

# Optimal investment under capital gains taxes

(not yet on arXiv—a slightly preliminary version is available from me)

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Metabief

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- In most countries, trading gains have to be taxed.
- Extensive financial economics literature, relatively few mathematical finance papers: Jouini, Koehl, Touzi (1999, 2000), Ben Tahar, Soner, Touzi (2007, 2010), Seifried (2010), Cai, Chen, Dai (2018), Bian, Chen, Dai, Qian (2021), ...

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- Core principles
  - (1) Tax systems are **realization based**: gains are taxed when assets are sold and not when gains arise on paper.
  - (2) “**Limited use of losses**”: losses do not trigger negative taxes, but can only be offset against gains in the same year or in the future
- How to model this?

## Exact tax basis in discrete time, following Dybvig & Koo (1996)

- Given: filtration  $(\mathcal{F}_t)_{t=0,\dots,T}$ , stock prices  $S_t^j$ , interest rate  $r_t$
- $N_{i,t,j} \in L_+^0(\mathcal{F}_t)$ : number of stocks of type  $j$  that are bought at time  $i$  and kept in the portfolio at least after trading at  $t \geq i$   
 $N_{i,i,j} \geq N_{i,i+1,j} \geq \dots \geq N_{i,T,j} = 0$

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- Bank account:  $\eta_t \in L^0(\mathcal{F}_t)$  number of €s after trading at  $t$
- Accumulated realized profits and losses:

$$G_t := \sum_{u=1}^t \left( \eta_{u-1} r_u + \sum_{i=0}^{u-1} \langle N_{i,u-1} - N_{i,u}, S_u - S_i \rangle \right)$$

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- $(\eta, N)$  is self-financing with initial capital  $x$  iff  $\eta_{-1} = x$  and

$$\eta_t - \eta_{t-1} = r_t \eta_{t-1} \mathbf{1}_{(t \geq 1)} + \left\langle \sum_{i=0}^{t-1} (N_{i,t-1} - N_{i,t}) - N_{t,t}, S_t \right\rangle - (\Pi_t - \Pi_{t-1})$$

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- $r_t \geq 0 \forall t \implies N \mapsto \eta_T =: V^\alpha(x, N)$  is concave.

# Utility maximization

## Assumption

Let  $U : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty\}$  be a function that takes the value  $-\infty$  on  $\mathbb{R}_- \setminus \{0\}$ , whose restriction to  $\mathbb{R}_+ \setminus \{0\}$  is  $\mathbb{R}$ -valued, nondecreasing, and concave, and that satisfies  $U(0) = \lim_{x \searrow 0} U(x)$ .

Consider the maximization problem

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## Proposition (Dimitrov and K. (2026+))

Let  $x > 0$  and  $\alpha \in [0, 1)$ . We have the equivalence:

$$u^\alpha(x) < \infty \quad \Leftrightarrow \quad u^0(x) < \infty$$

## Idea of proof.

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“ $\Rightarrow$ ”  $\forall N \exists \tilde{N}$  such that  $V^\alpha(x, \tilde{N}) \geq (1 - \alpha)^T V^0(x, N)$  if  $\text{RHS} \geq 0$   $\square$

## Theorem (Rásonyi and Stettner (2005): frictionless markets)

*No-arbitrage  $\implies \sup_N \mathbb{E}[U(V^0(x, N))]$  is attained (if it is finite)*

### Idea of proof.

- In discrete time frictionless models, no-arbitrage already implies that  $\{V^0(x, N) : N \text{ strategy}\} - L_+^0(\mathcal{F}_T)$  is closed in probability (Dalang, Morton, and Willinger (1990))
- In a one-period model,  $\exists L \in L_+^0(\mathcal{F}_1)$  with  $\mathbb{E}[L] < \infty$  s.t.

$$L \geq U^+(x + \langle \beta, S_1 - S_0 \rangle) \text{ a.s. } \forall \text{ strategies } \beta \text{ s.t. } x + \langle \beta, S_1 - S_0 \rangle \geq 0$$

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- In general,  $L \neq U^+(\hat{x} + \langle \hat{\beta}, S_1 - S_0 \rangle)$  for some  $\hat{x}, \hat{\beta}$ , thus, not directly extendable to multiperiod model.
- Rásonyi and Stettner (2005) develop dynamic programming principle  $\rightsquigarrow$  supremum is also attained in multiperiod model.

## With taxes: no-arbitrage $\nRightarrow$ supremum is attained

### Counterexample:

Let  $T = 3$ ,  $r > 0$ ,  $\Omega = \{\omega_{k,i,j} : k \in \mathbb{N}, i = 1, 2, j = 1, 2\}$ ,  $k$  is revealed at time 1,  $i$  at time 2,  $j$  at time 3

Stock price:  $S_0 = 1$ ,  $S_1 = 1 + r$

$$S_2(\omega_{k,1,j}) = (1+r)(1+r+1/k), \quad S_2(\omega_{k,2,j}) = (1+r)(1+r-1/k)$$

$$S_3(\omega_{k,1,1}) = (1+r)(1+r+1/k)(1+r+1/k), \quad S_3(\omega_{k,2,1}) = \dots$$

- **At time 0, stocks can be purchased on credit without risk, but losses can be realized and gains deferred.**
- Probabilities are chosen such that it is optimal to hold  $k$  shares between 1 and 2 and  $k/2$  shares between 2 and 3.

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 $\omega = \omega_{k,i,j}$  but  $k$  is not known at time 0

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- Supremum can only be approximated by  $k \rightarrow \infty$ .

## When is the set of attainable terminal wealth

$\{V^\alpha(x, N) : N \text{ strategy}\} - L_+^0(\mathcal{F}_T)$  closed in probability?

- If  $|\Omega| < \infty$ , then it is always closed (even NA is not needed).

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**Definition** The market model satisfies the **no unbounded non-substitutable investment with bounded risk (NUIBR)**

condition iff:  $\forall$  initial capital  $x \in \mathbb{R}_+ \exists$  reaction function  $R$  with  $V^\alpha(x, R(N)) \geq V^\alpha(x, N)$  a.s.  $\forall N$  s.t.  $\forall K \in \mathbb{R}_+$

$\text{conv} \left\{ \max_{i=0, \dots, T-1, j=1, \dots, d} R_{i,j}(N) : N \in \mathcal{N} \text{ with } V^\alpha(x, N) \geq -K \right\}$   
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is  $L^0$ -bounded.

“After eliminating trades that are not strictly required from the strategies, the remaining trading volumes cannot explