_{0<t\le T,\ \Delta N^2_t=1} \delta_{(t,(\underline{S}_t+\overline{S}_t)/2)} =: \mu^1 + \mu^2 + \mu^3 + \mu^4, \end{aligned} \tag{3.3}$$

where $\varepsilon > 0$ and N^1, N^2 are independent homogeneous Poisson processes also independent of $\underline{S}$ and $\overline{S}$. ν is defined analogously with Poisson processes independent of $(\underline{S}, \overline{S}, N^1, N^2)$. A downside jump of $\overline{S}$ always triggers a limit buy order execution (see $\mu^1 + \mu^2$), i.e. Assumption 3.1(ii) is satisfied. By contrast, a downside jump of $\underline{S}$ triggered by a cancelation of limit buy orders (cf. the event (v) in Subsection 1.1) does not lead to an execution of limit buy orders. An execution takes place if market sell orders eat into the book (cf. the event (ii) in Subsection 1.1). In the easiest case the events (v) and (ii) may be distinguished by downside jumps of $\underline{S}$ smaller resp. larger than some threshold jump size $\varepsilon > 0$. Note that by the manipulation with ε there are only finitely many favorable executions (i.e. at a price strictly below the current best ask). This means that Assumption 3.1(iii) is satisfied. In addition, if a downside jump of $\underline{S}$ coincides with an upward jump of $\overline{S}$, no execution occurs (cf. μ^2). As ν is defined analogously, this economically meaningful property ensures that favorable executions by μ and ν do not coincide, i.e. Assumption 3.1(v) is satisfied. Furthermore, there may appear executions not coming along with jumps of $\underline{S}$ or $\overline{S}$. They are modelled by N^1 and N^2 (whose jumps do not coincide with the jumps of $\underline{S}$ and $\overline{S}$ due to independence).

3.2 Approximation by real-world strategies

Elementary or real-world strategies are trading strategies that can be implemented by finitely many operations. Executed limit orders are not automatically renewed. The execution is modeled by the random measures μ and ν. Beyond that, the best-ask (bid) price can pass *continuously* through the limit price of a buy (sell) order placed by the small trader. This entails an execution as no buy (sell) order with limit price higher (smaller) than the best-ask (bid) can persist in the book. This "continuous execution" cannot be triggered by the σ-finite random measures μ and ν and has to be modeled separately.

Suppose at a stopping time T_1^B we place a limit buy order $\widehat{L}^B := (\theta^B, p^B, T_1^B, T_2^B)$ of size $\theta^B \in L^0_+(\mathcal{F}_{T_1^B})$ and price $p^B \in L^0_+(\mathcal{F}_{T_1^B})$ with $p^B < \overline{S}_{T_1^B}$ and if the order is not executed up to stopping time $T_2^B \geq T_1^B$ we cancel it. Define the stopping times

$$\begin{aligned} T^{\overline{S}} &:= \inf\{t \in (T_1^B, T_2^B] \mid \overline{S}_t \leq p^B\}, \\ T^{\mu} &:= \inf\{\tau_i \mid T_1^B < \tau_i \leq T_2^B, Y_i \leq p^B\}, \\ T^* &:= T^{\overline{S}} \wedge T^{\mu}. \end{aligned} \tag{3.4}$$

T^* models the time at which the limit buy order is executed. If at all, the trade takes place at price p^B. The portfolio process of the limit buy order $\widehat{L}^B$ is defined as

$$\varphi_t^0(\omega) = -\theta^B(\omega)p^B(\omega)1_{]\!]T^*,T]\!]}(\omega,t) \quad \text{and} \quad \varphi_t^1(\omega) \;=\; \theta^B(\omega)1_{]\!]T^*,T]\!]}(\omega,t). \tag{3.5}$$

In the following we show that any real-world strategy can be replicated by a general strategy $\mathfrak{S} = (M^B, M^L, L^B, L^S)$ satisfying $L^B = 0$ on $[\overline{S}_-, \infty)$ and $L^S = 0$ on $(-\infty, \underline{S}_-]$. Thus, on the level of general strategies the limit buy (sell) order is taken out before the best-ask (bid) passes and a "continuous execution" does not appear.

Assumption 3.9. *For all $(\omega, t) \in \Omega \times [0,T]$ we have that*

$$\begin{aligned} \mu(\omega, \{t\} \times \{\overline{S}_{t-}(\omega)\}) = 1 \quad &\Longrightarrow \quad \Delta\overline{S}_t(\omega) \leq 0 \\ \nu(\omega, \{t\} \times \{\underline{S}_{t-}(\omega)\}) = 1 \quad &\Longrightarrow \quad \Delta\underline{S}_t(\omega) \geq 0. \end{aligned}$$

Proposition 3.10. *The quadruple $\mathfrak{S} = (M^B, 0, L^B, 0)$ with*

$$\begin{aligned} M_t^B(\omega) &:= \theta^B(\omega)1_{]\!]T^*,T]\!]}(\omega,t)1_{\{T^{\overline{S}}<T^{\mu}\}}(\omega), \\ L^B(\omega,t,x) &:= \theta^B(\omega)1_{]\!]T_1^B,T^*\wedge T_2^B]\!]}(\omega,t)\left(1_{(x\leq p^B(\omega),\ \overline{S}_{t-}(\omega)>p^B(\omega))} + 1_{(x<p^B(\omega),\ \overline{S}_{t-}(\omega)=p^B(\omega))}\right) \end{aligned} \tag{3.6}$$

is a trading strategy in the sense of Definition 3.4. Under Assumption 3.9 it leads to the portfolio process given in (3.5).

Proof. M^B and L^B are obviously $\mathcal{P}$- resp. $\widetilde{\mathcal{P}}$-measurable (note that $\{T^{\overline{S}} < T^{\mu}\} \in \mathcal{F}_{T^*}$). To show that their portfolio process from Definition 3.5 coincides with (3.5), we have to distinguish four cases. If $T^{\overline{S}} < T^{\mu}$, the stocks are purchased by the market order strategy M^B at price $\overline{S}_{T^*} = p^B$ (Assumption 3.1(ii) ensures that there cannot be a downside jump of $\overline{S}$ at T^* if $T^{\overline{S}} < T^{\mu}$). If $T^{\overline{S}} = T^{\mu} \leq T_2^B$ and $\overline{S}_{T^*-} > p^B$, then $Y_i \leq \overline{S}_{T^*} \leq p^B$ (cf. Assumption 3.1(ii)) and the stocks are purchased by the limit order strategy L^B at price p^B (see the first indicator function in the definition of L^B). If $T^{\overline{S}} = T^{\mu} \leq T_2^B$ and $\overline{S}_{T^*-} = p^B$, then by Assumption 3.1(i) and Assumption 3.1(iv) we have that $Y_i < p^B$ and the stocks are purchased by L^B at price p^B (see the second indicator function in the definition of L^B). If $T^{\mu} < T^{\overline{S}}$, we have by Assumption 3.9 that $\overline{S}_{T^*-} > p^B$ and with $Y_i \leq p^B$ the stocks are again purchased by L^B considering the first indicator function in its definition. $\square$

To obtain the embedding (3.6) we separate the execution of the limit buy order triggered by $\overline{S}$ hitting the limit price p^B at no jump time from all other possible executions of the limit buy order (including the case that $\overline{S}$ jumps into $[0, p^B]$) and treat this "continuous execution" by market buy orders at the same price instead of limit orders, whereas all the other executions of limit buy orders are modeled by L^B and μ. The intuition behind this is that a limit buy order which is triggered by $\overline{S}$ hitting p^B at no jump time is superfluous. The asset can be purchased instead by a market order placed at the hitting time paying also the best-ask price. By this consideration, we gain much in tractability as we do not have to deal with the "continuous executions" of the limit orders.

In the case that $\overline{S}$ is continuous, the hitting time $T^{\overline{S}}$ of $[0, p^B]$ is a predictable stopping time, i.e. it possesses a sequence of announcing stopping times (cf. e.g. page 103 of [17]) and the conditional probability of an execution of a limit buy order tends to 1 while the limit price is approached by $\overline{S}$. But, in spite of the partial replacement by market orders, limit orders are even for continuous $\overline{S}$ not superfluous as they can e.g. be executed according to $\mu^3 + \mu^4$ given in Example 3.8. On the other hand, in models with a finite price grid (as introduced in Subsection 3.5) $\overline{S}$ becomes a finite variation pure jump process. Then, already on the level of real-world strategies "continuous execution" cannot happen as $T^{\overline{S}} < T^{\mu}$ does not occur (cf. Assumption 3.1(ii)), i.e. the market order part from (3.6) disappears.

Remark 3.11. *In the model of Smid [21] a limit buy order is only executed if the best-ask process $\overline{S}$ hits or jumps below the order's limit price p^B. If market orders can be submitted all the time, Smid proves that any wealth process from a strategy containing limit orders can be dominated by a wealth process generated solely by market orders (see Proposition 1 in [21]). Proposition 3.10 and our approach are in the same vein (if $\overline{S}$ possesses downward jumps, there is the difference that limit orders cannot be replaced by market orders alone, but other limit orders are needed as we want to obtain the same portfolio process and not just a larger one). If market orders can only be submitted at a discrete time grid, a situation that is not considered here, Smid shows that it is nevertheless necessary to use also limit orders in his model to obtain optimal portfolios, but in a numerical analysis the benefit from their usage turns out to be small (see [20]).*

However, our framework allows for further execution events. E.g., the limit buy order with price p^B can be executed by a market sell order and afterwards we still have $\overline{S} > p^B$. In the articles by Guilbaud and Pham [9] and Kühn and Stroh [12], limit buy order executions are mainly at prices below $\overline{S}$, cf. also the executions by μ_2, μ_3, and μ_4 in Example 3.8. These executions make limit orders worthwhile and compensate certain risks. Buy orders with limit prices below the best-ask are of course not affected at all by the replacement by market orders in Proposition 3.10 as these order strategies satisfy $L^B = 0$ on $[\overline{S}_-, \infty)$. Even if $\overline{S}$ and $\underline{S}$ are continuous processes these limit orders can be executed by μ_3 and μ_4 from Example 3.8.

The analysis of real-world limit sell orders is completely analogous and thus omitted. Due to the observation made in Proposition 3.10, the real-world limit buy order $\widehat{L}^B$ can be identified with the strategy $\mathfrak{S}$ from (3.6). This leads to the following definition.

Definition 3.12 (Real-world strategies)**.** *A trading strategy $\mathfrak{S}$ is called a real-world buying strategy if it can be written as finite linear combination of $(M^B, 0, L^B, 0)$, where the pair (M^B, L^B) is defined in (3.6), and $(M^B, 0, 0, 0)$, where $M^B = \theta^1 1_{]\!]T_1, T_2]\!]} + \theta^2 1_{[\![T_3]\!]}$, where T_1, T_2 are $[0, T]$-valued stopping times, T_3 is a $[0, T]$-valued predictable stopping time, and $\theta^1 \in L^0_+(\mathcal{F}_{T_1})$, $\theta^2 \in L^0_+(\mathcal{F}_{T_3})$. A real-world selling strategy is defined correspondingly. A trading strategy $\mathfrak{S}$ is called a real-world strategy if it can be written as the sum of a real-world buying strategy and a real-world selling strategy.*

For $\mathcal{F} \otimes \mathcal{B}([0, T])$-measurable real-valued processes X and Y let

$$d_{\mathrm{up}}(X, Y) := E\left(1 \wedge \sup_{t \in [0,T]} |X_t - Y_t|\right).$$

d_{up} metrizes the convergence "uniformly in probability" (up-convergence), cf. e.g. Protter [17].

Theorem 3.13 (Approximation by real-world strategies)**.** *For any trading strategy* $\mathfrak{S} = (M^B, M^S, L^B, L^S)$ *and any* $\varepsilon > 0$ *there exists a real-world strategy* $\mathfrak{S}^\varepsilon$ *s.t.*

$$d_{up}(\varphi^0(\mathfrak{S}^\varepsilon), \varphi^0(\mathfrak{S})) < \varepsilon \qquad \text{and} \qquad d_{up}(\varphi^1(\mathfrak{S}^\varepsilon), \varphi^1(\mathfrak{S})) < \varepsilon.$$

Put differently, the portfolio processes that can be generated by real-world strategies are dense w.r.t. the convergence "uniformly in probability" in the set of all portfolio processes.

Theorem 3.13 tells us that we can approximate the portfolio process resulting from strategies with possibly infinitely many limit prices and continuously varying order sizes by placing only finitely many orders. This is of the same flavor as the fact that (under certain assumptions) the stochastic integral of a predictable process can be approximated by the stochastic integrals of simple predictable processes.

3.3 Closedness of the strategy set

The possibility of approximating the portfolio processes in our model with real-world strategies alone would not make the model particularly useful, if the strategy set would not be closed in some sense. To proceed towards the closedness result, let us first recall the concept of a strictly consistent price process.

Definition 3.14. *An adapted* $(0, \infty)$*-valued process* $\widetilde{S} = (\widetilde{S}_t)_{t\in[0,T]}$ *is called a* strictly consistent price process *for the risky asset if there exists a probability measure* $\widetilde{P} \sim P$ *s.t.* $\widetilde{S}$ *is a càdlàg* $\widetilde{P}$*-martingale with*

$$\widetilde{S}_t \in (\underline{S}_t, \overline{S}_t), \ \forall t \in [0, T] \quad \text{and} \quad \widetilde{S}_{t-} \in (\underline{S}_{t-}, \overline{S}_{t-}), \ \forall t \in (0, T], \quad P\text{-a.s.}$$

The existence of $\widetilde{S}$ is equivalent to the existence of a strictly consistent price process in the sense of Definition 2.3 in Campi and Schachermayer [3].

Definition 3.15. *Define the following sets of stochastic processes*

$$\mathcal{P}_1 := \{X \text{ is a } [0, \infty]\text{-valued predictable process with } P(X_{\tau_i} \leq Y_i) = 1 \ \forall i \in \mathbb{N}\}$$
$$\mathcal{P}_2 := \{X \text{ is a } [0, \infty]\text{-valued predictable process with } P(X_{\sigma_i} \geq Z_i) = 1 \ \forall i \in \mathbb{N}\}$$

where $(\tau_i, Y_i)_{i\in\mathbb{N}}$ *and* $(\sigma_i, Z_i)_{i\in\mathbb{N}}$ *are the representations of* μ *resp.* ν *from Remark 3.7. Let* $\underline{X}$ *be the essential supremum of the functions in* $\mathcal{P}_1$ *taken w.r.t. the predictable* σ*-algebra on* $\Omega \times [0, T]$ *and the measure* $\widehat{M}_\mu$ *from Definition 2.3. Accordingly, let* $\overline{X}$ *be the essential infimum of the functions in* $\mathcal{P}_2$ *taken w.r.t. the predictable* σ*-algebra and the measure* $\widehat{M}_\nu$ *defined as in Definition 2.3.*

Assumption 3.16.

$$P(\underline{X}_{\tau_i} = Y_i) = 0 \quad \text{and} \quad P(\overline{X}_{\sigma_i} = Z_i) = 0 \quad \forall i \in \mathbb{N}.$$

$\underline{X}$ resp. $\overline{X}$ can be interpreted as the highest (resp. smallest) predictable limit price below (above) which a limit buy (resp. sell) order is *not* executed for sure. Assumption 3.16 says that at these boundary limit prices an execution is also not possible. It is needed for the following closedness result, cf. also Example 3.18.

Theorem 3.17 (Closedness of the strategy set)**.** *Let Assumption 3.16 be satisfied and suppose that there exists a strictly consistent price process for the risky asset in the sense of Definition 3.14. In addition, assume that $\underline{S}$ and $\overline{S}$ are semimartingales and $P(\inf_{t\in[0,T]} \underline{S}_t > 0) = 1$. Let $(\mathfrak{S}^n)_{n\in\mathbb{N}}$ be an admissible sequence of trading strategies with the same threshold level a and the same initial capital (η^0, η^1) for all n. If the sequence of associated portfolio processes $\big((\varphi^0(\mathfrak{S}^n), \varphi^1(\mathfrak{S}^n))\big)_{n\in\mathbb{N}}$ is a Cauchy sequence w.r.t. the convergence "uniformly in probability", then there exists an admissible trading strategy $\mathfrak{S}$ with threshold level a and initial capital (η^0, η^1) s.t. $\big((\varphi^0(\mathfrak{S}^n), \varphi^1(\mathfrak{S}^n))\big)_{n\in\mathbb{N}}$ converges uniformly in probability to the associated portfolio process $(\varphi^0(\mathfrak{S}), \varphi^1(\mathfrak{S}))$ of $\mathfrak{S}$.*

The following example shows that the assertion of Theorem 3.17 would not hold without Assumption 3.16.

Example 3.18. *Let $\underline{S} = 100$ and $\overline{S} = 101$. Furthermore let $t_0 \in (0,T)$ and let X be a random variable with distribution $0.5\delta_{100} + 0.5\lambda[100,101]$, where δ denotes a Dirac measure and λ denotes a uniform distribution. Consider the usual augmentation of the filtration generated by the stochastic process $X1_{[t_0,T]}$. Define $\mu_\omega := \delta_{(t_0,X(\omega))}$ and $\nu := 0$, i.e. at time t_0 limit buy orders with a limit price of X or higher are executed, whereas no limit sell orders are executed at all. Now consider the sequence of strategies $(M^{B,n}, M^{S,n}, L^{B,n}, L^{S,n})_{n\in\mathbb{N}}$ with $L^{S,n} = 0$, $M^{B,n} = 0$,*

$$L^{B,n}(\omega,t,x) := \begin{cases} n & \text{if } x \le 100 + e^{-n}, \\ -\ln(x-100) & \text{if } 100 + e^{-n} < x < 101, \\ 0 & \text{if } x \ge 101, \end{cases} \qquad \forall(\omega,t) \in \Omega \times [0,T], n \in \mathbb{N},$$

and $M^{S,n}_t := n1_{\{X=100\}}1_{(t_0,T]}(t)$. Thus the n-th strategy consists in buying $L^{B,n}(\omega,t,X(\omega))$ shares via limit order and selling the same amount via market order iff $X(\omega) = 100$. The only time of interest is of course the instant from t_0 to t_0+. We obtain

$$\Delta^+\varphi^{1,n}_{t_0} = n1_{\{100<X\le 100+e^{-n}\}} - \ln(X-100)1_{\{X>100+e^{-n}\}},$$

$$\begin{aligned} \Delta^+\varphi^{0,n}_{t_0} &= \int_X^{101} yL^{B,n}(t,dy) + 100n1_{\{X=100\}} \\ &= \int_{X\vee(100+e^{-n})}^{101} \left(\frac{-y}{y-100}\right) dy + 100n1_{\{X=100\}} \\ &= [-x - 100\ln(x-100)]^{101}_{X\vee(100+e^{-n})} + 100n1_{\{X=100\}} \\ &= (-1+e^{-n})1_{\{X=100\}} + (-1+e^{-n}-100n)1_{\{100<X\le 100+e^{-n}\}} \\ &\quad +(-101 + X + 100\ln(X-100))1_{\{X>100+e^{-n}\}}. \end{aligned}$$

If the initial positions are $(\eta^{0,n}, \eta^{1,n}) = (0,0)$, we obtain that stock positions are nonnegative and $\varphi^{0,n} + \varphi^{1,n}\underline{S} \ge -1$ for all $n \in \mathbb{N}$. Thus $(\varphi^{0,n}, \varphi^{1,n})_{n\in\mathbb{N}}$ are admissible portfolio processes with the joint threshold -1. In addition they converge uniformly in probability to

$$(\psi^0, \psi^1) = \big((-1,0)1_{\{X=100\}} + (X - 101 + 100\ln(X-100), -\ln(X-100))1_{\{X>100\}}\big)\, 1_{(t_0,T]}$$

(note, however, that the convergence is not uniform in a neighborhood of $X = 100$). Let us show that there exists no trading strategy $\mathfrak{S} = (M^B, M^S, L^B, L^S)$ s.t. $\big(\varphi^0(\mathfrak{S}), \varphi^1(\mathfrak{S})\big) = (\psi^0, \psi^1)$ up to evanescence. As L^B has to be predictable, $L^B(t_0, x)$ has to be chosen regardless of the outcome

of X. Thus, to buy the correct amount of shares at the right prices, on $\{X > 100\}$ market orders cannot be used and we must have $L^B(t_0, x) = -\ln(x - 100)$ for all $x \in (100, 101)$ up to a P-null set which does not depend on x by the required monotonicity. Again, by the monotonicity requirement, this implies that $P(L^B(t_0, 100) = \infty) = 1$. But, as $P(X = 100) > 0$, this violates the μ-integrability of L^B. Such a limit buy order strategy, possibly combined with $\Delta^+ M^S_{t_0} = \infty$ on $\{X = 100\}$, would not lead to a well-defined portfolio process.

The following example shows that we have to allow for double jumps of L^B and L^S in the price variable in order to obtain closedness.

Example 3.19. *[The need for double jumps of $x \mapsto L^B(t, x)$] Let $\mu = \delta_{(1,Y_1)}$ with $P(Y_1 \in A) = \frac{1}{2} 1_{(1 \in A)} + \frac{1}{2}\lambda(A \cap (0, 1))$ for all $A \in \mathcal{B}(\mathbb{R}_+)$, where λ is the Lebesgue-measure. As $\mu(\omega, \{t\} \times \{x\}) = 1$ means that all limit buy orders with limit price smaller* or equal *x are executed, an elementary limit buy order strategy with limit price p^B, entering in Definition 3.5 for the portfolio process, is represented by $L^B = 1_{[0,p^B]}$ (and not by $1_{[0,p^B)}$), i.e. by a function that is left-continuous in the price variable. Now consider the sequence of elementary strategies $L^{B,n} = 1_{[0,1-1/n]}$, $n \in \mathbb{N}$. The corresponding portfolio processes converge in up to $\varphi^0_t = -1_{\{t>1\}} 1_{\{Y_1<1\}}$ and $\varphi^1_t = 1_{\{t>1\}} 1_{\{Y_1<1\}}$. As $P(Y_1 = 1) > 0$, in Definition 3.5 (φ^0, φ^1) can only be generated by the right-continuous function $\widetilde{L}^B = 1_{[0,1)}$. $\widetilde{L}^B$ is a (non-elementary) limit order that pays the limit price 1, but in contrast to $1_{[0,1]}$ it is only buying if the execution boundary falls strictly below 1. Thus, in order to obtain a closed set of portfolio processes, limit order strategies should not be restricted from the first to left- or right-continuous functions.*

3.4 Arbitrage theory

Definition 3.20. *A limit order market model satisfies the condition of* no free lunch with vanishing risk (NFLVR) *if there does not exist a pair consisting of a nonnegative random variable f with $P(f > 0) > 0$ and a sequence of admissible strategies $(\mathfrak{S}^n)_{n\in\mathbb{N}}$ (with some thresholds $a_n > 0$) s.t. the associated portfolio processes given in Definition 3.5 satisfy $\varphi^0_0(\mathfrak{S}^n) = \varphi^1_0(\mathfrak{S}^n) = \varphi^1_T(\mathfrak{S}^n) = 0$ and $P(f \le \varphi^0_T(\mathfrak{S}^n) + 1/n) = 1$ for all $n \in \mathbb{N}$.*

(The definition makes sense if all stock positions are liquidated at $T-$ which is of course no restriction as T can be arbitrarily chosen. We follow here [3] to avoid special notations for trading at T, see Assumption 2.2 and Remark 4.2 therein)

Definition 3.21. *An adapted $[0, \infty)$-valued process $\widetilde{S} = (\widetilde{S}_t)_{t\in[0,T]}$ is called a shadow price process of the risky asset if there exists a probability measure $\widetilde{P} \sim P$ s.t. $\widetilde{S}$ is a càdlàg $\widetilde{P}$-martingale with*

$$\underline{S} \le \widetilde{S} \le \overline{S}, \quad \widetilde{S}_{\tau_i} \le Y_i, \quad \text{and} \quad \widetilde{S}_{\sigma_i} \ge Z_i, \quad \forall i \in \mathbb{N}, \tag{3.7}$$

where $(\tau_i, Y_i)_{i\in\mathbb{N}}$ and $(\sigma_i, Z_i)_{i\in\mathbb{N}}$ are introduced in Remark 3.7.

Any shadow price process is a consistent price process (cf. Definition 3.14 for its "strict version"), but not vice versa. Namely, not only trading with market orders but also trading with limit orders is at least as favorable as trading in the fictitious frictionless market with price process $\widetilde{S}$. Condition (3.7) can be seen as a generalization of condition (2.5) in [12], for the connection between the models see Example 6.2.

Theorem 3.22. *If there exists a shadow price process in the sense of Definition 3.21, then the limit order market model satisfies NFLVR.*

Proof. Let $\mathfrak{S}$ be an admissible strategy with threshold $a > 0$ satisfying $\varphi_0^0 = \varphi_0^1 = 0$. Consider the corresponding wealth process if stock positions are evaluated at the shadow price, i.e. $\widetilde{V} := \varphi^0 + \varphi^1 \widetilde{S}$. By Definition 3.5 and Lemma 8.2 in Muhle-Karbe [14] we obtain

$$\begin{aligned}
\widetilde{V} &= \varphi^1 \bullet \widetilde{S} + \varphi^0 + (\widetilde{S}_-, \widetilde{S}) \bullet \varphi^1 \\
&= \varphi^1 \bullet \widetilde{S} - (\overline{S}_- - \widetilde{S}_-, \overline{S} - \widetilde{S}) \bullet M^B - (\widetilde{S}_- - \underline{S}_-, \widetilde{S} - \underline{S}) \bullet M^S \\
&\quad + \int_{[0,\cdot)\times\mathbb{R}_+} \int_x^{\overline{S}_{s-}} (y - \widetilde{S}_s) L^B(s, dy) \mu(ds, dx) + \int_{[0,\cdot)\times\mathbb{R}_+} \int_{\underline{S}_{s-}}^x (y - \widetilde{S}_s) L^S(s, dy) \nu(ds, dx).
\end{aligned}$$

The admissibility of $\mathfrak{S}$ with threshold $a > 0$ and $\underline{S} \le \widetilde{S} \le \overline{S}$ imply that $\widetilde{V} \ge -a(1 + \widetilde{S})$. By (3.7), $\widetilde{V} - \varphi^1 \bullet \widetilde{S}$ is a nonincreasing process starting in zero. Putting together, we obtain that $\varphi^1 \bullet \widetilde{S} \ge -a(1+\widetilde{S})$ which implies that $\varphi^1 \bullet \widetilde{S}$ is a $\widetilde{P}$-local martingale (and not only a σ-martingale under $\widetilde{P}$). Therefore, the nonincreasing process $\widetilde{V} - \varphi^1 \bullet \widetilde{S}$ is bounded from below by the $\widetilde{P}$-local martingale $-a(1 + \widetilde{S}) - \varphi^1 \bullet \widetilde{S}$. This implies that $\widetilde{V} - \varphi^1 \bullet \widetilde{S}$ is a $\widetilde{P}$-local supermartingale. Consequently, $\widetilde{V}$ is a $\widetilde{P}$-local supermartingale, bounded from below by a $\widetilde{P}$-martingale and thus it is a $\widetilde{P}$-supermartingale with $E_{\widetilde{P}}(\widetilde{V}_T) \le 0$. If $\varphi_T^1 = 0$, we have $E_{\widetilde{P}}(\varphi_T^0) \le 0$. As this holds for any $\mathfrak{S}$ and we assume $\widetilde{P} \sim P$ the model satisfies NFLVR. $\square$

The converse does not hold in general. This is shown by the following simple two-period example (which is not based on the way of extending elementary strategies). The example shows that for a limit order market model satisfying NFLVR there need not exist a corresponding frictionless market that is at least as favorable and that is also arbitrage-free.

Example 3.23. *Let $T = 3$, $\nu = 0$ and $\mu = \delta_{(1,Y_1)}$, where Y_1 is uniformly distributed on $(0, 1)$. Let X be independent of Y_1 with $P(X = 0) = P(X = 1) = 1/2$. Define $\underline{S}_0 = \underline{S}_1 = 0$, $\underline{S}_2 = \underline{S}_3 = Y_1 + X$, $\overline{S}_0 = \overline{S}_1 = 2$, $\overline{S}_2 = \overline{S}_3 = Y_1 + X$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \sigma(Y_1)$, and $\mathcal{F}_2 = \mathcal{F}_3 = \sigma(Y_1, X)$.*

This means that limit sell orders are never executed and limit buy orders are executed at time 1 iff the deterministic limit price is greater or equal to Y_1. The investor can resell the stock position at time 2 by a market sell order at price $Y_1 + X$ (see the remark after Definition 3.20 why we choose $T = 3$ and not $T = 2$ for this actual two-period example).

If the investor buys the asset at time 1, he makes almost surely a loss on the event $\{X = 0\}$. This follows from the fact that with probability 1 only limit buy orders with limit price strictly above Y_1 are executed. This already shows no-arbitrage. Let us prove that the limit order market even satisfies NFLVR.

Let $(\mathfrak{S}^n)_{n\in\mathbb{N}}$ be a sequence of strategies with $\varphi_0^0(\mathfrak{S}^n) = \varphi_0^1(\mathfrak{S}^n) = \varphi_3^1(\mathfrak{S}^n) = 0$. We only need to consider the deterministic nonincreasing functions $L^{B,n}(1, \cdot)$ that model the purchases by limit orders at time 1. Then, the stock position is liquidated at time 2 at price $Y_1 + X$. Now, assume that $\varphi_3^0(\mathfrak{S}^n) \ge -1/n$. This implies that $\int_0^\infty y L^{B,n}(1, dy)\, dy \ge -1/n$. Namely, Y_1 can fall arbitrage close to 0 on the event $\{X = 0\}$. Then, a stock purchased at the limit price y can only be resold at a price close to 0. But, with integration by parts, we have that $\int_0^\infty y L^{B,n}(1, dy) = -\int_0^\infty L^{B,n}(y)\, dy$ and $\int_0^1 L^{B,n}(y)\, dy$ is the expected amount of purchased stocks. Thus $E_P(\varphi_3^0(\mathfrak{S}^n))$ tends to 0 for $n \to \infty$. This implies NFLVR.

On the other hand, by (3.7), for any shadow price process $\widetilde{S}$ we must have that $\widetilde{S}_2 - \widetilde{S}_1 \ge (Y_1 + X) - Y_1 = X$ which is a contradiction to the martingale property under an equivalent measure $\widetilde{P}$. Thus, a shadow price cannot exist.

Remark 3.24. *To the best of our knowledge, there does not exist a counterexample of this kind for models with proportional transaction costs in the literature. In Section 3 of Schachermayer [19] there is given an example of an arbitrage-free model that does not possess a consistent* *price system, but the example is based on the existence of a "free lunch", a (uniformly bounded) sequence of terminal wealths approximating in probability an arbitrage. Even such a sequence cannot exist in Example 3.23 as to approximate a gain there has to be an $\varepsilon > 0$ s.t. $L^{B,n}(1,\varepsilon) \geq \varepsilon$ for infinitely many $n \in \mathbb{N}$ which is on the over hand a contradiction to the condition that the limiting random variable should be nonnegative.*

To find handy additional assumptions for limit order market models under which the converse of Theorem 3.22 can be proven goes beyond the scope of this article and is thus left to future research.

3.5 A finite tick size

At the high-frequency level a finite tick size $\delta > 0$, leading to a price grid $\{0, \delta, 2\delta, \ldots\}$, becomes an important issue, see the model and the discussion in Guilbaud and Pham [9]. In this case μ and ν satisfy

$$\mu(\omega, [0,T] \times (\mathbb{R}_+ \setminus \{0, \delta, 2\delta, \ldots\})) = \nu(\omega, [0,T] \times (\mathbb{R}_+ \setminus \{0, \delta, 2\delta, \ldots\})) = 0 \tag{3.8}$$

for all $\omega \in \Omega$, i.e. their state space is actually discrete. It can also be assumed that $\underline{S}$ and $\overline{S}$ are $\{0, \delta, 2\delta, \ldots\}$-valued. Denote by $\mathfrak{T}^\delta$ the subset of trading strategies from Definition 3.4 for which $x \mapsto L^B(\omega, t, x)$ (resp. $x \mapsto L^S(\omega, t, x)$) is left-continuous (resp. right-continuous) and constant between grid points $0, \delta, 2\delta, \ldots$.

Theorem 3.25. *(i) For a strategy from $\mathfrak{T}^\delta$ the approximating real-world strategies in Theorem 3.13 can be chosen to contain only $\{0, \delta, 2\delta, \ldots\}$-valued limit prices.*

(ii) Assume that (3.8), all assumptions of Theorem 3.17 except for Assumption 3.16, and

$$\mathcal{F}_{\tau_i} = \sigma(\mathcal{F}_{\tau_i -}, Y_i) \quad \textit{and} \quad \mathcal{F}_{\sigma_i} = \sigma(\mathcal{F}_{\sigma_i -}, Z_i), \quad \forall i \in \mathbb{N}, \tag{3.9}$$

hold. Then, the set of portfolio processes generated by strategies from $\mathfrak{T}^\delta$ is closed in the same sense as in Theorem 3.17.

Note that (3.9) is e.g. satisfied if $\mathcal{F}_{t\in[0,T]}$ is generated by the marked point processes $(\tau_i, Y_i)_{i\in\mathbb{N}}$, $(\sigma_i, Z_i)_{i\in\mathbb{N}}$ from Remark 3.7 and by $(\underline{S}, \overline{S})$, whose possible jumps at τ_i and σ_i depend only on Y_{τ_i} resp. Z_{σ_i}, and the execution times of limit buy and sell orders do not coincide. Thus it is satisfied in the model of Guilbaud and Pham [9].

The proof of Theorem 3.25 can be found at the end of Section 5. After the proof of Theorem 3.17 it only remains to show that in a discrete state space Assumption 3.16 is not needed for Lemma 5.2 (i.e. the unpleasant effect from Example 3.18 cannot occur).

4 Proof of Theorem 3.13: Approximation by real-world strategies

Let $\mathfrak{S} = (M^B, M^S, L^B, L^S)$. By linearity of $(\varphi^0(\mathfrak{S}), \varphi^1(\mathfrak{S}))$ in $\mathfrak{S}$, it is sufficient to approximate $(M^B, 0, 0, 0)$, $(0, M^S, 0, 0)$, $(0, 0, L^B, 0)$, and $(0, 0, 0, L^S)$ separately. The assertion for $(M^B, 0, 0, 0)$ and $(0, M^S, 0, 0)$ holds by Guasoni, Lépinette, and Rásonyi [8]. See their Theorem A.10 and note that for denseness w.r.t. d_{up} it is not necessary that $\underline{S}$ and $\overline{S}$ are locally bounded, but it is sufficient that they are prelocally bounded (a property that any càdlàg process satisfies). It remains to prove the assertion for $(0, 0, L^B, 0)$. The proof for $(0, 0, 0, L^S)$ is analog.

Let us firstly give a short outline of the proof. We take a limit buy order strategy $L^B \in \mathcal{L}^B$ and cut its support off at a *simple* predictable price process bounded away from $[\overline{S}_-, \infty)$ (see

L^δ and Lemma 4.3). For L^δ the disturbing "continuous execution" triggered by $T^{\overline{S}}$ from (3.4) is excluded. Then, we discretize L^δ in the price variable x for fixed (ω, t) which leads to $L^{\delta,m}$ (see Figure 1 and Lemma 4.5). Lemma 4.6 and Lemma 4.7 show that the convergence of the limit buy order strategy implies the convergence of the number of purchased assets and the corresponding costs. Finally, by standard monotone class arguments, we can discretize time (see Theorem 4.8).

Definition 4.1. *Denote by $(x_k)_{k\in\mathbb{N}}$ a sequence running through $\mathbb{Q}_+$. We define the σ-finite measure $\widetilde{M}_\mu$ on $\widetilde{\mathcal{O}}$ by*

$$\widetilde{M}_\mu(A) := M_\mu(A) + \widehat{M}_\mu \otimes \sum_{k=1}^{\infty} 2^{-k}\delta_{x_k}\left(A \cap \mathit{supergraph}(\underline{X})\right), \quad \forall A \in \widetilde{\mathcal{O}}.$$

Note that M_μ is a σ-finite measure on $\widetilde{\mathcal{O}}$ by Definition 2.2(iii). The σ-finiteness of $\widehat{M}_\mu$ on $\widetilde{\mathcal{O}}$ follows from the representation of μ in Remark 3.7.

In the following for any $\delta > 0$ denote by $\overline{S}^\delta$ the canonical simple predictable process constructed on page 57 in Protter [17] with

$$P(\sup_{t\in[0,T]} |\overline{S}_t^\delta - \overline{S}_{t-}| > \delta) < \delta.$$

Denote by τ^δ the first time that $\overline{S}^\delta$ departs farther than δ from $\overline{S}_-$, i.e.

$$\tau^\delta := \inf\{t > 0 \mid |\overline{S}_t^\delta - \overline{S}_{t-}| > \delta\}, \tag{4.10}$$

and note that at time τ^δ the processes are still not more than δ apart as they are both left-continuous.

Definition 4.2 (δ-cut off)**.** *For any $\delta > 0$ and $L^B \in \mathcal{L}^B$ let us denote by L^δ the function defined by*

$$L^\delta(\omega, t, x) := L^B(\omega, t, x) 1_{\{x \le \overline{S}_t^\delta(\omega) - 3\delta\}} 1_{[\![0,\tau^\delta\wedge T]\!]}(\omega, t).$$

Lemma 4.3. *For any $\delta > 0$, we have that $L^\delta \in \mathcal{L}^B$. Furthermore, there exists a sequence $(\delta_i)_{i\in\mathbb{N}} \subset \mathbb{R}_+ \setminus \{0\}$ with $\delta_i \to 0$ for $i \to \infty$ s.t. $(L^{\delta_i})_{i\in\mathbb{N}}$ converges $\widetilde{M}_\mu$-a.e. to L.*

Proof. As $\overline{S}^\delta$ and $1_{[\![0,\tau^\delta]\!]}$ are predictable, we have that L^δ is $\widetilde{\mathcal{P}}$-measurable. Integrability follows immediately from $L^\delta \le L^B$ and the other requirements for L^δ being in $\mathcal{L}^B$ are also obviously satisfied.

Put $\delta_i := 2^{-i}$. By the lemma of Borel-Cantelli the events $\{\sup_{t\in[0,T]} |\overline{S}_t^{2^{-i}} - \overline{S}_{t-}| > 2^{-i}\}$, $i = 1, 2, \dots$ occur only finitely often on a set N^c with $P(N^c) = 1$. Thus, for any $\omega \in N^c$ there exists an $i_0(\omega)$ s.t. $|\overline{S}_t^{2^{-i}}(\omega) - \overline{S}_{t-}(\omega)| \le 2^{-i}$ for all $i \ge i_0(\omega)$, $t \in [0,T]$ and thus $\tau^{2^{-i}}(\omega) = \infty$. Consequently, for all $\omega \in N^c$, $t \in [0,T]$, and $x < \overline{S}_{t-}(\omega)$ we have that

$$1_{\{x \le \overline{S}_t^{2^{-i}}(\omega) - 3\cdot 2^{-i}\}} 1_{[\![0,\tau^{2^{-i}}\wedge T]\!]}(\omega, t) = 1 \quad \text{and thus} \quad L^{2^{-i}}(\omega, t, x) = L^B(\omega, t, x)$$

for $i \ge i_0(\omega) \vee \left(1 - \log_2((\overline{S}_{t-}(\omega) - x)/4)\right)$. For $\omega \in N^c$, $t \in [0,T]$, and $x \ge \overline{S}_{t-}(\omega)$ we obtain

$$1_{\{x \le \overline{S}_t^{2^{-i}}(\omega) - 3\cdot 2^{-i}\}} = 0 \qquad \text{for } i \ge i_0(\omega).$$

By assumption $L^B \in \mathcal{L}^B$ and thus $L^B(\omega, t, x) = 0$ if $x \ge \overline{S}_{t-}(\omega)$. Therefore, $L^{2^{-i}}$ converges to L^B pointwise on $N^c \times [0,T] \times \mathbb{R}_+$ and thus $\widetilde{M}_\mu$-a.e. $\square$

We proceed by discretizing L^δ in the price variable. Fix any $m \in \mathbb{N}$ and divide $(0, m]$ into dyadic intervals $((l-1)2^{-m}, l2^{-m}]$ for $l = 1, \ldots, m2^m$. Now we want to approximate $x \mapsto L^\delta(\omega, t, x)$ by a left-continuous step function $L^{\delta,m}$, which is constant between two points of the dyadic grid. For each interval we check if there exists a point x in this interval s.t. $L^\delta(\omega, t, x-) > L^\delta(\omega, t, x+)$. If this is the case, we fix the price $x^*_{l,m}(\omega, t)$ for which this "jump" is the largest and let our function take the value of $L^\delta(\omega, t, x^*_{l,m}(\omega, t))$ for the whole interval. When the largest jump is attained at different prices (which can only be finitely many), we take the smallest of these prices. If there is no "jump", we just set $x^*_{l,m}(\omega, t) = (l-1)2^{-m}$, i.e. for the interval we take the value of L^δ at the left boundary. It is advisable to have a look at Figure 1 to grasp the basic idea of the definitions below, which are, however, complicated by technical problems. Especially, the formal definition has to ensure that $L^{\delta,m}(\omega, t, x)$ is only infinite if $L^\delta(\omega, t, x)$ is infinite.

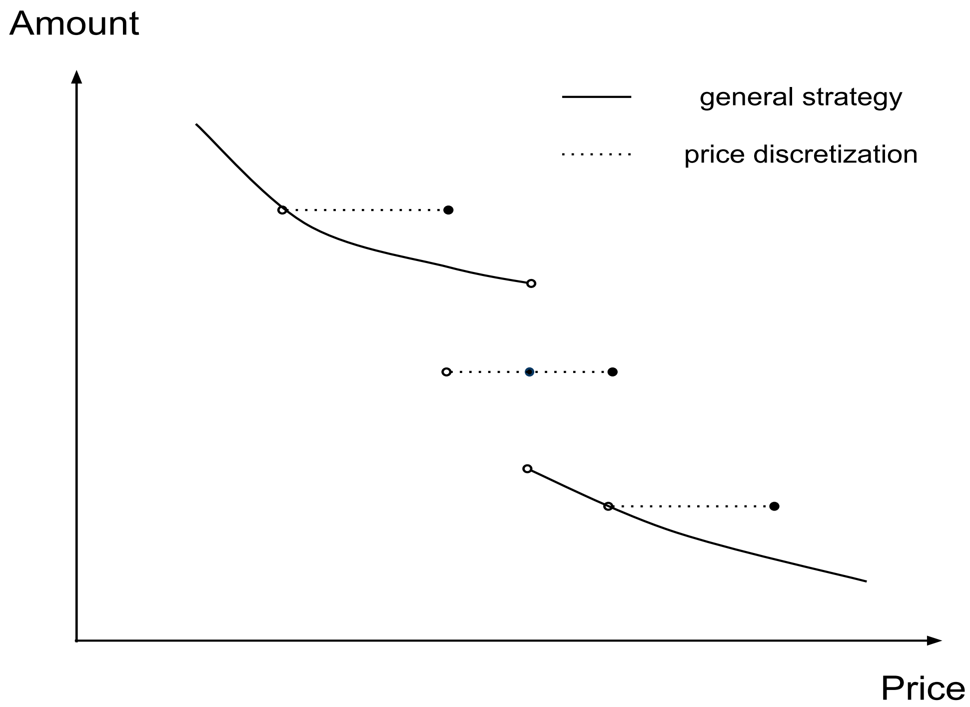

Figure 1: Illustration how L^δ is approximated by $L^{\delta,m}$.

For any $\delta > 0$, $m \in \mathbb{N}$, and $l \in \{1, \ldots, m2^m\}$ we define

$$x^*_{l,m}(\omega, t) := \begin{cases} \min\left\{\operatorname{argmax}_{x\in((l-1)2^{-m},l2^{-m}]} \left(L^\delta(\omega,t,x-) - L^\delta(\omega,t,x+)\right)\right\} \\ \qquad \text{if } L^\delta(\omega,t,(l-1)2^{-m}) < \infty \text{ and } \sup_x \left(L^\delta(\omega,t,x-) - L^\delta(\omega,t,x+)\right) > 0, \\ (l-1)2^{-m} \\ \qquad \text{if } L^\delta(\omega,t,(l-1)2^{-m}) < \infty \text{ and } \sup_x \left(L^\delta(\omega,t,x-) - L^\delta(\omega,t,x+)\right) = 0, \\ \inf\{x \in \mathbb{R}_+ \mid L^\delta(\omega,t,x) < \infty\} \\ \qquad \text{if } L^\delta(\omega,t,(l-1)2^{-m}) = \infty \text{ and } L^\delta(\omega,t,l2^{-m}) < \infty, \\ (l-1)2^{-m} \\ \qquad \text{if } L^\delta(\omega,t,(l-1)2^{-m}) = \infty \text{ and } L^\delta(\omega,t,l2^{-m}) = \infty. \end{cases}$$

Definition 4.4 (1/m-price discretization)**.** *Let $\delta > 0$ and $m \in \mathbb{N}$. For any $l \in \{1, \dots, m2^m\}$ we define*

$$L^{\delta,m}(\omega,t,x) := \sum_{l=1}^{m2^m} \theta_t^{\delta,l,m}(\omega) 1_{\{(l-1)2^{-m}<x\leq l2^{-m}\}} + L^\delta(\omega,t,0)1_{\{x=0\}},$$

where

$$\theta_t^{\delta,l,m}(\omega) := \begin{cases} L^\delta(\omega,t,x^*_{l,m}(\omega,t)), & \text{if } L^\delta(\omega,t,x^*_{l,m}(\omega,t)) < \infty, \\ L^\delta(\omega,t,l2^{-m}), & \text{otherwise.} \end{cases}$$

Lemma 4.5. *For any $\delta > 0$ and $m \geq [-\log_2(\delta)] + 1 =: m_0$, where $[x] := \max\{k \in \mathbb{N}_0 \mid k \leq x\}$, we have that $L^{\delta,m} \in \mathcal{L}^B$. Furthermore $\sup_{m\in\{m_0,m_0+1,\dots\}} L^{\delta,m}$ is μ-integrable and $(L^{\delta,m})_{m\in\{m_0,m_0+1,\dots\}}$ converges to L^δ $\widetilde{M}_\mu$-a.e.*

Proof. Step 1: By Lemma 4.3, L^δ is $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}_+)$-measurable. In addition, we observe that for all $l = 1,\dots,m2^m$ the process $(\omega,t)\mapsto x^*_{l,m}(\omega,t)$ is predictable. This is the case because the location of the largest jump $L^\delta(\omega,t,x-)-L^\delta(\omega,t,x+)$ for $x \in ((l-1)2^m, l2^m]$ can be expressed by suprema and pointwise limits of distances between elements of $\{(\omega,t)\mapsto L^\delta(\omega,t,q) \mid q \in \mathbb{Q}_+\}$ (the detailed proof which makes use of the monotonicity of $x \mapsto L^\delta(\omega,t,x)$ is straightforward but somewhat tedious and left to the reader). Consequently, $(\omega,t)\mapsto L^\delta(\omega,t,x^*_{l,m}(\omega,t))$ is a composition of the $\mathcal{P}/(\mathcal{P}\otimes\mathcal{B}(\mathbb{R}_+))$-measurable function $(\omega,t)\mapsto(\omega,t,x^*_{l,m}(\omega,t))$ and the $(\mathcal{P}\otimes\mathcal{B}(\mathbb{R}_+))/\mathcal{B}(\overline{\mathbb{R}}_+)$-measurable function $(\omega,t,x)\mapsto L^\delta(\omega,t,x)$ and thus $\mathcal{P}/\mathcal{B}(\overline{\mathbb{R}}_+)$-measurable, i.e. predictable.

The monotonicity of $L^{\delta,m}$ follows immediately from the monotonicity of $x \mapsto L^\delta(\omega,t,x)$. Moreover, by construction of $L^{\delta,m}$, the largest x for which $L^{\delta,m}(\omega,t,x) > 0$ holds, can only exceed the largest x for which $L^\delta(\omega,t,x) > 0$ holds by at most 2^{-m}. Thus, we have that

$$L^{\delta,m} = 0 \quad \text{on} \quad \{(\omega,t,x)\in\Omega\times[0,T]\times\mathbb{R}_+ \mid x > \overline{S}_{t-}(\omega)-\delta\} \quad \forall m \geq m_0. \tag{4.11}$$

Consequently, part (ii) of Definition 3.3 is satisfied.

Step 2: Let us now show that $\sup_{m\in\{m_0,m_0+1,\dots\}} L^{\delta,m}$ is μ-integrable. Let $(\omega,t,x)\in\Omega\times[0,T]\times\mathbb{R}_+$ s.t. $L^\delta(\omega,t,x) < \infty$.

Case 1: $L^\delta(x,t,x-) < \infty$, i.e. there exists $\varepsilon > 0$ s.t. $L^\delta(x,t,x-\varepsilon) < \infty$. We have that $x^*_{l_m,m}(\omega,t) \geq x-\varepsilon$ for all m up to finitely many, where $l_m = l_m(x)$ satisfy $(l_m-1)2^{-m} < x \leq l_m 2^{-m}$. In addition, we have that $L^{\delta,m}(\omega,t,x) < \infty$ for any $m \in \mathbb{N}$.

Case 2: $L^\delta(x,t,x-) = \infty$. Then, $x^*_{l_m,m}(\omega,t) = x$ for all $m \in \mathbb{N}$.

Thus, in both cases we arrive at

$$\sup_{m\in\{m_0,m_0+1,\dots\}} L^{\delta,m}(\omega,t,x) < \infty. \tag{4.12}$$

Together this implies that (4.12) holds M_μ-a.e. as $\{L^\delta = \infty\}$ is a M_μ-null set. In addition, we have that $\mu(\omega,[0,T]\times(\overline{S}_{t-}(\omega)-\delta,\infty)) < \infty$ (Assumption 3.1(iii) combined with the fact that $\overline{S}$ is càdlàg). Due to (4.11), this already implies that $\sup_{m\in\{m_0,m_0+1,\dots\}} L^{\delta,m} < \infty$ is μ-integrable.

Step 3: Let us now deal with the *convergence* part of the lemma. Fix a $(\omega,t,x)\in\widetilde{\Omega}$ with $L^\delta(\omega,t,x) < \infty$.

Case 1: $L^\delta(\omega,t,x-) = L^\delta(\omega,t,x+) < \infty$.

For any $\varepsilon > 0$ there exists a constant $c_\varepsilon(\omega,t,x) > 0$ s.t. for all $y \in (x-c_\varepsilon, x+c_\varepsilon)$ it holds that $|L^\delta(\omega,t,y) - L^\delta(\omega,t,x)| < \varepsilon$. Thus, for all m large enough s.t. $((l_m-1)2^{-m}, l_m 2^{-m}] \subset (x-c_\varepsilon, x+c_\varepsilon)$ we have that $|L^{\delta,m}(\omega,t,x) - L^\delta(\omega,t,x)| < \varepsilon$.

Case 2: $L^\delta(\omega,t,x-) > L^\delta(\omega,t,x+)$.

Clearly, this implies $x^*_{l_m,m}(\omega,t) = x$ for all m large enough, thus $L^\delta(\omega,t,x) = L^{\delta,m}(\omega,t,x)$ holds for all m large enough.

The case differentiation above yields the convergence for all $(\omega,t,x) \in \widetilde{\Omega}$ s.t. $L^\delta(\omega,t,x) < \infty$. It remains to show that $\{(\omega,t,x) \in \widetilde{\Omega} \mid L^\delta(\omega,t,x) = \infty\}$ is a $\widetilde{M}_\mu$-null set. It is clear that the set is a M_μ-null set as L^δ is μ-integrable. However, we still have to verify that for all $q \in \mathbb{Q}_+$

$$\left(\widehat{M}_\mu \otimes \delta_q\right)\left(\{L^\delta = \infty\} \cap \mathrm{supergraph}(\underline{X})\right) = 0,$$

i.e. $\widehat{M}_\mu(A_q) = 0$, where $A_q := \{(\omega,t) \in \Omega \times [0,T] \mid \underline{X}(\omega,t) < q \text{ and } L^\delta(\omega,t,q) = \infty\}$.

Assume that $\widehat{M}_\mu(A_q) > 0$. Then the predictable process $\widetilde{X}_t(\omega) := q 1_{A_q}(\omega,t) + \underline{X}_t(\omega) 1_{A_q^c}(\omega,t)$ is *not* another version (besides $\underline{X}$) of the essential supremum introduced in Definition 3.15. Consequently, there exists an $i \in \mathbb{N}$ with

$$P\left(\{\omega \in \Omega \mid Y_i(\omega) < q \text{ and } (\omega,\tau_i(\omega)) \in A_q\}\right) = P\left(Y_i < \widetilde{X}_{\tau_i}\right) > 0,$$

which would imply by the monotonicity of $y \mapsto L^\delta(\omega,t,y)$ that $P(\{\omega \in \Omega \mid L^\delta(\omega,\tau_i(\omega),Y_i(\omega)) = \infty\}) > 0$. But, this is a contradiction to the μ-integrability of L^δ.

□

Lemma 4.6. *Let $(H^n)_{n\in\mathbb{N}}$ be a sequence of $\overline{\mathbb{R}}$-valued and $\widetilde{\mathcal{F}}$-measurable functions that converges M_μ-a.e. to an $\overline{\mathbb{R}}$-valued and $\widetilde{\mathcal{F}}$-measurable function H. Suppose there exists an $\overline{\mathbb{R}}$-valued and $\widetilde{\mathcal{F}}$-measurable function K, which is μ-integrable and dominates $(H^n)_{n\in\mathbb{N}}$, i.e. $|H^n| \le K$ M_μ-a.e. for all $n \in \mathbb{N}$. Then $(H^n)_{n\in\mathbb{N}}$ and H are μ-integrable and $(\int_{[0,\cdot)\times\mathbb{R}_+} H^n d\mu)_{n\in\mathbb{N}}$ converges to $\int_{[0,\cdot)\times\mathbb{R}_+} H d\mu$ uniformly in probability.*

Proof. Let $N \in \widetilde{\mathcal{F}}$ with $M_\mu(N) = 0$ and $H^n \to H$, $|H^n| \le K$ on N^c. By Fubini's theorem for transition kernels we obtain that $\mu(\omega, N_\omega) = 0$ for P-a.a. $\omega \in \Omega$. By dominated convergence we obtain that

$$\int_{[0,T]\times\mathbb{R}_+} |H^n(\omega,s,x) - H(\omega,s,x)|\, \mu(\omega,ds,dx) \to 0, \quad n \to \infty,$$

for all $\omega \in \Omega$ with $\mu(\omega,N_\omega) = 0$ and $\int K d\mu(\omega,\cdot) < \infty$. As K is assumed to be μ-integrable, we have that $P(\int K d\mu < \infty) = 1$ and thus $(\int_{[0,\cdot)\times\mathbb{R}_+} H^n d\mu)_{n\in\mathbb{N}}$ converges to $\int_{[0,\cdot)\times\mathbb{R}_+} H d\mu$ uniformly in probability. □

Lemma 4.7. *Let $(L^{B,n})_{n\in\mathbb{N}}$ and L^B be in $\mathcal{L}^B$. Furthermore, assume that $(L^{B,n})_{n\in\mathbb{N}}$ converges $\widetilde{M}_\mu$-a.e. towards L^B and that $\sup_{n\in\mathbb{N}} L^{B,n}$ is μ-integrable. Then for $n \to \infty$*

$$\int_{[0,\cdot)\times\mathbb{R}_+} \int_x^\infty y L^{B,n}(s,dy)\mu(ds,dx) \to \int_{[0,\cdot)\times\mathbb{R}_+} \int_x^\infty y L^B(s,dy)\mu(ds,dx),$$

uniformly in probability.

Similarly, let $(L^{S,n})_{n\in\mathbb{N}}$ and L^S be in $\mathcal{L}^S$. Furthermore assume that $(L^{S,n})_{n\in\mathbb{N}}$ converges $\widetilde{M}_\nu$-a.e. towards L^S and that $\sup_{n\in\mathbb{N}} L^{S,n}$ is ν-integrable. Then for $n \to \infty$

$$\int_{[0,\cdot)\times\mathbb{R}_+} \int_0^x y L^{S,n}(s,dy)\nu(ds,dx) \to \int_{[0,\cdot)\times\mathbb{R}_+} \int_0^x y L^S(s,dy)\nu(ds,dx),$$

uniformly in probability.

Note that the convergence has to hold $\widetilde{M}_\mu$-a.e., it is not sufficient to assume convergence only M_μ-a.e..

Proof. We only prove the first part of the lemma, because the proof of the second part is completely analog. Let $\widetilde{N}$ be a $\widetilde{M}_\mu$-null set s.t. $(L^{B,n})_{n\in\mathbb{N}}$ converges pointwise towards L^B on $\widetilde{N}^c$.

Step 1: Let us show that

$$H^n(\omega,t,x) := \int_x^\infty y L^{B,n}(\omega,t,dy) = \int_x^{\overline{S}_{t-}(\omega)} y L^{B,n}(\omega,t,dy)$$

converges pointwise to

$$H(\omega,t,x) := \int_x^\infty y L^{B}(\omega,t,dy) = \int_x^{\overline{S}_{t-}(\omega)} y L^{B}(\omega,t,dy)$$

for all $(\omega,t,x)\in N^c$, where

$$N := \widetilde{N} \cup \operatorname{subgraph}(\underline{\mathrm{X}}) \cup \bigcup_{q\in\mathbb{Q}_+} \{(\omega,t,x)\in\widetilde{\Omega} \mid (\omega,t,q)\in\widetilde{N}\cap\operatorname{supergraph}(\underline{X})\} \cup \{L^B=\infty\}.$$

Fix any $(\omega,t,x)\in N^c$. For any $\varepsilon>0$ choose $K\in\mathbb{N}$ and $y_1<\ldots<y_K$ in $\mathbb{Q}_+$ s.t. $x=:y_0<y_1$, $y_K\geq \overline{S}_{t-}(\omega)$ and $y_i-y_{i-1}<\varepsilon$ for all $i\in\{1,\ldots,K\}$. As $(\omega,t,x)\notin\operatorname{subgraph}(\underline{\mathrm{X}})$, and $y_i>x$ for $i\geq 1$, we have that $(\omega,t,y_i)\in\operatorname{supergraph}(\underline{\mathrm{X}})$ for $i\geq 1$. Thus, we have that $L^{B,n}(\omega,t,y_i)\to L^B(\omega,t,y_i)$ for $n\to\infty$. For any strategy $\widetilde{L}^B$ we have that

$$\begin{aligned}
\sum_{i=1}^K y_{i-1}\left(\widetilde{L}^B(\omega,t,y_i)-\widetilde{L}^B(\omega,t,y_{i-1})\right) &\geq \int_x^{\overline{S}_{t-}(\omega)} y\widetilde{L}^B(\omega,t,dy)\\
&= \int_{y_0}^{y_K} y\widetilde{L}^B(\omega,t,dy)\\
&\geq \sum_{i=1}^K y_i\left(\widetilde{L}^B(\omega,t,y_i)-\widetilde{L}^B(\omega,t,y_{i-1})\right)
\end{aligned}$$

(Note that $y\mapsto\widetilde{L}^B(\omega,t,y)$ is nonincreasing). By $L^{B,n}(\omega,t,y_i)\to L^B(\omega,t,y_i)$ for all $i=0,\ldots,K$ as $n\to\infty$ this implies that

$$\begin{aligned}
&\liminf_{n\to\infty}\int_x^{\overline{S}_{t-}(\omega)} yL^{B,n}(\omega,t,dy)\\
\geq\ & \sum_{i=1}^K y_i\left(L^B(\omega,t,y_i)-L^B(\omega,t,y_{i-1})\right)\\
\geq\ & \sum_{i=1}^K (y_{i-1}+\varepsilon)\left(L^B(\omega,t,y_i)-L^B(\omega,t,y_{i-1})\right)\\
\geq\ & -\varepsilon L^B(\omega,t,y_0)+\sum_{i=1}^K y_{i-1}\left(L^B(\omega,t,y_i)-L^B(\omega,t,y_{i-1})\right)\\
\geq\ & -\varepsilon L^B(\omega,t,y_0)+\sum_{i=1}^K \int_{y_{i-1}}^{y_i} yL^B(\omega,t,dy)\\
=\ & -\varepsilon L^B(\omega,t,x)+\int_x^{\overline{S}_{t-}(\omega)} yL^B(\omega,t,dy).
\end{aligned}$$

Since ε can be chosen arbitrarily small and $L^B(\omega,t,x)<\infty$ by construction of N, this yields

$$\liminf_{n\to\infty} H^n(\omega,t,x)\geq H(\omega,t,x).$$

Analogously, we obtain that $\limsup_{n\to\infty} H^n(\omega,t,x)\leq H(\omega,t,x)$ and thus

$$H^n(\omega,t,x)\to H(\omega,t,x)\quad \forall(\omega,t,x)\in N^c.$$

Step 2: Let us show that $M_\mu(N)=0$. By $\widetilde{M}_\mu(\widetilde{N})=0$, we have that $M_\mu(\widetilde{N})=0$ and $\widehat{M}_\mu(\{(\omega,t)\mid \exists q\in\mathbb{Q}_+ \text{ s.t. } (\omega,t,q)\in(\widetilde{N}\cap\text{supergraph}(\underline{X}))=0$. In addition, we use that $M_\mu(\text{subgraph}(\underline{X}))=0$ and $M_\mu(\{L^B=\infty\})=0$ to arrive at

$$\begin{aligned}M_\mu(N) &\leq M_\mu(\widetilde{N})+M_\mu(\text{subgraph}(\underline{X}))\\ &+ M_\mu(\{(\omega,t,x)\in\widetilde{\Omega}\mid \exists q\in\mathbb{Q}_+ \text{ s.t. } (\omega,t,q)\in\widetilde{N}\cap\text{supergraph}(\underline{X})\})+M_\mu(\{L^B=\infty\})\\ &= 0+0+0+0=0.\end{aligned}$$

Now note that M_μ-a.e. we have

$$0\geq H^n(\omega,t,x)=\int_x^{\overline{S}_{t-}(\omega)} yL^{B,n}(\omega,t,dy)\geq -\sup_{t\in[0,T]}\overline{S}_{t-}(\omega)\sup_{n\in\mathbb{N}}L^{B,n}(\omega,t,x),$$

i.e. $(H^n)_{n\in\mathbb{N}}$ is dominated by $\sup_{t\in[0,T]}\overline{S}_{t-}\sup_{n\in\mathbb{N}}L^{B,n}$, which is clearly μ-integrable since $\sup_{n\in\mathbb{N}}L^{B,n}$ is μ-integrable by assumption. Thus an application of Lemma 4.6 completes the proof.

□

So far, we have already shown that the portfolio process of L^B can be approximated by the portfolio process of $L^{\delta,m}$. $L^{\delta,m}$ is piecewise constant in the price variable, but not in time. In the following theorem $L^{\delta,m}$ (and thus also L^B) is approximated by $\widehat{L}$ which is piecewise constant both in the price and the time variable.

Theorem 4.8. *For any $\varepsilon>0$ and any $L^B\in\mathcal{L}^B$ there exist $A^\varepsilon\in\mathcal{F}$, $\delta>0$, $m\in\mathbb{N}$, and nonnegative simple predictable processes $\widehat{\xi}^0,\widehat{\xi}^1,\ldots,\widehat{\xi}^{m2^m}$ vanishing on $\Omega\times\{0\}$ s.t. $P(A^\varepsilon)\geq 1-\varepsilon$ and*

$$\widehat{\xi}^l=0,\qquad \text{on}\qquad \{(\omega,t)\in\Omega\times[0,T]\mid \overline{S}_{t-}(\omega)-\delta\leq l2^{-m}\}.$$

Furthermore for

$$\widehat{L}(\omega,t,x):=\sum_{l=0}^{m2^m}\widehat{\xi}^l_t(\omega)1_{\{x\leq l2^{-m}\}}$$

we have that $\widehat{L}\in\mathcal{L}^B$ and for every $\omega\in A^\varepsilon$

$$\sup_{t\in[0,T]}\Big|\int_{[0,t)\times\mathbb{R}_+}L^B(\omega,s,x)\mu(\omega,ds,dx)-\int_{[0,t)\times\mathbb{R}_+}\widehat{L}(\omega,s,x)\mu(\omega,ds,dx)\Big|<\varepsilon$$

and

$$\sup_{t\in[0,T]}\Big|\int_{[0,t)\times\mathbb{R}_+}\int_x^\infty yL^B(\omega,s,dy)\mu(\omega,ds,dx)-\int_{[0,t)\times\mathbb{R}_+}\int_x^\infty y\widehat{L}(\omega,s,dy)\mu(\omega,ds,dx)\Big|<\varepsilon.$$

Proof. Step 1: Let $\varepsilon > 0$. By Lemma 4.3, Lemma 4.5, Lemma 4.6, Lemma 4.7, and the fact that the up-convergence is metrizable, it is possible to choose at first a $\delta > 0$ small enough and afterwards an $m \in \mathbb{N}$ large enough s.t. there exists a set $U \in \mathcal{F}$ s.t. $P(U) \geq 1 - \frac{\varepsilon}{3}$ and for all $\omega \in U$

$$\sup_{t\in[0,T]} \left| \int_{[0,t)\times\mathbb{R}_+} L^B(\omega, s, x)\mu(\omega, ds, dx) - \int_{[0,t)\times\mathbb{R}_+} L^{\delta,m}(\omega, s, x)\mu(\omega, ds, dx) \right| < \frac{\varepsilon}{2} \quad \text{and} \tag{4.13}$$

$$\sup_{t\in[0,T]} \left| \int_{[0,t)\times\mathbb{R}_+} \int_x^\infty yL^B(\omega, s, dy)\mu(\omega, ds, dx) - \int_{[0,t)\times\mathbb{R}_+} \int_x^\infty yL^{\delta,m}(\omega, s, dy)\mu(\omega, ds, dx) \right| < \frac{\varepsilon}{2}$$

holds. Furthermore, if we choose δ at least as small as $\frac{\varepsilon}{3}$ by the definition of $\overline{S}^\delta$ there exists a set $V \in \mathcal{F}$ s.t. $P(V) \geq 1 - \frac{\varepsilon}{3}$ and for all $\omega \in V$

$$\sup_{t\in[0,T]} |\overline{S}^\delta_t(\omega) - \overline{S}_{t-}(\omega)| \leq \delta \quad \text{and} \quad \tau^\delta(\omega) = \infty. \tag{4.14}$$

Finally, m can be chosen large enough s.t. $m > -\log_2(\delta)$.

Step 2: For any $\delta > 0$ we decompose μ into the executions triggered by the jumps of $\overline{S}$ with sizes lying in $[-\delta, 0)$ and the rest. More precisely, let

$$\mu = \mu^{1,\delta} + \mu^{2,\delta} \tag{4.15}$$

with $\mu^{1,\delta} \perp \mu^{2,\delta}$ and $\mu^{1,\delta}(\{t\} \times \{x\}) = 1$ iff $x = \overline{S}_t$ and $\Delta\overline{S}_t \in [-\delta, 0)$. Note that by (i), (iii), and (iv) of Assumption 3.1 and as $\overline{S}$ is càdlàg, $\mu^{2,\delta}$ is a finite random measure. By contrast, $\mu^{1,\delta}$ is in general infinite. Orders with limit prices below $\overline{S}_- - \delta$ cannot be executed by $\mu^{1,\delta}$.

Define $\xi^l_t(\omega) := \theta^{\delta,l,m}_t(\omega) - \theta^{\delta,l+1,m}_t(\omega)$ for all $l = 1, \ldots, m2^m - 1$, $\xi^{m2^m}_t(\omega) := \theta^{\delta,m2^m,m}_t(\omega)$, and $\xi^0_t(\omega) := L^{\delta,m}(\omega, t, 0) - \theta^{\delta,1,m}_t(\omega)$, where $\theta^{\delta,l,m}$, $l = 1, \ldots, m2^m$ are introduced in Definition 4.4. In addition define

$$A^l_t(\omega) := \mu^{2,\delta}(\omega, [0, t] \times [0, l2^{-m}])$$

for $l = 0, \ldots, m2^m$. Observe that we can use these processes to specify a representation of the shares bought and the cash payments resulting from strategy $L^{\delta,m}$ by

$$\begin{aligned}
\int_{[0,t)\times\mathbb{R}_+} L^{\delta,m}(s, x)\mu(ds, dx) &= \int_{[0,t)\times\mathbb{R}_+} L^{\delta,m}(s, x)\mu^{2,\delta}(ds, dx) \\
&= \sum_{l=0}^{m2^m} \xi^l \bullet A^l_{t-} \qquad \text{and}
\end{aligned}$$

$$\begin{aligned}
-\int_{[0,t)\times\mathbb{R}_+} \int_x^\infty yL^{\delta,m}(s, dy)\mu(ds, dx) &= -\int_{[0,t)\times\mathbb{R}_+} \int_x^\infty yL^{\delta,m}(s, dy)\mu^{2,\delta}(ds, dx) \\
&= \sum_{l=0}^{m2^m} l2^{-m}\xi^l \bullet A^l_{t-}.
\end{aligned}$$

By Assumption 3.1(vi), we have that $A^l_0 = 0$. Thus different conventions for the integral w.r.t. A at 0 do not matter. Note that we can replace μ by $\mu^{2,\delta}$ as by construction we have that $L^{\delta,m} = 0$ on $[\overline{S}_- - \delta, \infty)$. Since $L^{\delta,m}$ is μ-integrable, the integrability of any ξ^l w.r.t. A^l is satisfied. As $\mu^{2,\delta}(\omega, \cdot)$ is a finite measure for any $\omega \in \Omega$, there exists a probability measure $Q \sim P$ s.t.

$$E_Q\left[A^l_{T-}\right] < \infty \quad \text{and} \quad E_Q\left[\int_0^{T-} \xi^l \, dA^l\right] < \infty.$$

Then it is well-known (and provable by the monotone class theorem) that the predictable process ξ^l can be approximated by a simple predictable process $\widetilde{\xi}^l$ in the sense that

$$E_Q\left[\int_0^{T-} |\xi^l - \widetilde{\xi}^l| dA^l\right] \tag{4.16}$$

gets arbitrarily small. As ξ^l is nonnegative, $\widetilde{\xi}^l$ can be chosen to be nonnegative as well. By $A_0^l = 0$, $\widetilde{\xi}^l$ can be chosen to vanish on $\Omega \times \{0\}$. Since $L^1(Q)$-convergence implies convergence in Q- resp. P-probability, $\widetilde{\xi}^l$ can be chosen s.t. on a set $U^l \in \mathcal{F}$ with $P(U^l) \geq 1 - \frac{\varepsilon}{3(m2^m+1)}$ it holds that

$$\int_0^{T-} |\xi^l - \widetilde{\xi}^l| dA^l < \frac{\varepsilon}{2m(m2^m+1)}. \tag{4.17}$$

Define the process

$$\widehat{\xi}^l := \widetilde{\xi}^l 1_{\{\overline{S}^\delta - 2\delta > l2^{-m}\}} 1_{[\![0,\tau^\delta]\!]}$$

which is simple predictable as $\overline{S}^\delta$ and $\widetilde{\xi}^l$ are simple predictable. By $A_0^l = 0$, we can choose $\widetilde{\xi}_0^l = 0$ and thus $\widehat{\xi}_0^l = 0$. By construction of $L^{\delta,m}$, we also have for ξ^l that $\xi^l = 0$ on $\{\overline{S}^\delta - 2\delta < l2^{-m}\} \cup]\!]\tau^\delta, T]\!]$. Thus, (4.17) implies

$$\int_0^{T-} |\xi^l - \widehat{\xi}^l| dA^l < \frac{\varepsilon}{2m(m2^m+1)} \tag{4.18}$$

on U^l. In addition, we have

$$\widehat{\xi}^l = 0 \qquad \text{on } \{(\omega, t) \in \Omega \times [0,T] \mid \overline{S}_{t-}(\omega) - \delta \leq l2^{-m}\}. \tag{4.19}$$

Now (4.18) clearly implies that on U^l it holds that

$$\sup_{t\in[0,T]} |\xi^l \bullet A_{t-}^l - \widehat{\xi}^l \bullet A_{t-}^l| < \frac{\varepsilon}{2m(m2^m+1)},$$

and because $l2^{-m} \leq m$ we also have

$$\sup_{t\in[0,T]} |l2^{-m}\xi^l \bullet A_{t-}^l - l2^{-m}\widehat{\xi}^l \bullet A_{t-}^l| < \frac{\varepsilon}{2(m2^m+1)}$$

on U^l. Hence on $\bigcap_{l=0}^{m2^m} U^l$ we arrive at

$$\sup_{t\in[0,T]} \left| \int_{[0,t)\times\mathbb{R}_+} L^{\delta,m}(\omega,s,x)\mu(\omega,ds,dx) - \int_{[0,t)\times\mathbb{R}_+} \widehat{L}(\omega,s,x)\mu(\omega,ds,dx)\right| < \frac{\varepsilon}{2m} \quad \text{and} \tag{4.20}$$
$$\sup_{t\in[0,T]} \left| \int_{[0,t)\times\mathbb{R}_+} \int_x^\infty yL^{\delta,m}(\omega,s,dy)\mu(\omega,ds,dx) - \int_{[0,t)\times\mathbb{R}_+} \int_x^\infty y\widehat{L}(\omega,s,dy)\mu(\omega,ds,dx)\right| < \frac{\varepsilon}{2}.$$

Now we only have to make certain that (4.13), (4.14), (4.19), and (4.20) all hold on the same set A^ε, which is easily achieved by setting $A^\varepsilon := U \cap V \cap \left(\bigcap_{l=0}^{m2^m} U^l\right)$. □

Let $\widehat{\xi}^0, \widehat{\xi}^1, \ldots, \widehat{\xi}^{m2^m}$ be the nonnegative simple predictable processes vanishing on $\Omega \times \{0\}$ from Theorem 4.8. Note that any $\widehat{\xi}^l$ can be written as a finite sum of terms of the form $\widehat{\xi}^{l,i} 1_{]\!]T_1^{l,i}, T_2^{l,i}]\!]}$, where $T_1^{l,i} \leq T_2^{l,i}$ are stopping times and $\widehat{\xi}^{l,i} > 0$ is $\mathcal{F}_{T_1^{l,i}}$-measurable. To finish the proof of Theorem 3.13 it is sufficient to show that the portfolio process of $\mathfrak{S} =$

$(0,0,\widehat{\xi}^{l,i}1_{]\!]T_1^{l,i},T_2^{l,i}]\!]}1_{\{x\le l2^{-m}\}},0)$ can be approximated uniformly in probability by portfolio processes of real-world trading strategies. We define the sequence of stopping times

$$\overline{\tau}_0 := T_1^{l,i}, \qquad \overline{\tau}_j := \inf\left\{t > \overline{\tau}_{j-1} : \mu^{2,\delta}\left(\{t\}\times[0,l2^{-m}]\right) > 0\right\}, \quad j\in\mathbb{N},$$

where $\mu^{2,\delta}$ refers to the finite measure defined in (4.15). Thus, we get $P(\overline{\tau}_j \ge T_2^{l,i}) \uparrow 1$ as $j\to\infty$. Let $\varepsilon>0$. There exists a $K\in\mathbb{N}$ s.t. $P(\overline{\tau}_K \ge T_2^{l,i}) \ge 1-\varepsilon$. By Theorem 4.8 we have that

$$]\!]T_1^{l,i},T_2^{l,i}]\!] \subset \{(\omega,t)\in\Omega\times[0,T] \mid \overline{S}_{t-}(\omega)-\delta > l2^{-m}\}. \tag{4.21}$$

(4.21) implies that on $\{\overline{\tau}_K \ge T_2^{l,i}\}\times[0,T]\times\mathbb{R}_+$ it holds that

$$\begin{aligned}
&\widehat{\xi}^{l,i}1_{]\!]T_1^{l,i},T_2^{l,i}]\!]}1_{\{x\le l2^{-m}\}} = \sum_{j=1}^{K}\widehat{\xi}^{l,i}1_{]\!]\overline{\tau}_{j-1}\wedge T_2^{l,i},\overline{\tau}_j\wedge T_2^{l,i}]\!]}1_{\{x\le l2^{-m}\}}\\
&= \sum_{j=1}^{K}\widehat{\xi}^{l,i}1_{]\!]\overline{\tau}_{j-1}\wedge T_2^{l,i},\overline{\tau}_j\wedge T_2^{l,i}]\!]}\left(1_{\{x\le l2^{-m},\overline{S}_{t-}>l2^{-m}\}} + 1_{\{x<l2^{-m},\overline{S}_{t-}=l2^{-m}\}}\right).
\end{aligned} \tag{4.22}$$

The strategy in the second line of (4.22) is a real-world buying strategy with $M^B=0$ in the sense of (3.6). Namely, T^* describing the execution time of the order placed at time $\overline{\tau}_{j-1}\wedge T_2^{l,i}$, as defined in (3.4), satisfies $T^*=\overline{\tau}_j$ on $\{\overline{\tau}_j\le T_2^{l,i}\}$, again by (4.21) which makes an execution triggered by $\mu^{1,\delta}$ impossible. With $\varepsilon\to 0$, the strategies in (4.22) and thus their portfolio processes coincide on a set with probability tending to one. Together with Theorem 4.8 and the triangle inequality of d_{up} this proves Theorem 3.13.

5 Proof of Theorem 3.17: Closedness of the strategy set

In the whole section let the assumptions of Theorem 3.17 hold and let $(\varphi^{0,n},\varphi^{1,n})_{n\in\mathbb{N}}$ with $\varphi^{0,n}:=\varphi^0(\mathfrak{S}^n)$ and $\varphi^{1,n}=\varphi^1(\mathfrak{S}^n)$ be an up-Cauchy sequence where $(\mathfrak{S}^n)_{n\in\mathbb{N}} = (M^{B,n},M^{S,n},L^{B,n},L^{S,n})_{n\in\mathbb{N}}$ is an a-admissible sequence of trading strategies.

Since the space of làdlàg functions (also called regulated functions) mapping from $[0,T]$ to $\mathbb{R}$ is complete w.r.t. the supremum norm, there exist predictable làdlàg processes ψ^0 and ψ^1 s.t. $(\varphi^{0,n})_{n\in\mathbb{N}}$ converges uniformly in probability to ψ^0 and $(\varphi^{1,n})_{n\in\mathbb{N}}$ converges uniformly in probability to ψ^1. By going to a subsequence of $(\mathfrak{S}^n)_{n\in\mathbb{N}}$ we can assume w.l.o.g. that $\left(\varphi^{0,n},\varphi^{1,n}\right)_{n\in\mathbb{N}}$ even converges (component wise) P-a.s. uniformly on $[0,T]$ to (ψ^0,ψ^1).

We have to show that (ψ^0,ψ^1) can be generated by some admissible strategy $\mathfrak{S}$. To get briefly an overview the reader may directly pass to the Proof of Theorem 3.17. Before, several lemmas are stated which are needed for this proof.

Lemma 5.1. *Let $\widehat{\tau}^0:=0$ and for $k,n\in\mathbb{N}$ define the stopping times*

$$\begin{aligned}
\widehat{\tau}^{k,n} &:= \inf\{t>0 \mid |\varphi_t^{0,n}|>k\}\wedge\inf\{t>0 \mid |\varphi_t^{1,n}|>k\}\\
&\quad\wedge\inf\{t>0 \mid \int_{[0,t]\times\mathbb{R}_+}1_{[\underline{S}_s,\overline{S}_s)}(x)\mu(ds,dx)+\int_{[0,t]\times\mathbb{R}_+}1_{(\underline{S}_s,\overline{S}_s]}(x)\nu(ds,dx)>k\}\wedge T,\\
\widehat{\tau}^k &:= \inf_{n\in\mathbb{N}}\widehat{\tau}^{k,n}.
\end{aligned}$$

There exists a probability measure Q equivalent to P s.t. for all $k\in\mathbb{N}$ there is a $K_k\in\mathbb{R}_+$ with

$$E_Q\left[\mathrm{var}(\varphi^{0,n})_{\widehat{\tau}^k}+\mathrm{var}(\varphi^{1,n})_{\widehat{\tau}^k}\right]\le K_k, \qquad \forall n\in\mathbb{N}.$$

Furthermore, $(\widehat{\tau}^k)_{k\in\mathbb{N}}$ *is an increasing sequence of stopping times with* $P(\widehat{\tau}^k = T) \to 1$, $k \to \infty$, *i.e. it is localizing.*

The idea of the proof is to reduce the result to a similar result of Campi and Schachermayer [3] for proportional transaction costs. To do so we have to separate those limit order executions which are more favorable than trades with market orders at the same time.

Proof. Let $\widehat{\sigma}^0 := 0$ and

$$\widehat{\sigma}^i := \inf\left\{ t > \widehat{\sigma}^{i-1} \mid \int_{[0,t]\times\mathbb{R}_+} 1_{[\underline{S}_s,\overline{S}_s)}(x)\mu(ds,dx) + \int_{[0,t]\times\mathbb{R}_+} 1_{(\underline{S}_s,\overline{S}_s]}(x)\nu(ds,dx) \geq i \right\}, \quad i \in \mathbb{N}.$$

$\widehat{\tau}^k$ and $\widehat{\sigma}^i$ are debuts of optional sets and thus stopping times by the usual conditions (cf. e.g. Theorem 4.30 of [10]). Note that we have $\widehat{\sigma}^i \geq \widehat{\tau}^k$ for all $i > k$. From the definition of $\widehat{\tau}^k$ and the observation that $\varphi^{0,n}_{t\wedge\widehat{\tau}^k} = \varphi^{0,n}_{\widehat{\tau}^k}$, $\varphi^{1,n}_{t\wedge\widehat{\tau}^k} = \varphi^{1,n}_{\widehat{\tau}^k}$ for all $t \geq \widehat{\tau}^k$, we see that $|\Delta^+(\varphi^{0,n}_{\cdot\wedge\widehat{\tau}^k})_t|, |\Delta^+(\varphi^{1,n}_{\cdot\wedge\widehat{\tau}^k})_t| \leq 2k$ for all $t \in [0,T]$ and thus

$$\sum_{i=0}^{\infty} \left(\Delta^+\text{var}\left(\varphi^{0,n}_{\cdot\wedge\widehat{\tau}^k}\right)_{\widehat{\sigma}^i} + \Delta^+\text{var}\left(\varphi^{1,n}_{\cdot\wedge\widehat{\tau}^k}\right)_{\widehat{\sigma}^i} \right) \leq 4k(k+1). \tag{5.23}$$

For any $(\varphi^{0,n}, \varphi^{1,n})$ and each $i = 1, 2, \ldots, k+1$ we define a self-financing, admissible portfolio process in the sense of Campi and Schachermayer (see [3] for details) with initial endowment $\varphi^{0,n,k,i}_0 = \varphi^{1,n,k,i}_0 = k$ and threshold level a by

$$\begin{aligned} \varphi^{0,n,k,i} &:= k1_{[\![0,\widehat{\sigma}^{i-1}\wedge\widehat{\tau}^k]\!]} + \varphi^{0,n}1_{]\!]\widehat{\sigma}^{i-1}\wedge\widehat{\tau}^k,\widehat{\sigma}^i\wedge\widehat{\tau}^k]\!]} - a1_{]\!]\widehat{\sigma}^i\wedge\widehat{\tau}^k,T]\!]}, \\ \varphi^{1,n,k,i} &:= k1_{[\![0,\widehat{\sigma}^{i-1}\wedge\widehat{\tau}^k]\!]} + \varphi^{1,n}1_{]\!]\widehat{\sigma}^{i-1}\wedge\widehat{\tau}^k,\widehat{\sigma}^i\wedge\widehat{\tau}^k]\!]} - a1_{]\!]\widehat{\sigma}^i\wedge\widehat{\tau}^k,T]\!]}. \end{aligned}$$

By construction $(\varphi^{0,n,k,i}, \varphi^{1,n,k,i}) = (\varphi^{0,n}, \varphi^{1,n})$ on $]\!]\widehat{\sigma}^{i-1}\wedge\widehat{\tau}^k, \widehat{\sigma}^i\wedge\widehat{\tau}^k]\!]$. Thus, $(\varphi^{0,n,k,i}, \varphi^{1,n,k,i})$ is certainly a-admissible. If $\widehat{\sigma}^{i-1}(\omega) < \widehat{\tau}^k(\omega)$, the change of the portfolio from (k,k) to $(\varphi^{0,n}_{\widehat{\sigma}^{i-1}(\omega)}(\omega), \varphi^{1,n}_{\widehat{\sigma}^{i-1}(\omega)}(\omega))$ at time $\widehat{\sigma}^{i-1}(\omega)$ is clearly self-financing. In addition, for $\widehat{\sigma}^{i-1}(\omega) \geq \widehat{\tau}^k(\omega)$ we have $(\widehat{\sigma}^{i-1}(\omega)\wedge\widehat{\tau}^k(\omega), \widehat{\sigma}^i(\omega)\wedge\widehat{\tau}^k(\omega)] = \emptyset$ and a possible change from (k,k) to $(-a,-a)$ is also self-financing. Furthermore, on $]\!]\widehat{\sigma}^{i-1}\wedge\widehat{\tau}^k, \widehat{\sigma}^i\wedge\widehat{\tau}^k]\!]$ no favorable executions of limit orders can influence the portfolio process (remember that a limit order executed at stopping time $\widehat{\sigma}^i$ only shows up in the portfolio process immediately after $\widehat{\sigma}^i$). While there may be executions of limit orders on $]\!]\widehat{\sigma}^{i-1}\wedge\widehat{\tau}^k, \widehat{\sigma}^i\wedge\widehat{\tau}^k[\![$, the prices paid by the small investor are at most as favorable as in the model with proportional transaction costs. If e.g. a limit buy order of size $\theta^B(\omega)$ with limit price $p^B(\omega)$ is executed at time $T^*(\omega)$ with $\widehat{\sigma}^{i-1}(\omega) < T^*(\omega) < \widehat{\sigma}^i(\omega)$ we know by construction that $\overline{S}_{T^*}(\omega) \leq p^B(\omega)$. Hence, the investor would be at least as well of just buying the amount $\theta^B(\omega)$ at time $T^*(\omega)$ with a market order at price $\overline{S}_{T^*}(\omega)$. Thus, $(\varphi^{0,n,k,i}, \varphi^{1,n,k,i})$ is indeed a self-financing portfolio process in a model with proportional transaction costs and price processes $\underline{S}$ and $\overline{S}$ if it is allowed to "throw away" assets. More precisely, if we translate $\{\underline{S}, \overline{S}\}$ with $P(\inf_{t\in[0,T]} \underline{S}_t > 0) = 1$ into the càdlàg bid-ask process

$$\Pi := \begin{pmatrix} 1 & \overline{S} \\ \frac{1}{\underline{S}} & 1 \end{pmatrix}$$

used in [3], then it is straightforward to show that $\widehat{V}^{n,k,i} := (\varphi^{0,n,k,i}, \varphi^{1,n,k,i})$ is a self-financing, admissible portfolio process with threshold a in the sense of Definition 2.7 in [3]. Therefore we can apply Lemma 3.2 in [3] to derive the existence of a probability measure $Q \sim P$ and a constant $C > 0$ s.t. for all $k, n \in \mathbb{N}$ and all $i = 1, \dots, k+1$

$$E_Q[\mathrm{var}(\varphi^{0,n,k,i})_T] \leq C(k+a) \quad \text{and} \quad E_Q[\mathrm{var}(\varphi^{1,n,k,i})_T] \leq C(k+a). \tag{5.24}$$

Right from the definition of $(\varphi^{0,n,k,i}, \varphi^{1,n,k,i})$ it follows that for all $n \in \mathbb{N}$ and $i = 1, 2, \dots, k+1$

$$\begin{aligned}
\mathrm{var}(\varphi^{0,n}_{\cdot \wedge \widehat{\tau}^k})_{\widehat{\sigma}^i} - \mathrm{var}(\varphi^{0,n}_{\cdot \wedge \widehat{\tau}^k})_{\widehat{\sigma}^{i-1}+} &= \mathrm{var}(\varphi^{0,n,k,i})_{\widehat{\sigma}^i} - \mathrm{var}(\varphi^{0,n,k,i})_{\widehat{\sigma}^{i-1}+}, \\
\mathrm{var}(\varphi^{1,n}_{\cdot \wedge \widehat{\tau}^k})_{\widehat{\sigma}^i} - \mathrm{var}(\varphi^{1,n}_{\cdot \wedge \widehat{\tau}^k})_{\widehat{\sigma}^{i-1}+} &= \mathrm{var}(\varphi^{1,n,k,i})_{\widehat{\sigma}^i} - \mathrm{var}(\varphi^{1,n,k,i})_{\widehat{\sigma}^{i-1}+}.
\end{aligned} \tag{5.25}$$

Combining (5.24) and (5.25) we obtain

$$\begin{aligned}
E_Q[\mathrm{var}(\varphi^{0,n}_{\cdot \wedge \widehat{\tau}^k})_{\widehat{\sigma}^i} - \mathrm{var}(\varphi^{0,n}_{\cdot \wedge \widehat{\tau}^k})_{\widehat{\sigma}^{i-1}+}] &\leq E_Q[\mathrm{var}(\varphi^{0,n,k,i})_T] \leq C(k+a), \\
E_Q[\mathrm{var}(\varphi^{1,n}_{\cdot \wedge \widehat{\tau}^k})_{\widehat{\sigma}^i} - \mathrm{var}(\varphi^{1,n}_{\cdot \wedge \widehat{\tau}^k})_{\widehat{\sigma}^{i-1}+}] &\leq E_Q[\mathrm{var}(\varphi^{1,n,k,i})_T] \leq C(k+a).
\end{aligned}$$

and thus

$$\begin{aligned}
& E_Q\left[\sum_{i=1}^{\infty}\left(\mathrm{var}(\varphi^{0,n}_{\cdot \wedge \widehat{\tau}^k})_{\widehat{\sigma}^i} - \mathrm{var}(\varphi^{0,n}_{\cdot \wedge \widehat{\tau}^k})_{\widehat{\sigma}^{i-1}+}\right) + \sum_{i=1}^{\infty}\left(\mathrm{var}(\varphi^{1,n}_{\cdot \wedge \widehat{\tau}^k})_{\widehat{\sigma}^i} - \mathrm{var}(\varphi^{1,n}_{\cdot \wedge \widehat{\tau}^k})_{\widehat{\sigma}^{i-1}+}\right)\right] \\
\leq \;& (k+1)2C(k+a).
\end{aligned} \tag{5.26}$$

By combining (5.23) and (5.26) the first part of the lemma is proven.

As discussed at the beginning of the section there exists a set $N \in \mathcal{F}$ s.t. $P(N) = 0$ and s.t. $\left(\varphi^{i,n}(\omega)\right)_{n \in \mathbb{N}}$ converges towards $\psi^i(\omega)$ uniformly on $[0, T]$ for all $\omega \in N^C$, $i = 0, 1$. Fix any $\omega \in N^C$. Remember that any làdlàg function is bounded on a compact interval. Thus, there exists an $n_0(\omega)$ s.t. for all $n \in \mathbb{N}$ we have

$$\sup_{t \in [0,T]} |\varphi^{i,n}_t(\omega)| \leq \left(\bigvee_{j=1}^{n_0(\omega)} \sup_{t \in [0,T]} |\varphi^{i,j}_t(\omega)|\right) \vee \left(\sup_{t \in [0,T]} |\psi^i_t(\omega)| + 1\right) < \infty, \quad i = 0, 1.$$

Hence, we have that $P\left(\sup_{n \in \mathbb{N}} \sup_{t \in [0,T]} |\varphi^{i,n}_t| < \infty\right) = 1$, $i = 0, 1$. By Assumption 3.1 (iii) we also have $P\left(\int_{[0,t] \times \mathbb{R}_+} 1_{[\underline{S}_s, \overline{S}_s)}(x) \mu(ds, dx) + \int_{[0,t] \times \mathbb{R}_+} 1_{(\underline{S}_s, \overline{S}_s]}(x) \nu(ds, dx) < \infty\right) = 1$. Hence, $(\widehat{\tau}^k)_{k \in \mathbb{N}}$ is localizing. $\square$

Lemma 5.2. *We have*

$$M_\mu\left(\{(\omega, t, x) \in \widetilde{\Omega} \mid \sup_{n \in \mathbb{N}} L^{B,n}(\omega, t, x) = \infty\}\right) = 0 \qquad \textit{and}$$
$$M_\nu\left(\{(\omega, t, x) \in \widetilde{\Omega} \mid \sup_{n \in \mathbb{N}} L^{S,n}(\omega, t, x) = \infty\}\right) = 0,$$

i.e. $(L^{B,n})_{n \in \mathbb{N}}$ *is* M_μ*-a.e. bounded and* $(L^{S,n})_{n \in \mathbb{N}}$ *is* M_ν*-a.e. bounded.*

Proof. We only deal with $(L^{B,n})_{n \in \mathbb{N}}$ as the assertion regarding $(L^{S,n})_{n \in \mathbb{N}}$ can be proved similarly. Each $L^{B,n}$ is μ-integrable and hence it holds that $L^{B,n} < \infty$ M_μ-almost everywhere. Thus, we can ignore the beginning of the sequence. Define the set A by

$$A := \{(\omega, t, x) \in \widetilde{\Omega} \mid \limsup_{n \to \infty} L^{B,n}(\omega, t, x) = \infty\} \in \widetilde{\mathcal{P}}$$

and for any $\varepsilon > 0$ let

$$B_\varepsilon := \text{supergraph}(\underline{X} + \varepsilon) = \{(\omega, t, x) \in \widetilde{\Omega} \mid \underline{X}(\omega, t) + \varepsilon < x\} \in \widetilde{\mathcal{P}}.$$

From the monotonicity of $x \mapsto L^{B,n}(\omega, t, x)$ it follows that

$$\begin{aligned}
\widehat{N}_\varepsilon &:= \{(\omega, t) \mid \exists x \in \mathbb{R}_+ \text{ s.t. } (\omega, t, x) \in A \cap B_\varepsilon\} \\
&= \{(\omega, t) \mid \exists x \in \mathbb{Q}_+ \text{ s.t. } (\omega, t, x) \in A \cap B_\varepsilon\} \\
&= \bigcup_{q \in \mathbb{Q}_+} \{(\omega, t) \mid (\omega, t, q) \in A \cap B_\varepsilon\} = \bigcup_{q \in \mathbb{Q}_+} (A \cap B_\varepsilon)_q
\end{aligned}$$

and we have that $\widehat{N}_\varepsilon \in \mathcal{P}$. Define

$$N_\varepsilon := \left\{(\omega, t, x) \in \widetilde{\Omega} \mid (\omega, t) \in \widehat{N}_\varepsilon,\ \underline{X}(\omega, t) \le x \le \underline{X}(\omega, t) + \frac{\varepsilon}{2}\right\} \in \widetilde{\mathcal{P}}.$$

Step 1: Let us show that $M_\mu(N_\varepsilon) = 0$. We have P-a.s.

$$\begin{aligned}
&\lim_{n\to\infty} \sup_{t\in[0,T)} |\Delta^+\varphi_t^{0,n}(\omega) - \Delta^+\psi_t^0(\omega)| = 0, \\
&\lim_{n\to\infty} \sup_{t\in[0,T)} |\Delta^+\varphi_t^{1,n}(\omega) - \Delta^+\psi_t^1(\omega)| = 0.
\end{aligned} \tag{5.27}$$

Now (5.27) implies for all (τ_i, Y_i), $i \in \mathbb{N}$, introduced in Remark 3.7, it holds P-a.s. on $\tau_i < T$

$$\lim_{n\to\infty} \left(\Delta^+ M_{\tau_i}^{B,n} - \Delta^+ M_{\tau_i}^{S,n} + L^{B,n}(\tau_i, Y_i) - \sum_{j=1}^{\infty} L^{S,n}(\sigma_j, Z_j) 1_{\{\tau_i = \sigma_j\}} \right) = \Delta^+\psi_{\tau_i}^1.$$

This yields that for $n \to \infty$ and M_μ-a.a. $(\omega, t, x) \in \widetilde{\Omega}$ with $t < T$ we have

$$\Delta^+ M_t^{B,n}(\omega) - \Delta^+ M_t^{S,n}(\omega) + L^{B,n}(\omega, t, x) - \int_{\{t\}\times\mathbb{R}_+} L^{S,n}(\omega, s, z)\nu(\omega, ds, dz) \to \Delta^+\psi_t^1(\omega). \tag{5.28}$$

By Assumption 3.1 (i) we have $\underline{S}_t(\omega) \le x \le \overline{S}_t(\omega)$ for M_μ-a.a. $(\omega, t, x) \in \widetilde{\Omega}$. Combining this with Assumption 3.1 (v) implies that limit sell orders can M_μ-a.e. only be executed if $x = \overline{S}_t(\omega)$. By Assumption 3.1 (i) for ν, in the latter case no limit sell order with limit price above x is executed. Thus, we have M_μ-a.e.

$$\begin{aligned}
&\Delta^+\varphi_t^{0,n}(\omega) \\
= \;& -\overline{S}_t(\omega)\Delta^+ M_t^{B,n}(\omega) + \underline{S}_t(\omega)\Delta^+ M_t^{S,n}(\omega) + \int_x^\infty y L^{B,n}(\omega, t, dy) \\
& + \int_{\{t\}\times\mathbb{R}_+} \int_0^z y L^{S,n}(\omega, s, dy)\nu(\omega, ds, dz) \\
\le \;& -x\Delta^+ M_t^{B,n}(\omega) + x\Delta^+ M_t^{S,n}(\omega) - x\left(L^{B,n}(\omega, t, x) - L^{B,n}(\omega, t, x + \frac{\varepsilon}{2})\right) \\
& - \left(x + \frac{\varepsilon}{2}\right) L^{B,n}(\omega, t, x + \frac{\varepsilon}{2}) + x\int_{\{t\}\times\mathbb{R}_+} L^{S,n}(\omega, s, z)\nu(\omega, ds, dz) \\
= \;& x\left(-\Delta^+ M_t^{B,n}(\omega) + \Delta^+ M_t^{S,n}(\omega) - L^{B,n}(\omega, t, x) + \int_{\{t\}\times\mathbb{R}_+} L^{S,n}(\omega, s, z)\nu(\omega, ds, dz)\right) \\
& - \frac{\varepsilon}{2} L^{B,n}(\omega, t, x + \frac{\varepsilon}{2}).
\end{aligned}$$

Now (5.28) implies that the term in the last but one line converges to $-x\Delta^+\psi_t^1(\omega)$ for M_μ-a.a. $(\omega,t,x)\in\widetilde{\Omega}$ whereas by the same arguments $\Delta^+\varphi_t^{0,n}(\omega)$ converges to $\Delta^+\psi_t^{0,n}(\omega)$ for M_μ-a.a. $(\omega,t,x)\in\widetilde{\Omega}$. Putting together, this implies that

$$\limsup_{n\to\infty} L^{B,n}(\omega,t,x+\frac{\varepsilon}{2})<-\infty,\quad \text{for } M_\mu\text{-a.a. } (\omega,t,x)\in\widetilde{\Omega}.$$

By the implication $(\omega,t,x)\in N_\varepsilon \implies (\omega,t,x+\varepsilon/2)\in A$, we arrive at $M_\mu(N_\varepsilon)=0$.

Step 2: Let us show that $\widehat{M}_\mu(\widehat{N}_\varepsilon)=0$. Let $Z:=\underline{X}1_{\widehat{N}_\varepsilon^c}+(\underline{X}+\frac{\varepsilon}{2})1_{\widehat{N}_\varepsilon}$. Z is predictable and Step 1 implies that $M_\mu(\text{subgraph}(Z))=0$. By the definition of the essential supremum we obtain $Z\le\underline{X}$ $\widehat{M}_\mu$-a.e. and thus $\widehat{M}_\mu(\widehat{N}_\varepsilon)=0$.

Step 3: We obtain that

$$M_\mu(A\cap B_\varepsilon)\le M_\mu(\widehat{N}_\varepsilon\times\mathbb{R}_+)=\widehat{M}_\mu(\widehat{N}_\varepsilon)=0.$$

Note that for $r\to\infty$ we get supergraph$(\underline{X}+\frac{1}{r})\uparrow$ supergraph$(\underline{X})$, which yields

$$M_\mu(A\cap\text{supergraph}(\underline{X}))=\lim_{r\to\infty}M_\mu(A\cap B_{\frac{1}{r}})=0.$$

As $M_\mu(A\cap\text{subgraph}(\underline{X}))=0$ always holds and $M_\mu(A\cap\text{graph}(\underline{X}))=0$ by Assumption 3.16 we arrive at

$$M_\mu(A)=M_\mu(A\cap\text{supergraph}(\underline{X}))+M_\mu(A\cap\text{subgraph}(\underline{X}))+M_\mu(A\cap\text{graph}(\underline{X}))=0.$$

$\square$

Lemma 5.3. *There exists a probability measure $\widetilde{Q}\sim P$ s.t. for all $k\in\mathbb{N}$ there is a $\widetilde{K}_k\in\mathbb{R}_+$ with*

$$E_{\widetilde{Q}}\left[M^{B,n}_{\widehat{\tau}^k}+\int_{[0,\widehat{\tau}^k)\times\mathbb{R}_+}L^{B,n}(s,x)\mu(ds,dx)\right]\le\widetilde{K}_k,\quad\forall n\in\mathbb{N}$$

and

$$E_{\widetilde{Q}}\left[M^{S,n}_{\widehat{\tau}^k}+\int_{[0,\widehat{\tau}^k)\times\mathbb{R}_+}L^{S,n}(s,x)\nu(ds,dx)\right]\le\widetilde{K}_k,\quad\forall n\in\mathbb{N},$$

where the stopping times $\widehat{\tau}^k$ are defined in Lemma 5.1.

Proof. For any $A\in\mathcal{B}([0,T])\otimes\mathcal{B}(\mathbb{R}_+)$ define $\widetilde{\mu}(A):=\mu(A\cap\{(t,x)\in[0,T]\times\mathbb{R}_+\mid x<\overline{S}_t\})$, $\mu^{\overline{S}}(A)=\mu(A\cap\{(t,x)\in[0,T]\times\mathbb{R}_+\mid x=\overline{S}_t\})$. Clearly $\mu^{\overline{S}}\perp\widetilde{\mu}$ and by Assumption 3.1 (i) we furthermore know that $\mu=\widetilde{\mu}+\mu^{\overline{S}}$. Let $\widetilde{\nu}$ and $\nu^{\underline{S}}$ be defined similarly. Note that by Assumption 3.1 (iii) we get that $\widetilde{\mu}$ and $\widetilde{\nu}$ are P-a.s. finite measures.

An important observation regarding $\mu^{\overline{S}}$ and $\nu^{\underline{S}}$ is that the limit order executions that are triggered by these measures are at most as favorable to the investor as trading by market orders. This yields

$$\begin{aligned}\int_{\{t\}\times\mathbb{R}_+}L^{B,n}(s,x)\mu^{\overline{S}}(ds,dx)\ &\le\ \frac{\underline{S}_t}{\underline{S}_t-\overline{S}_t}\int_{\{t\}\times\mathbb{R}_+}L^{B,n}(s,x)\mu^{\overline{S}}(ds,dx)\\ &\quad+\frac{1}{\underline{S}_t-\overline{S}_t}\int_{\{t\}\times\mathbb{R}_+}\int_x^\infty yL^{B,n}(s,dy)\mu^{\overline{S}}(ds,dx)\quad(5.29)\end{aligned}$$

and

$$\frac{\underline{S}_t}{\underline{S}_t - \overline{S}_t} \int_{\{t\}\times\mathbb{R}_+} L^{S,n}(s,x)\nu^{\underline{S}}(ds,dx) - \frac{1}{\underline{S}_t - \overline{S}_t} \int_{\{t\}\times\mathbb{R}_+} \int_0^x y L^{S,n}(s,dy)\nu^{\underline{S}}(ds,dx) \le 0. \quad (5.30)$$

In the equation for $\varphi^{1,n}$ in Definition 3.5 we isolate $M^{S,n}$ and plug it into the equation for $\varphi^{0,n}$. In this equation we isolate the terms with $M^{B,n}$ and the two integrals w.r.t. $\mu^{\overline{S}}$. Then, an application of (5.29) and (5.30) yields the follows estimation

$$\begin{aligned} & \Delta^+ M_t^{B,n} + \int_{\{t\}\times\mathbb{R}_+} L^{B,n}(s,x)\mu^{\overline{S}}(ds,dx) \\ \le \quad & \frac{\Delta^+ \mathrm{var}(\varphi^{0,n})_t + \underline{S}_t \left(\Delta^+ \mathrm{var}(\varphi^{1,n})_t + \int_{\{t\}\times\mathbb{R}_+} L^{B,n}(s,x)\widetilde{\mu}(ds,dx)\right)}{\overline{S}_t - \underline{S}_t} \\ + \quad & \frac{\int_{\{t\}\times\mathbb{R}_+} \int_0^x y L^{S,n}(s,dy)\widetilde{\nu}(ds,dx)}{\overline{S}_t - \underline{S}_t}. \end{aligned} \quad (5.31)$$

The impact of the limit orders on the portfolio process is limited to the $t+$ jumps, hence for the time when there are no such jumps we arrive with similar but simpler calculations as above at

$$M_{\widehat{\tau}^k}^{B,n} - \sum_{t<\widehat{\tau}^k} \Delta^+ M_t^{B,n} \le \frac{\mathrm{var}(\varphi^{0,n})_{\widehat{\tau}^k} + \sup_{t\in[0,T]} \underline{S}_t \mathrm{var}(\varphi^{1,n})_{\widehat{\tau}^k}}{\inf_{t\in[0,T]}(\overline{S}_t - \underline{S}_t)}. \quad (5.32)$$

Putting (5.31) and (5.32) together we arrive at

$$\begin{aligned} & M_{\widehat{\tau}^k}^{B,n} + \int_{[0,\widehat{\tau}^k)\times\mathbb{R}_+} L^{B,n}(s,x)\mu(ds,dx) \\ \le \quad & \frac{2\mathrm{var}(\varphi^{0,n})_{\widehat{\tau}^k} + 2\sup_{t\in[0,T]} \underline{S}_t \mathrm{var}(\varphi^{1,n})_{\widehat{\tau}^k} + \sup_{t\in[0,T]} \underline{S}_t \int_{[0,T)\times\mathbb{R}_+} \sup_{m\in\mathbb{N}} L^{B,m}(s,x)\widetilde{\mu}(ds,dx)}{\inf_{t\in[0,T]}(\overline{S}_t - \underline{S}_t)} \\ + \quad & \frac{\sup_{t\in[0,T]} \overline{S}_t \int_{[0,T)\times\mathbb{R}_+} \sup_{m\in\mathbb{N}} L^{S,m}(s,x)\widetilde{\nu}(ds,dx)}{\inf_{t\in[0,T]}(\overline{S}_t - \underline{S}_t)} \\ + \quad & \int_{[0,T)\times\mathbb{R}_+} \sup_{m\in\mathbb{N}} L^{B,m}(s,x)\widetilde{\mu}(ds,dx), \qquad \forall n\in\mathbb{N}. \end{aligned}$$

By Lemma 5.2 we know that $\sup_{m\in\mathbb{N}} L^{B,m}$ and $\sup_{m\in\mathbb{N}} L^{S,m}$ are M_μ-a.e. resp. M_ν-a.e. finite. Hence, because $\widetilde{\mu}(\omega,\cdot)$ and $\widetilde{\nu}(\omega,\cdot)$ are for P-a.a. ω finite measures on $[0,T]\times\mathbb{R}_+$, we conclude that $P(\int_{[0,T)\times\mathbb{R}_+} \sup_{m\in\mathbb{N}} L^{B,m}(s,x)\widetilde{\mu}(ds,dx) < \infty) = 1$ and $P(\int_{[0,T)\times\mathbb{R}_+} \sup_{m\in\mathbb{N}} L^{S,m}(s,x)\widetilde{\nu}(ds,dx) < \infty) = 1$. We obtain the existence of a random variable Z with $P(0\le Z<\infty)=1$ and

$$M_{\widehat{\tau}^k}^{B,n} + \int_{[0,\widehat{\tau}^k)\times\mathbb{R}_+} L^{B,n}(s,x)\mu(ds,dx) \le Z\left(\mathrm{var}(\varphi^{0,n})_{\widehat{\tau}^k} + \mathrm{var}(\varphi^{1,n})_{\widehat{\tau}^k} + 1\right) \quad \forall n\in\mathbb{N}.$$

Similarly we can show that there exists a $\widetilde{Z}$ with $P(0\le\widetilde{Z}<\infty)=1$ and

$$M_{\widehat{\tau}^k}^{S,n} + \int_{[0,\widehat{\tau}^k)\times\mathbb{R}_+} L^{S,n}(s,x)\nu(ds,dx) \le \widetilde{Z}\left(\mathrm{var}(\varphi^{0,n})_{\widehat{\tau}^k} + \mathrm{var}(\varphi^{1,n})_{\widehat{\tau}^k} + 1\right) \quad \forall n\in\mathbb{N}.$$

By Lemma 5.1 we know that there exist a probability measure $Q \sim P$ (independent of k) and a $K_k \in \mathbb{R}_+$ s.t.

$$E_Q\left[\operatorname{var}(\varphi^{0,n})_{\widehat{\tau}^k} + \operatorname{var}(\varphi^{0,n})_{\widehat{\tau}^k} + 1\right] \le K_k + 1 \quad \forall n \in \mathbb{N}.$$

As $(Z \vee \widetilde{Z} \vee 1)^{-1}$ is a bounded random variable we can define a new measure $\widetilde{Q}$ by

$$\frac{d\widetilde{Q}}{dQ} = \frac{1}{E_Q\left[(Z \vee \widetilde{Z} \vee 1)^{-1}\right]} \frac{1}{Z \vee \widetilde{Z} \vee 1}.$$

We have for all $n \in \mathbb{N}$

$$\begin{aligned}
& E_{\widetilde{Q}}\left[\left(M^{B,n}_{\widehat{\tau}^k} + \int_{[0,\widehat{\tau}^k)\times\mathbb{R}_+} L^{B,n}(s,x)\mu(ds,dx)\right) \vee \left(M^{S,n}_{\widehat{\tau}^k} + \int_{[0,\widehat{\tau}^k)\times\mathbb{R}_+} L^{S,n}(s,x)\nu(ds,dx)\right)\right] \\
\le\ & E_{\widetilde{Q}}\left[(Z \vee \widetilde{Z})\left(\operatorname{var}(\varphi^{0,n})_{\widehat{\tau}^k} + \operatorname{var}(\varphi^{1,n})_{\widehat{\tau}^k} + 1\right)\right] \\
\le\ & \frac{1}{E_Q\left[(Z \vee \widetilde{Z} \vee 1)^{-1}\right]} E_Q\left[\frac{Z \vee \widetilde{Z}}{Z \vee \widetilde{Z} \vee 1}\left(\operatorname{var}(\varphi^{0,n})_{\widehat{\tau}^k} + \operatorname{var}(\varphi^{1,n})_{\widehat{\tau}^k} + 1\right)\right] \\
\le\ & \frac{1}{E_Q\left[(Z \vee \widetilde{Z} \vee 1)^{-1}\right]}(K_k + 1) =: \widetilde{K}_k.
\end{aligned}$$

□

Lemma 5.4. *Under the conditions of Theorem 3.17, both the total numbers of purchased stocks* $(M^{B,n} + \int_{[0,\cdot)\times\mathbb{R}_+} L^{B,n}_s\, d\mu_s)_{n\in\mathbb{N}}$ *and the total numbers of sold stocks* $(M^{S,n} + \int_{[0,\cdot)\times\mathbb{R}_+} L^{S,n}_s\, d\nu_s)_{n\in\mathbb{N}}$ *are up-Cauchy sequences.*

Proof. Of course it is sufficient to prove only the first part of the assertion as the second one is completely analog. Assume that $(\varphi^{0,n}, \varphi^{1,n})_{n\in\mathbb{N}}$ is an up-Cauchy sequence.

Step 1: Let us consider the corresponding discounted wealth processes if stock positions are evaluated at the best-bid price $\underline{S}$ and the numeraire is the spread $\overline{S} - \underline{S}$, i.e.

$$\widehat{V}^n := \frac{\varphi^{0,n}}{\overline{S} - \underline{S}} + \frac{\varphi^{1,n}\underline{S}}{\overline{S} - \underline{S}}.$$

The stock evaluation and the choice of the numeraire simplify the calculations. Namely, sales by market orders do not change the wealth process and the purchase of one share by a market order reduces the discounted wealth by one unit. Note that $(\widehat{V}^n)_{n\in\mathbb{N}}$ is again up-Cauchy and the processes $\frac{1}{\overline{S}-\underline{S}}$ and $\frac{\underline{S}}{\overline{S}-\underline{S}}$ are again semimartingales by $P\left(\inf\{\overline{S}_t - \underline{S}_t \mid t \in [0,T]\} > 0\right) = 1$ and Itô's formula. By Definition 3.5 and Lemma 8.2 in Muhle-Karbe [14] we obtain

$$\begin{aligned}
\widehat{V}^n =\ & \widehat{V}^n_0 + \varphi^{0,n} \bullet \left(\frac{1}{\overline{S} - \underline{S}}\right) + \varphi^{1,n} \bullet \left(\frac{\underline{S}}{\overline{S} - \underline{S}}\right) - M^{B,n} \\
& + \int_{[0,\cdot)\times\mathbb{R}_+} \int_x^{\overline{S}_{s-}} \frac{y - \underline{S}_s}{\overline{S}_s - \underline{S}_s} L^{B,n}(s,dy)\mu(ds,dx) + \int_{[0,\cdot)\times\mathbb{R}_+} \int_{\underline{S}_{s-}}^x \frac{y - \underline{S}_s}{\overline{S}_s - \underline{S}_s} L^{S,n}(s,dy)\nu(ds,dx).
\end{aligned}$$

Note that $L^{B,n}(s,x) = 0$ for $x \ge \overline{S}_{s-}$ and $L^{S,n}(s,x) = 0$ for $x \le \underline{S}_{s-}$. Let $\mu = \mu^{1,\delta} + \mu^{2,\delta}$ be the decomposition from (4.15). In the following, executed limit buy orders with limit price close to the best-ask are charged at the best-ask. The process $A^{\delta,n}$ is the corresponding error term and formally defined by

$$
\begin{aligned}
&\int_{[0,t)\times\mathbb{R}_+}\int_x^{\overline{S}_{s-}} \frac{y-\underline{S}_s}{\overline{S}_s-\underline{S}_s} L^{B,n}(s,dy)\mu(ds,dx)\\
&=\int_{[0,t)\times\mathbb{R}_+}\int_{\overline{S}_{s-}-\delta}^{\overline{S}_{s-}} \frac{y-\underline{S}_s}{\overline{S}_s-\underline{S}_s} L^{B,n}(s,dy)\mu^{1,\delta}(ds,dx)\\
&\quad+\int_{[0,t)\times\mathbb{R}_+}\int_x^{\overline{S}_{s-}} \frac{y-\underline{S}_s}{\overline{S}_s-\underline{S}_s} L^{B,n}(s,dy)\mu^{2,\delta}(ds,dx)\\
&=-\int_{[0,t)\times\mathbb{R}_+} L_s^{B,n}\,d\mu_s^{1,\delta}+\int_{[0,t)\times\mathbb{R}_+}\int_x^{\overline{S}_{s-}} \frac{y-\underline{S}_s}{\overline{S}_s-\underline{S}_s} L^{B,n}(s,dy)\mu^{2,\delta}(ds,dx)+A_t^{\delta,n}.
\end{aligned}
$$

$A^{\delta,n}$ is nonincreasing and

$$
|A_T^{\delta,n}|\leq \frac{\delta\int_{[0,T]} L_s^{B,n}d\mu_s}{\inf\{\overline{S}_t-\underline{S}_t \mid t\in[0,T]\}}. \tag{5.33}
$$

Analogously, we define $\nu^{1,\delta},\nu^{2,\delta}$ by $\nu=\nu^{1,\delta}+\nu^{2,\delta}$, $\nu^{1,\delta}\perp\nu^{2,\delta}$, and $\nu^{1,\delta}(\{t\}\times\{x\})=1$ iff $x=\underline{S}_t$ and $\Delta\underline{S}_t\in(0,\delta]$. Again, $\nu^{2,\delta}$ is a finite random measure. The process $B^{\delta,n}$ is the error term when limit sell orders with limit price close to the best-bid are charged at the best-bid. Formally, it is defined by

$$
\begin{aligned}
&\int_{[0,t)\times\mathbb{R}_+}\int_{\underline{S}_{s-}}^{x} \frac{y-\underline{S}_s}{\overline{S}_s-\underline{S}_s} L^{S,n}(s,dy)\nu(ds,dx)\\
&=\int_{[0,t)\times\mathbb{R}_+}\int_{\underline{S}_{s-}}^{x} \frac{y-\underline{S}_s}{\overline{S}_s-\underline{S}_s} L^{S,n}(s,dy)\nu^{2,\delta}(ds,dx)+B_t^{\delta,n}.
\end{aligned}
$$

$B^{\delta,n}$ is nonincreasing and

$$
|B_T^{\delta,n}|\leq \frac{\delta\int_{[0,T]} L_s^{S,n}d\mu_s}{\inf\{\overline{S}_t-\underline{S}_t \mid t\in[0,T]\}}. \tag{5.34}
$$

We arrive at

$$
\begin{aligned}
\widehat{V}^n &= \widehat{V}_0^n+\varphi^{0,n}\bullet\left(\frac{1}{\overline{S}-\underline{S}}\right)+\varphi^{1,n}\bullet\left(\frac{\underline{S}}{\overline{S}-\underline{S}}\right)-M^{B,n}-\int_{[0,\cdot)\times\mathbb{R}_+} L^{B,n}\,d\mu_s^{1,\delta}\\
&\quad+\int_{[0,\cdot)\times\mathbb{R}_+}\int_x^{\overline{S}_{s-}} \frac{y-\underline{S}_s}{\overline{S}_s-\underline{S}_s} L^{B,n}(s,dy)\mu^{2,\delta}(ds,dx)\\
&\quad+\int_{[0,\cdot)\times\mathbb{R}_+}\int_{\underline{S}_{s-}}^{x} \frac{y-\underline{S}_s}{\overline{S}_s-\underline{S}_s} L^{S,n}(s,dy)\nu^{2,\delta}(ds,dx)+A^{\delta,n}+B^{\delta,n}
\end{aligned}
$$

and thus

$$
\begin{aligned}
&M^{B,n} + \int_{[0,\cdot)\times\mathbb{R}_+} L^{B,n}\, d\mu_s \\
&= -\widehat{V}^n + \widehat{V}_0^n + \varphi^{0,n} \bullet \left(\frac{1}{\overline{S} - \underline{S}}\right) + \varphi^{1,n} \bullet \left(\frac{\underline{S}}{\overline{S} - \underline{S}}\right) + \int_{[0,\cdot)\times\mathbb{R}_+} L^{B,n}\, d\mu_s^{2,\delta} \\
&+ \int_{[0,\cdot)\times\mathbb{R}_+} \int_x^{\overline{S}_{s-}} \frac{y - \underline{S}_s}{\overline{S}_s - \underline{S}_s} L^{B,n}(s,dy)\mu^{2,\delta}(ds,dx) \\
&+ \int_{[0,\cdot)\times\mathbb{R}_+} \int_{\underline{S}_{s-}}^{x} \frac{y - \underline{S}_s}{\overline{S}_s - \underline{S}_s} L^{S,n}(s,dy)\nu^{2,\delta}(ds,dx) + A^{\delta,n} + B^{\delta,n}.
\end{aligned} \tag{5.35}
$$

Step 2: Now let $\varepsilon > 0$. As $\widehat{V}^n$, $\widehat{V}_0^n$, $\varphi^{0,n} \bullet \left(\frac{1}{\overline{S}-\underline{S}}\right)$, and $\varphi^{1,n} \bullet \left(\frac{\underline{S}}{\overline{S}-\underline{S}}\right)$ are up-Cauchy sequences, there exists an $n_1 \in \mathbb{N}$ s.t.

$$
\begin{aligned}
P\Bigg(&\left|-(\widehat{V}_t^n - \widehat{V}_t^m) + (\widehat{V}_0^n - \widehat{V}_0^m) + (\varphi^{0,n} - \varphi^{0,m}) \bullet \left(\frac{1}{\overline{S} - \underline{S}}\right)_t + (\varphi^{1,n} - \varphi^{1,m}) \bullet \left(\frac{\underline{S}}{\overline{S} - \underline{S}}\right)_t\right| \\
&\leq \frac{\varepsilon}{4},\ \forall t \in [0,T]\Bigg) \geq 1 - \frac{\varepsilon}{4}, \quad \forall n,m \geq n_1.
\end{aligned} \tag{5.36}
$$

By Lemma 5.3 the sequences $(\int_{[0,T]} L_s^{B,n} d\mu_s)_{n\in\mathbb{N}}$ and $(\int_{[0,T]} L_s^{S,n} d\nu_s)_{n\in\mathbb{N}}$ are stochastically bounded. Thus, by (5.33) and (5.34), there exists a $\delta > 0$ s.t.

$$
P\left(|A_T^{\delta,n} + B_T^{\delta,n}| \leq \frac{\varepsilon}{4}\right) \geq 1 - \frac{\varepsilon}{4}, \quad \forall n \in \mathbb{N}. \tag{5.37}
$$

We fix this δ. As $\mu^{2,\delta}$ and $\nu^{2,\delta}$ are finite random measures the remaining terms on the rhs of (5.35) are up-Cauchy sequences by Lemma 5.2 and by Lemma 4.7 (formally not applied to μ and ν, but to $\mu^{2,\delta}$ and $\nu^{2,\delta}$ under which $\sup_{n\in\mathbb{N}} L^{B,n}$ resp. $\sup_{n\in\mathbb{N}} L^{S,n}$ are integrable). Thus, there exists an $n_2 \in \mathbb{N}$ s.t.

$$
\begin{aligned}
P\Bigg(&\Bigg|\int_{[0,t)\times\mathbb{R}_+} (L^{B,n} - L^{B,m})\, d\mu_s^{2,\delta} + \int_{[0,t)\times\mathbb{R}_+} \int_x^{\overline{S}_{s-}} \frac{y - \underline{S}_s}{\overline{S}_s - \underline{S}_s} (L^{B,n} - L^{B,m})(s,dy)\mu^{2,\delta}(ds,dx) \\
&+ \int_{[0,t)\times\mathbb{R}_+} \int_{\underline{S}_{s-}}^{x} \frac{y - \underline{S}_s}{\overline{S}_s - \underline{S}_s} (L^{S,n} - L^{S,m})(s,dy)\nu^{2,\delta}(ds,dx)\Bigg| \leq \frac{\varepsilon}{4},\ \forall t \in [0,T]\Bigg) \geq 1 - \frac{\varepsilon}{4},
\end{aligned}
$$

for all $n,m \geq n_2$. Combining this with (5.36), (5.37), and (5.35), we arrive at

$$
P\left(\left|M_t^{B,n} + \int_{[0,t)\times\mathbb{R}_+} L^{B,n}\, d\mu_s - M_t^{B,m} - \int_{[0,t)\times\mathbb{R}_+} L^{B,m}\, d\mu_s\right| \leq \varepsilon,\ \forall t \in [0,T]\right) \geq 1 - \varepsilon
$$

for all $n,m \geq n_1 \vee n_2$. Thus, $(M^{B,n} + \int_{[0,\cdot)\times\mathbb{R}_+} L_s^{B,n} d\mu_s)_{n\in\mathbb{N}}$ is a Cauchy sequence w.r.t. the convergence "uniformly in probability". □

Lemma 5.5. *Let S be an adapted càdlàg-process and let $(A^n)_{n\in\mathbb{N}}$ be a sequence of predictable processes of finite variation for which there exist a measure $Q \sim P$ and a constant $K > 0$ s.t.*

$$
E_Q[\mathrm{var}(A^n)_T] \leq K, \qquad \forall n \in \mathbb{N},
$$

and let A be a predictable process of finite variation s.t.

$$\sup_{t\in[0,T]} |A_t^n - A_t| \to 0, \quad \textit{in probability}, \quad \textit{for } n \to \infty.$$

Then it holds that

$$\sup_{t\in[0,T]} |((S_-, S) \bullet A^n)_t - ((S_-, S) \bullet A)_t| \to 0, \quad \textit{in probability}, \quad \textit{for } n \to \infty.$$

Proof. By linearity of the integral w.r.t. the integrator and by the assumption that the limit A is itself of finite variation, it follows from Proposition A.1 ii) in [8] and Fatou's lemma that $E_Q[\mathrm{var}(A)] \leq K$ holds. Thus it is sufficient to prove the result for $A = 0$.

The well-known equivalence between convergence in probability of a sequence of random variables to zero and that any subsequence of this sequence contains a subsubsequence converging almost surely to zero implies that it is sufficient to prove the assertion under the assumption that $\sup_{t\in[0,T]} |A_t^n| \to 0$, P-a.s. for $n \to \infty$.

Furthermore, by a stopping argument, we may suppose that there exist constants $C_1 > 0$ and $C_2 > 0$ s.t. $\sup_{t\in[0,T]} |A_t^n| \leq C_1$ for all $n \in \mathbb{N}$ and $|S| \leq C_2$.

We start similarly to the proof of Theorem A.9 iii) in [8]. By Theorem A.9 ii) in [8] it holds for all $t \in [0,T]$ and $n \in \mathbb{N}$ that

$$\sup_{t\in[0,T]} |((S_-, S) \bullet A^n)_t| \leq \mathrm{var}(A^n)_T \sup_{t\in[0,T]} |S_t|. \tag{5.38}$$

For any $m \in \mathbb{N}$ define the sequence of stopping times $T_0^m, T_1^m, \ldots$ by

$$T_0^m := 0 \qquad \text{and} \qquad T_{i+1}^m := \inf\left\{ t > T_i^m \mid |S_t - S_{T_k}| > \frac{1}{m} \right\},$$

and let

$$S^m := \sum_{i=0}^{\infty} S_{T_i^m} 1_{[\![T_i^m, T_{i+1}^m[\![}.$$

For every m by construction of S^m it is possible to find a constant $\alpha_m \in \mathbb{N}$ and a set $B_m \in \mathcal{F}$ s.t. S^m consist only of α_m steps or less on B_m and it holds that $Q(B_m^c) \leq 2^{-m}$. Now (5.38) and the linearity of the integral w.r.t.the integrand yield for any $m_0 \in \mathbb{N}$ and any $m \geq m_0$

$$\begin{aligned}
&E_Q\left[\sup_{t\in[0,T]} |((S_-, S) \bullet A^n)_t| 1_{\bigcap_{m\geq m_0} B_m}\right] \\
&\leq E_Q\left[\sup_{t\in[0,T]} |((S_- - S_-^m, S - S^m) \bullet A^n)_t + ((S_-^m, S^m) \bullet A^n)_t| 1_{\bigcap_{m\geq m_0} B_m}\right] \\
&\leq \frac{1}{m} E_Q\left[\mathrm{var}(A^n)_T\right] + E_Q\left[\sup_{t\in[0,T]} |((S_-^m, S^m) \bullet A^n)_t| 1_{\bigcap_{m\geq m_0} B_m}\right].
\end{aligned}$$

By assumption of the lemma, by construction of S^m, α_m, and B_m, as well as by the assumptions at the beginning of the proof this implies for all $m \geq m_0$ that

$$E_Q\left[\sup_{t\in[0,T]} |((S_-, S) \bullet A^n)_t| 1_{\bigcap_{m\geq m_0} B_m}\right] \leq \frac{1}{m} K + 2C_2 \alpha_m E_Q\left[\sup_{t\in[0,T]} |A_t^n| 1_{\bigcap_{m\geq m_0} B_m}\right].$$

For any fixed m the second term on the right-hand side of the equation goes to zero as n goes to infinity, by dominated convergence. Therefore, for any $\varepsilon > 0$ and any $m_0 \in \mathbb{N}$ there exists an $n_0(\varepsilon, m_0) \in \mathbb{N}$ s.t. for all $n \geq n_0(\varepsilon, m_0)$ it holds that

$$E_Q\left[\sup_{t\in[0,T]} |((S_-, S) \bullet A^n)_t| 1_{\bigcap_{m\geq m_0} B_m}\right] \leq \varepsilon.$$

Hence for any $m_0 \in \mathbb{N}$ we have that

$$\sup_{t\in[0,T]} |((S_-, S) \bullet A^n)_t| 1_{\bigcap_{m\geq m_0} B_m} \to 0, \qquad \text{in probability}$$

as n goes to infinity. Note that by a Borel-Cantelli argument we have that $Q(\limsup_m B_m^c) = 0$, which implies that $(\bigcap_{m\geq m_0} B_m)_{m_0\in\mathbb{N}}$ is an increasing sequence with $P(\liminf_m B_m) = 1$. Therefore, it also holds that

$$\sup_{t\in[0,T]} |((S_-, S) \bullet A^n)_t| \to 0, \qquad \text{in probability}$$

as n goes to infinity. □

Lemma 5.6. *Assume that the sequence $(L^{B,n})_{n\in\mathbb{N}}$ (as introduced at the beginning of the section) converges $\widetilde{M}_\mu$-a.e. towards a $\widetilde{\mathcal{P}}$-measurable function $L^B : \widetilde{\Omega} \to \overline{\mathbb{R}}_+$. Then, there exists a $\widehat{L}^B \in \mathcal{L}^B$ s.t. $L^B = \widehat{L}^B$ holds $\widetilde{M}_\mu$-a.e.. The analog assertion holds for limit sell order strategies.*

Note that we do not yet have that $(L^{B,n})_{n\in\mathbb{N}}$ converges $\widetilde{M}_\mu$-a.e. (later on we build convex combinations to achieve this). The lemma only tells us that *if* the sequence converges, then the limit function can be chosen as valid limit order strategy.

Proof. Denote by $\widetilde{N}$ a $\widetilde{M}_\mu$-null set s.t. $(L^{B,n})_{n\in\mathbb{N}}$ converges pointwise to L^B on $\widetilde{N}^c$, by $(x_k)_{k\in\mathbb{N}}$ a sequence running through $\mathbb{Q}_+$, and by $\left(\widetilde{N} \cap \text{supergraph}(\underline{X})\right)_{x_k}$ the x_k-section of $\widetilde{N} \cap \text{supergraph}(\underline{X})$, i.e. the set $\left\{(\omega, t) \in \Omega \times [0, T] \mid (\omega, t, x_k) \in \widetilde{N} \cap \text{supergraph}(\underline{X})\right\} \in \mathcal{P}$. Define

$$\overline{N} := \bigcup_{k=1}^{\infty} \left(\widetilde{N} \cap \text{supergraph}(\underline{X})\right)_{x_k}$$

and note that from $\left(\widetilde{N} \cap \text{supergraph}(\underline{X})\right)_{x_k} \times \{x_k\} \subset \widetilde{N} \cap \text{supergraph}(\underline{X})$ and

$$\left(\left(\widetilde{N} \cap \text{supergraph}(\underline{X})\right)_{x_k} \times \{x_k\}\right) \cap \text{supergraph}(\underline{X}) = \left(\widetilde{N} \cap \text{supergraph}(\underline{X})\right)_{x_k} \times \{x_k\}$$

it follows that $\widehat{M_\mu}\left(\left(\widetilde{N} \cap \text{supergraph}(\underline{X})\right)_{x_k}\right) = 0$ and thus also $\widehat{M_\mu}(\overline{N}) = 0$, i.e. $M_\mu(\overline{N} \times \mathbb{R}_+) = 0$. Hence, we arrive at $\widetilde{M}_\mu(\overline{N} \times \mathbb{R}_+) = 0$. By definition of $\overline{N}$, for all $(\omega, t, x) \in \left(\overline{N} \times \mathbb{R}_+\right)^c \cap \text{supergraph}(\underline{X}) \cap (\Omega \times [0, T] \times \mathbb{Q}_+)$ we know that $\left(L^{B,n}(\omega, t, x)\right)_{n\in\mathbb{N}}$ converges to $L^B(\omega, t, x)$. By assumption, for all n, $x \mapsto L^{B,n}(\omega, t, x)$ is nonincreasing for all $(\omega, t) \in \Omega \times [0, T]$ and $L^{B,n}(\omega, t, x) = L^{B,n}(\omega, t, x)1_{\{x<\overline{S}_{t-}(\omega)\}}$ for all $(\omega, t, x) \in \widetilde{\Omega}$. The latter implies that

$$L^B(\omega, t, x) = L^B(\omega, t, x)1_{\{x<\overline{S}_{t-}(\omega)\}} \quad \forall(\omega, t, x) \in \widetilde{N}^c. \tag{5.39}$$

We proceed by defining the function $\widehat{L}^B$ by

$$\widehat{L}^B(\omega,t,x) := \infty 1_{\{x\le \underline{X}_t(\omega),\ x<\overline{S}_{t-}(\omega)\}} 1_{\overline{N}^c}(\omega,t)$$
$$+\text{median}\left\{L^B(\omega,t,x),\ \sup_{x<x_k} L^B(\omega,t,x_k),\ \inf_{x_k<x} L^B(\omega,t,x_k)\right\} 1_{\{\underline{X}_t(\omega)<x<\overline{S}_{t-}(\omega)\}} 1_{\overline{N}^c}(\omega,t).$$

As L^B is $\widetilde{\mathcal{P}}$-measurable and $L(\cdot,\cdot,x_k)$, $k\in\mathbb{N}$, are $\mathcal{P}$-measurable, $\widehat{L}^B$ is expressed as supremum, median, and sum of $\widetilde{\mathcal{P}}$-measurable functions and thus itself $\widetilde{\mathcal{P}}$-measurable. For $(\omega,t,x),(\omega,t,y)\in\widetilde{\Omega}$ with $x<y$ for which both $L^{B,n}(\omega,t,x)\to L^B(\omega,t,x)$ and $L^{B,n}(\omega,t,y)\to L^B(\omega,t,y)$ hold for $n\to\infty$, we have

$$L^B(\omega,t,x) = \lim_{n\to\infty} L^{B,n}(\omega,t,x) \ge \lim_{n\to\infty} L^{B,n}(\omega,t,y) = L^B(\omega,t,y). \tag{5.40}$$

Hence, for $(\omega,t,x)\in\widetilde{N}^c\cap(\overline{N}\times\mathbb{R}_+)^c\cap\text{supergraph}(\underline{X})$ it holds that

$$\sup_{x<x_k} L^B(\omega,t,x_k) \le L^B(\omega,t,x) \le \inf_{x_k<x} L^B(\omega,t,x_k).$$

Together with (5.39) we obtain $\widehat{L}^B = L^B$ on $\widetilde{N}^c\cap(\overline{N}\times\mathbb{R}_+)^c\cap\text{supergraph}(\underline{X})$. By the construction of $\widetilde{M}_\mu$ and Assumption 3.16, we already know that $\text{subgraph}(\underline{X})\cup\text{graph}(\underline{X})$ is a $\widetilde{M}_\mu$-null set. Hence, the set of points on which we set $\widehat{L}^B$ to the value ∞ is in any case not relevant for the question whether $L^B=\widehat{L}^B$ $\widetilde{M}_\mu$-a.e. or not (though it does play a role to assure monotonicity of course, which is supposed to hold for *all* $(\omega,t)\in\Omega\times[0,T]$). Furthermore, we have seen above that $\widetilde{M}_\mu(\overline{N}\times\mathbb{R}_+)=0$ and consequently $L^B=\widehat{L}^B$ holds $\widetilde{M}_\mu$-almost everywhere.

Let us verify that $x\mapsto\widehat{L}^B(\omega,t,x)$ is nonincreasing for all $(\omega,t)\in\Omega\times[0,T]$, which is part (i) of Definition 3.3. On $\left(\overline{N}\times\mathbb{R}_+\right)^c\cap\text{supergraph}(\underline{X})$ we have

By definition of $\widehat{L}^B$ via the median and since $\sup_{x<x_k} L^B(\omega,t,x_k)\le\inf_{x_k<x} L^B(\omega,t,x_k)$ on $\left(\overline{N}\times\mathbb{R}_+\right)^c\cap\text{supergraph}(\underline{X})$ by (5.40) we get

$$\sup_{x<x_k} L^B(\omega,t,x_k) \le \widehat{L}^B(\omega,t,x) \le \inf_{x_k<x} L^B(\omega,t,x_k).$$

This yields for all $(\omega,t,x),(\omega,t,y)\in\left(\overline{N}\times\mathbb{R}_+\right)^c\cap\text{supergraph}(\underline{X})$ with $x<y$ that

$$\widehat{L}^B(\omega,t,x) \ge \sup_{x<x_k} L^B(\omega,t,x_k) \ge \inf_{x_k<y} L^B(\omega,t,x_k) \ge \widehat{L}^B(\omega,t,y).$$

Moreover for all $(\omega,t)\in\overline{N}^c$ we have that $\widehat{L}^B(\omega,t,x)=\infty$ for all $x\le\underline{X}_t(\omega)$. Therefore, the monotonicity of $x\mapsto\widehat{L}^B(\omega,t,x)$ on $\mathbb{R}_+$ is established for all $(\omega,t)\in\overline{N}^c$. For $(\omega,t)\in\overline{N}$ we have $\widehat{L}^B(\omega,t,\cdot)\equiv 0$ and the monotonicity is trivially satisfied. $\widehat{L}^B$ satisfies part (ii) of Definition 3.3 by construction.

We complete the proof by checking that part (iii) of Definition 3.3 holds, i.e. that $\widehat{L}^B$ is μ-integrable. By the $\widetilde{M}_\mu$-a.e. convergence of $(L^{B,n})_{n\in\mathbb{N}}$ to $\widehat{L}^B$, Fatou's lemma, and Lemma 5.3, there exist a probability measure $\widetilde{Q}\sim P$ and $\widetilde{K}_k\in\mathbb{R}_+$ s.t.

$$E_{\widetilde{Q}}\left[\int_{[0,\widehat{\tau}^k)\times\mathbb{R}_+}\widehat{L}^B(t,x)\mu(dt,dx)\right] \le \liminf_{n\to\infty} E_{\widetilde{Q}}\left[\int_{[0,\widehat{\tau}^k)\times\mathbb{R}_+} L^{B,n}(t,x)\mu(dt,dx)\right] \le \widetilde{K}_k, \quad \forall k\in\mathbb{N},$$

where $\widehat{\tau}^k$ refers to the stopping time defined in Lemma 5.1. As $(\widehat{\tau}^k)_{k\in\mathbb{N}}$ is a localizing sequence we get $P(\int_{[0,T)\times\mathbb{R}_+}\widehat{L}^B(t,x)\mu(dt,dx)<\infty)=1$ and thus μ-integrability of $\widehat{L}^B$. □

Proof of Theorem 3.17. Our goal is to find a *limit strategy* $\mathfrak{S} = (M^B, M^S, L^B, L^S)$ for the sequence $(\mathfrak{S}^n)_{n\in\mathbb{N}}$ which satisfies $\big(\varphi^0(\mathfrak{S}), \varphi^1(\mathfrak{S})\big) = \big(\psi^0, \psi^1\big)$, where (ψ^0, ψ^1) is the predictable làdlàg limit process of $(\varphi^{0,n}, \varphi^{1,n})_{n\in\mathbb{N}}$ introduced at the beginning of this section. We deal with the limit orders first. We apply Lemma 9.8.1 (which is a Komlós type theorem) and Remark 9.8.2 in [7] twice (first w.r.t. the limit buy orders $L^{B,n}$ and the σ-finite measure $\widetilde{M}_\mu$, then w.r.t. the limit sell orders $L^{S,n}$ and the σ-finite measure $\widetilde{M}_\nu$, where we build convex combinations of the convex combinations chosen for the limit buy orders), which yields that there exist $\widetilde{\mathcal{P}}$-measurable $\overline{\mathbb{R}}_+$-valued functions L^B and L^S and a sequence of (finite) convex combinations $\widehat{\mathfrak{S}}^n \in \operatorname{conv}(\mathfrak{S}^n, \mathfrak{S}^{n+1}, \dots)$ s.t. $(\widehat{L}^{B,n})_{n\in\mathbb{N}}$ converges $\widetilde{M}_\mu$-a.e. to L^B and $(\widehat{L}^{S,n})_{n\in\mathbb{N}}$ converges $\widetilde{M}_\nu$-a.e. to L^S. Note that by a convex combination of strategies $\mathfrak{S}^n$ we mean a quadruple $(\widehat{M}^{B,n}, \widehat{M}^{S,n}, \widehat{L}^{B,n}, \widehat{L}^{S,n})$ where $\widehat{M}^{B,n} \in \operatorname{conv}\{M^{B,n}, M^{B,n+1}, \dots\}$ and so forth, where we use the same weights for $\widehat{M}^{B,n}, \widehat{M}^{S,n}, \widehat{L}^{B,n}$, and $\widehat{L}^{S,n}$. The associated portfolio process in Definition 3.5 of a finite convex combination of trading strategies is just the convex combination of the respective associated portfolio processes and a convex combination of a-admissible trading strategies is again a-admissible. Since the convex combinations were taken of trading strategies for which $\big(\varphi^0(\mathfrak{S}^n), \varphi^1(\mathfrak{S}^n)\big)_{n\in\mathbb{N}}$ converges P-a.s. uniformly on $[0,T]$ to (ψ^0, ψ^1) this also holds for $(\widehat{\mathfrak{S}}^n)_{n\in\mathbb{N}}$. Thus, we can assume that *w.l.o.g.* already the original sequence $(\mathfrak{S}^n)_{n\in\mathbb{N}}$ satisfies

$$L^{B,n} \to L^B, \quad \widetilde{M}_\mu - \text{a.e.} \quad \text{and} \quad L^{S,n} \to L^S, \quad \widetilde{M}_\nu - \text{a.e.}, \quad \text{for } n \to \infty.$$

Then, we apply Lemma 5.6 and obtain that w.l.o.g. $L^B \in \mathcal{L}^B$ and $L^S \in \mathcal{L}^S$. Given L^B and L^S we are in the position to present also the remaining market order part of our candidate for a limit strategy. Namely, by Lemma 5.4, stating that accumulated purchases and sells converge separately, there exist predictable processes

$$M^B := \lim_{n\to\infty}\left(M^{B,n} + \int_{[0,\cdot)\times\mathbb{R}_+} L^{B,n}(s,x)\mu(ds,dx)\right) - \int_{[0,\cdot)\times\mathbb{R}_+} L^B(s,x)\mu(ds,dx) \tag{5.41}$$

$$M^S := \lim_{n\to\infty}\left(M^{S,n} + \int_{[0,\cdot)\times\mathbb{R}_+} L^{S,n}(s,x)\nu(ds,dx)\right) - \int_{[0,\cdot)\times\mathbb{R}_+} L^S(s,x)\nu(ds,dx),$$

where the convergence holds "uniformly in probability". We need to check though that M^B and M^S are nondecreasing. To avoid repeating ourselves, we only examine M^B. Remember that $(L^{B,n})_{n\in\mathbb{N}}$ converges $\widetilde{M}_\mu$-a.e. to L^B. Thus, P-a.s. $\big(L^{B,n}(\omega)\big)_{n\in\mathbb{N}}$ converges μ_ω-a.e. to $L^B(\omega)$. In addition, the convergence in (5.41) holds P-a.s. uniformly in t on a subsequence. Let $A \in \mathcal{F}$ be a corresponding exceptional null set and $\omega \notin A$. For $t_1, t_2 \in [0,T]$ (possibly depending on ω) with $t_1 \le t_2$, the monotonicity of $M^{B,n}$ and an application of Fatou's lemma yield that

$$\begin{aligned}
&M^B_{t_2}(\omega) - M^B_{t_1}(\omega)\\
&\ge \liminf_{n\to\infty}\left(M^{B,n}_{t_2}(\omega) - M^{B,n}_{t_1}(\omega) + \int_{[t_1,t_2)\times\mathbb{R}_+} L^{B,n}(\omega,s,x)\mu(\omega,ds,dx)\right)\\
&\quad - \int_{[t_1,t_2)\times\mathbb{R}_+} L^B(\omega,s,x)\mu(\omega,ds,dx)\\
&\ge \liminf_{n\to\infty}\left(\int_{[t_1,t_2)\times\mathbb{R}_+} L^{B,n}(\omega,s,x)\mu(\omega,ds,dx)\right) - \int_{[t_1,t_2)\times\mathbb{R}_+} L^B(\omega,s,x)\mu(\omega,ds,dx)\\
&\ge 0.
\end{aligned} \tag{5.42}$$

Therefore, the candidate $\mathfrak{S} := (M^B, M^S, L^B, L^S)$ for our limit strategy is a valid trading strategy in the sense of Definition 3.4. Right from the definition of M^B and M^S we get that $M^B - M^S = -\eta^1 + \lim_{n\to\infty} \varphi^{1,n} - \int_{[0,\cdot)\times\mathbb{R}_+} L^B(s,x)\mu(ds,dx) + \int_{[0,\cdot)\times\mathbb{R}_+} L^S(s,x)\nu(ds,dx)$ and thus up-$\lim_{n\to\infty} \varphi^{1,n} = \varphi^1(\mathfrak{S})$. It remains to verify that $\varphi^0(\mathfrak{S}) = \psi^0$.

Main step: Let us show that $\varphi^{0,n} \to \varphi^0$ uniformly in probability where $\varphi^0 := \varphi(\mathfrak{S})$. If we are able to show the convergence for the buy and the sell order terms separately, we are done. The idea is to account executed limit buy orders with limit prices close to the best-ask as market buy orders (in the limit they can indeed turn into market orders as Example 6.1 shows, by contrast, executed limit orders "away" from the best-ask price remain limit orders in the limit strategy as there are only finitely many execution times). For $\delta > 0$ let $\mu = \mu^{1,\delta} + \mu^{2,\delta}$ be the decomposition from (4.15). We have

$$\begin{aligned}&\left| \int_{[0,\cdot)\times\mathbb{R}_+} \int_x^{\overline{S}_{s-}} y L^{B,n}(s,dy)\mu^{1,\delta}(ds,dx) - \int_{[0,\cdot)\times\mathbb{R}_+} \overline{S}_{s-} L^{B,n}(s,x)\mu^{1,\delta}(ds,dx) \right| \\ &\leq \delta \int_{[0,T)\times\mathbb{R}_+} L^{B,n}(s,x)\mu^{1,\delta}(ds,dx), \quad \forall n \in \mathbb{N}. \end{aligned} \tag{5.43}$$

Let $\varepsilon > 0$. It follows from Lemma 5.3 that $(\int_{[0,T)\times\mathbb{R}_+} L^{B,n}(s,x)\mu^{1,\delta}(ds,dx))_{n\in\mathbb{N}}$ is P-stochastically bounded. Together with $P(\int_{[0,T)\times\mathbb{R}_+} L^B(s,x)\mu^{1,\delta}(ds,dx) < \infty) = 1$, we obtain the existence of a $\delta > 0$ s.t.

$$d_{\mathrm{up}}\left(\int_{[0,\cdot)\times\mathbb{R}_+} \int_x^{\overline{S}_{s-}} y L^{B,n}(s,dy)\mu^{1,\delta}(ds,dx), (\overline{S}_-, \overline{S}) \bullet \int_{[0,\cdot)\times\mathbb{R}_+} L^{B,n}(s,x)\mu^{1,\delta}(ds,dx) \right) \leq \frac{\varepsilon}{4}$$

for all $n \in \mathbb{N}$ and

$$d_{\mathrm{up}}\left(\int_{[0,\cdot)\times\mathbb{R}_+} \int_x^{\overline{S}_{s-}} y L^B(s,dy)\mu^{1,\delta}(ds,dx), (\overline{S}_-, \overline{S}) \bullet \int_{[0,\cdot)\times\mathbb{R}_+} L^B(s,x)\mu^{1,\delta}(ds,dx) \right) \leq \frac{\varepsilon}{4}.$$

We fix this δ. By Lemma 5.2 and Lemma 4.7 applied to $\mu^{2,\delta}$, there exists an $n_1 \in \mathbb{N}$ with

$$d_{\mathrm{up}}\left(\int_{[0,\cdot)\times\mathbb{R}_+} \int_x^{\overline{S}_{s-}} y L^{B,n}(s,dy)\mu^{2,\delta}(ds,dx), \int_{[0,\cdot)\times\mathbb{R}_+} \int_x^{\overline{S}_{s-}} y L^B(s,dy)\mu^{2,\delta}(ds,dx) \right) \leq \frac{\varepsilon}{4}$$

for all $n \geq n_1$. By Lemma 5.2 and Lemma 4.6 applied to $\mu^{2,\delta}$, we know that $(\int_{[0,\cdot)\times\mathbb{R}_+} L^{B,n}(s,x)\mu^{2,\delta}(ds,dx))_{n\in\mathbb{N}}$ converges to $\int_{[0,\cdot)\times\mathbb{R}_+} L^B(s,x)\mu^{2,\delta}(ds,dx)$ uniformly in probability. This implies by definition of M^B that $(M^{B,n} + \int_{[0,\cdot)\times\mathbb{R}_+} L^{B,n}(s,x)\mu^{1,\delta}(ds,dx))_{n\in\mathbb{N}}$ converges to $M^B + \int_{[0,\cdot)\times\mathbb{R}_+} L^B(s,x)\mu^{1,\delta}(ds,dx)$ uniformly in probability. From Lemma 5.3 and Lemma 5.5 it follows the existence of an n_2 s.t.

$$\begin{aligned} d_{\mathrm{up}}\Bigg(&(\overline{S}_-, \overline{S}) \bullet (M^{B,n} + \int_{[0,\cdot)\times\mathbb{R}_+} L^{B,n}(s,x)\mu^{1,\delta}(ds,dx)), \\ &(\overline{S}_-, \overline{S}) \bullet (M^B + \int_{[0,\cdot)\times\mathbb{R}_+} L^B(s,x)\mu^{1,\delta}(ds,dx)) \Bigg) \leq \frac{\varepsilon}{4}. \end{aligned}$$

Altogether we obtain by the triangle inequality

$$d_{\text{up}}\left((\overline{S}_-,\overline{S})\bullet M^{B,n}+\int_{[0,\cdot)\times\mathbb{R}_+}\int_x^{\overline{S}_{s-}} yL^{B,n}(s,dy)\mu(ds,dx),\right.$$
$$\left.(\overline{S}_-,\overline{S})\bullet M^{B}+\int_{[0,\cdot)\times\mathbb{R}_+}\int_x^{\overline{S}_{s-}} yL^{B}(s,dy)\mu(ds,dx)\right)\leq\varepsilon.$$

As the corresponding result holds for sell orders we obtain that $\varphi^{0,n}\to\varphi^0$ in up and we are done. □

Proof of Theorem 3.25. An inspection of the proof of Theorem 3.13 reveals (i).

For (ii) we only have to show that under the assumptions of the theorem we do not need Assumption 3.16 in the proof of Lemma 5.2. Then, the proof of Theorem 3.17 reveals that the limiting strategy is just as its approximating sequence from $\mathfrak{T}^\delta$.

We have to analyze the neutralization of purchases by limit buy orders and sells by market orders (and vice versa) which are executed at the same time and the same price. This may lead to an explosion of the limit order strategies while the portfolio processes converge.

Step 1: For a market sell order strategy M^S define

$$A_t^1:=\sum_{\tau_i\leq t}1_{\{Y_i=\underline{X}_{\tau_i}=\underline{S}_{\tau_i}\}}\Delta^+M^S_{\tau_i},\quad t\geq 0,$$

with $\underline{X}$ from Definition 3.15. Let us show that the sells by A^1 can be perfectly compensated by purchases with limit orders. Define $A_t^2:=\sum_{\tau_i\leq t}1_{\{Y_i=\underline{X}_{\tau_i}\}}$ and let B^1 and B^2 be the predictable compensators of A^1 and A^2 (w.l.o.g. A^1 is locally integrable, otherwise we can apply the arguments under an equivalent measure). The càdlàg paths of B^1 and B^2 can be identified with measures on $[0,T]$ and by Proposition I.3.13 in [11] there exists a predictable version of the density

$$C_t=\frac{dB_t^1}{dB_t^2}.$$

By (3.9) we can write $\Delta^+M^S_{\tau_1}1_{\{\underline{X}_{\tau_1}=\underline{S}_{\tau_1}\}}=\sum_{k=0}^\infty U_k1_{\{Y_1=\delta k\}}$ for some sequence of $\mathcal{F}_{\tau_1-}$-measurable random variables $(U_k)_{k\in\mathbb{N}_0}$. For all bounded predictable processes H we have

$$E\left(\int_0^{\tau_1\wedge T}H_tC_t\,dB_t^2\right)=E\left(\int_0^{\tau_1\wedge T}H_tC_t\,dA_t^2\right)=E\left(H_{\tau_1}C_{\tau_1}1_{\{\tau_1\leq T,\ Y_1=\underline{X}_{\tau_1}\}}\right)$$

and

$$\begin{aligned}E\left(\int_0^{\tau_1\wedge T}H_tC_t\,dB_t^2\right)&=E\left(\int_0^{\tau_1\wedge T}H_t\,dB_t^1\right)=E\left(\int_0^{\tau_1\wedge T}H_t\,dA_t^1\right)\\&=E\left(H_{\tau_1}1_{\{\tau_1\leq T,\ Y_1=\underline{X}_{\tau_1}\}}\sum_{k=0}^\infty U_k1_{\{\underline{X}_{\tau_1}=\delta k\}}\right).\end{aligned}$$

Putting together we obtain that

$$E\left(H_{\tau_1}C_{\tau_1}1_{\{\tau_1\leq T,\ Y_1=\underline{X}_{\tau_1}\}}\right)=E\left(H_{\tau_1}1_{\{\tau_1\leq T,\ Y_1=\underline{X}_{\tau_1}\}}\sum_{k=0}^\infty U_k1_{\{\underline{X}_{\tau_1}=\delta k\}}\right).$$

C_{τ_1} and $\sum_{k=0}^{\infty} U_k 1_{\{\underline{X}_{\tau_1}=\delta k\}}$ are $\mathcal{F}_{\tau_1-}$-measurable, H_{τ_1} runs through all bounded $\mathcal{F}_{\tau_1-}$-measurable random variables, and the assertion is analog for τ_i, $i > 1$. Therefore, we arrive at

$$P\left(C_{\tau_i} 1_{\{\tau_i \le T,\ Y_i=\underline{X}_{\tau_i}\}} = \Delta^+ M^S_{\tau_i} 1_{\{\tau_i \le T,\ Y_i=\underline{X}_{\tau_i}=\underline{S}_{\tau_i}\}}\right) = 1 \quad \text{for all } i \in \mathbb{N}. \tag{5.44}$$

Step 2: Let $(\varphi^0(\mathfrak{S}^n), \varphi^1(\mathfrak{S}^n))_{n\in\mathbb{N}}$ be the sequence from the beginning of the section. Apply step 1 to the processes

$$A^{1,n}_t := \sum_{\tau_i \le t} 1_{\{Y_i=\underline{X}_{\tau_i}=\underline{S}_{\tau_i}\}} \min\left\{L^{B,n}(\tau_i, \underline{X}_{\tau_i}) - L^{B,n}(\tau_i, \underline{X}_{\tau_i}+\delta), \Delta^+ M^S_{\tau_i}\right\}, \quad n \in \mathbb{N},$$

and define the modified strategies $\widetilde{L}^{B,n}(\omega,t,x) := L^{B,n}(\omega,t,x) - 1_{\{x=\underline{X}_t(\omega)\}} C^n_t(\omega)$ with the predictable processes C^n from (5.44) and

$$\widetilde{M}^{S,n}_t := M^{S,n}_t - \sum_{\tau_i < t} 1_{\{Y_i=\underline{X}_{\tau_i}=\underline{S}_{\tau_i}\}} \min\left\{L^{B,n}(\tau_i, \underline{X}_{\tau_i}) - L^{B,n}(\tau_i, \underline{X}_{\tau_i}+\delta), \Delta^+ M^S_{\tau_i}\right\}.$$

This means that from the original strategy only the *surplus* of limit order purchases at the lowest price on the event $\underline{X}_{\tau_i} = \underline{S}_{\tau_i}$ above market sell orders on $\underline{X}_{\tau_i} = \underline{S}_{\tau_i}$ is actually realized. The modified strategies lead to the same portfolio processes and satisfy (cf. Assumption 3.1(vi))

$$P\Big(\Delta^+ \widetilde{M}^{S,n}_{\tau_i} 1_{\{\tau_i<T,\ Y_i=\underline{X}_{\tau_i}=\underline{S}_{\tau_i}\}} > 0 \ \text{ and } \\ (\widetilde{L}^{B,n}(\tau_i, \underline{X}_{\tau_i}) - \widetilde{L}^{B,n}(\tau_i, \underline{X}_{\tau_i}+\delta)) 1_{\{\tau_i<T,\ Y_i=\underline{X}_{\tau_i}=\underline{S}_{\tau_i}\}} > 0\Big) = 0,$$

i.e. there are no transactions cancelling each other out. Then, from (5.27) it follows as in the proof of Lemma 5.2 that

$$M_\mu\left(\left\{(\omega,t,x) \mid x = \underline{X}_t(\omega),\ \sup_{n\in\mathbb{N}} (\widetilde{L}^{B,n}(\omega,t,x) - \widetilde{L}^{B,n}(\omega,t,x+\delta)) = \infty\right\}\right) = 0 \tag{5.45}$$

(note that on the set $\{\underline{S}_{\tau_i} < Y_i = \underline{X}_{\tau_i}\}$ buying and reselling by market orders leads to a loss of at least $\delta > 0$ per share). With (5.45) we can proceed as in the proof of Lemma 5.2 without accessing Assumption 3.16 and arrive at $M_\mu(\sup_{n\in\mathbb{N}} \widetilde{L}^{B,n} = \infty) = 0$.

□

6 Examples

We give an example of a sequence of limit buy order strategies whose portfolio processes converge to the portfolio process of a market buy order strategy. An inspection of the proof of Theorem 3.17 reveals that this phenomenon cannot occur if the execution measure μ is finite. Namely, for Example 6.1 the last inequality of (5.42), which holds by Fatou's lemma, is strict. On the other hand, for a finite μ we have $\int_{[0,\cdot)\times\mathbb{R}_+} L^B(s,x)\mu(ds,dx) = \lim_{n\to\infty}\left(\int_{[0,\cdot)\times\mathbb{R}_+} L^{B,n}(s,x)\mu(ds,dx)\right)$ and thus also $M^B = \lim_{n\to\infty} M^{B,n}$.

This means that although the set of portfolio processes attainable by limit and market orders is closed by Theorem 3.17, the set of portfolio processes attainable by limit orders alone is not closed.

Example 6.1. *Assume that* $X = (X_t)_{t\in[0,T]}$ *is a Lévy process with infinitely many downward jumps, i.e.* $\lim_{n\to\infty} \overline{\mu}\big((-\infty,-1/n]\big) = \overline{\mu}\big((-\infty,0)\big) = \infty$, *where* $\overline{\mu}$ *is the Lévy measure of* X. *Now let us suppose that the best-ask price* $\overline{S}$ *is modeled as exponential-Lévy, i.e.* $\overline{S}_t = \overline{S}_0 \exp(X_t)$. *Consider the limit buy order strategies satisfying*

$$L^{B,n}(t,x) \;=\; (t-\varphi^{1,n}_t)1_{\left\{x\le \overline{S}_{t-}-\frac{1}{n}\overline{S}_{t-}\right\}} \quad \text{where } \varphi^{1,n}_0 = \varphi^{0,n}_0 = 0,$$

i.e. limit prices are slightly below the best-ask price and directly after a successful execution at time t *the total number of bought assets is* t *(that is* $\varphi^{1,n}_t = t$*).* $L^{B,n}$ *and* $\varphi^{1,n}$ *are obviously well-defined with* $0 \le \varphi^{1,n}_t \le t$ *as for every* n *and every path there are only finitely many executions. Let us show that the associated portfolio processes* $(\varphi^{0,n},\varphi^{1,n})_{n\in\mathbb{N}}$ *converge to* $\varphi^0_t := -\int_0^t \overline{S}_s\,ds$ *and* $\varphi^1_t := t$ *uniformly in probability.* (φ^0,φ^1) *is generated by the market buy order strategy* $M^B_t = t$. *Let* $Z_1,\ldots,Z_m$ *be i.i.d. exponential random variables with parameter* 1. *A well-known limit result for maxima tells us that*

$$P\left(Z_k \le \ln(m)+x,\ k=1,\ldots,m\right) \;\to\; \exp(-\exp(-x)), \quad m\to\infty, \quad \forall x\in\mathbb{R}. \tag{6.46}$$

By $\Delta\overline{S} = \overline{S}_-(\exp(\Delta X)-1)$, *an execution of* $L^{B,n}$ *is triggered by jumps* $\Delta X \le \ln(1-1/n)$. *The interarrival times of these jumps are i.i.d. exponentially distributed random variables with parameter* $\overline{\mu}\Big(\big(-\infty,\ln(1-1/n)\big]\Big)$ *and thus we have for any* $\varepsilon>0$

$$\begin{aligned}
&P(\text{“time between two successive executions of } L^{B,n} \text{ is always smaller than } \varepsilon\text{”})\\
&\ge P\left(Z_k < \varepsilon\overline{\mu}\Big(\big(-\infty,\ln(1-1/n)\big]\Big),\ k=1,\ldots,\left[2T\overline{\mu}\Big(\big(-\infty,\ln(1-1/n)\big]\Big)\right]\right)\\
&-P\left(\sum_{k=1}^{[2T\overline{\mu}((-\infty,\ln(1-1/n)])]} Z_k < T\overline{\mu}\Big(\big(-\infty,\ln(1-1/n)\big]\Big)\right),
\end{aligned} \tag{6.47}$$

where $[x] := \max\{k\in\mathbb{N}_0 \mid k\le x\}$. *Put* $m := [2T\overline{\mu}((-\infty,\ln(1-1/n)])]$. *By (6.47), (6.46), the law of large numbers, and the fact that* $(\overline{\mu}((-\infty,\ln(1-1/n)]))_{n\in\mathbb{N}}$ *converges faster to infinity than* $(\ln(2T\overline{\mu}((-\infty,\ln(1-1/n)])))_{n\in\mathbb{N}}$, *it follows that for any* $\varepsilon>0$ *there exists an* n_0 *s.t. for all* $n\ge n_0$

$$\begin{aligned}
&P\Big(|\varphi^{1,n}_t - \varphi^1_t| \le \varepsilon, \quad \forall t\in[0,T]\Big) \\
&\ge P(\text{“time between two successive executions of } L^{B,n} \text{ is always smaller than } \varepsilon\text{”}) \ge 1-\varepsilon.
\end{aligned} \tag{6.48}$$

It remains to show that $(\varphi^{0,n})_{n\in\mathbb{N}}$ *converges to* φ^0. *We have that* $\operatorname{var}(\varphi^{1,n})_T \le T$ *for all* $n\in\mathbb{N}$. *Consequently, we can apply Lemma 5.5 and conclude that* $\big((\overline{S}_-,\overline{S}) \bullet \varphi^{1,n}\big)_{n\in\mathbb{N}}$ *converges to* $(\overline{S}_-,\overline{S})\bullet\varphi^1 = -\varphi^0$ *uniformly in probability.*

Let $\varepsilon>0$. *Due to the up-convergence there exists an* $n_0\in\mathbb{N}$ *s.t.*

$$P\Big(|(\overline{S}_-,\overline{S})\bullet\varphi^{1,n}_t + \varphi^0_t| \le \varepsilon/3, \quad \forall t\in[0,T]\Big) \;\ge\; 1-\varepsilon/3, \quad \forall n\ge n_0. \tag{6.49}$$

For any $\delta>0$ *and* $t\in[0,T]$ *we have that*

$$\begin{aligned}
\left|\varphi^{0,n}_t + (\overline{S}_-,\overline{S})\bullet\varphi^{1,n}_t\right| \;&=\; \left|\int_{[0,t)\times\mathbb{R}_+}\int_x^\infty (y-\overline{S}_s)L^{B,n}(s,dy)\mu(ds,dx)\right|\\
&\le\; \delta\int_{[0,T]\times\mathbb{R}_+}\int_x^\infty L^{B,n}(s,dy)\mu^{1,\delta}(ds,dx)\\
&+\; \sup_{s\in[0,T]}|\overline{S}_s| \sup_{s\in[0,T]}|\varphi^{1,n}_s-\varphi^1_s|\mu^{2,\delta}([0,T]\times\mathbb{R}_+) =: I(n)+II(n),
\end{aligned}$$

where the decomposition $\mu = \mu^{1,\delta} + \mu^{2,\delta}$ *is defined after equation (4.15). As* $\int_{[0,T]\times\mathbb{R}_+}\int_{[x,\infty)} L^{B,n}(s,dy)\mu(ds,dx) \le T$ *for all* $n \in \mathbb{N}$*, we choose* $\delta := \varepsilon/(3T)$ *to obtain* $P(I(n) \le \varepsilon/3) = 1$ *for all* $n \in \mathbb{N}$*. We fix this* δ *and observe that* $P(\mu^{2,\delta}([0,T]\times\mathbb{R}_+) < \infty) = 1$*. Thus, by (6.48) applied to some appropriate* $\widetilde{\varepsilon} > 0$*, there exists an* $n_1 \in \mathbb{N}$ *s.t.* $P(II(n) \le \varepsilon/3) \ge 1-(2\varepsilon)/3$ *for all* $n \ge n_1$*. Combining this with (6.49) we arrive at*

$$P(|\varphi_t^{0,n} - \varphi_t^0| \le \varepsilon, \quad \forall t \in [0,T]) \ \ge\ 1-\varepsilon, \quad \forall n \ge n_0 \vee n_1.$$

Example 6.2. *In [9] and [12] small trader models are considered in which limit buy orders can only be placed at the current best-bid* $\underline{S}$ *and limit sell orders only at the current best-ask price* $\overline{S}$ *(or one tick above resp. below it). As* $\underline{S}$ *and* $\overline{S}$ *move continuously in time, it is interesting to see how these strategies can be approximated by real-world strategies with piecewise constant limit prices (and order sizes). For this purpose let us embed the model from Definition 1 in [12] into the more general framework of the current article (for the model in [9] the arguments are similar). Restricting to limit buy orders this yields strategies of the form*

$$L^B(\omega,t,x) := \widetilde{L}_t^B(\omega)1_{[0,\underline{S}_t(\omega)]}(x), \tag{6.50}$$

where $\underline{S}$ *is assumed to be continuous and the nonnegative predictable process specifying the size of the limit buy order at the best-bid is now denoted by* $\widetilde{L}^B$*. The execution measure* μ *is given by the condition that* $\mu(\omega,\{t\}\times\{x\}) = 1$ *iff* $t = \tau_i(\omega)$ *for some* $i \in \mathbb{N}$ *and* $x = \underline{S}_t(\omega)$*, where* τ_i *are the jump times of a counting process (e.g. a homogeneous Poisson process independent of* $\underline{S}$ *and* $\overline{S}$*) modeling the arrival times of noise traders. Theorem 3.13 yields that (6.50) can be approximated by real-world strategies. The approximating limit buy order strategy* $L^{\delta,m}$ *from Definition 4.4 is given by* $L^{\delta,m}(\omega,t,x) = \widetilde{L}_t^B(\omega)1_{[0,p^m(\omega,t)]}(x)1_{[\![0,\tau^\delta]\!]}(\omega,t)$*, where* $p^m(\omega,t) := \max\{l2^{-m} \mid (l-1)2^{-m} \le \underline{S}_t(\omega) \wedge (\overline{S}_t^\delta - 3\delta)\}$ *and* τ^δ *is defined in (4.10) for small* $\delta > 0$*. This means that* p^m *usually lies slightly above the current best-bid price and is bounded away from the best-ask. For fixed* (ω,t)*,* $\xi_t^l(\omega)$ *defined in the proof of Theorem 4.8 is only positive for at most one* l *in this example (namely* l *has to satisfy* $p^m(\omega,t) = l2^{-m}$*). Then, the approximating simple predictable* $\widetilde{\xi}^l$ *from (4.16) can be chosen to satisfy this as well. Thus we arrive at a single simple predictable limit order whose limit price dominates* $\underline{S}$ *"most of the time" and the order size approximates* $\widetilde{L}^B$*.*

7 Conclusion

We provide a mathematical framework to model continuous time trading of a small investor in limit order markets. Starting with quite arbitrary best bid and best ask price processes $\underline{S}$ and $\overline{S}$ at which market orders can be executed, the execution of limit orders is modelled by integer-valued random measures μ and ν which are consistent with $\underline{S}$ and $\overline{S}$. The general limit buy (sell) order strategies are predictable processes with values in the set of nonincreasing demand (nondecreasing supply) functions – not necessarily left- or right-continuous in the price variable. Accumulated market orders are nondecreasing predictable processes. The strategy set possesses the desirable properties that it is closed (under the convergence "uniformly in probability" of the portfolio process) and any attainable portfolio process can be approximated by portfolio processes from elementary strategies uniformly in probability. An interesting observation is that if the best-bid or the best-ask price processes possess infinitely many jumps on compact time intervals, then a sequence of limit order strategies can turn into a market order strategy when tending to the limit.

Trading strategies with values in an infinite-dimensional space also appear in idealized (frictionless) bond markets. The existence of tradable bonds for all maturities leads to a continuum of assets and thus the portfolio is infinite-dimensional. However, here the economically meaningful set of measure-valued portfolio strategies turns out to be too small as it is not complete and one has to extend it to strategies taking values in the set of linear functionals acting on the set of bond price curves (see [2], [6], and the references therein).

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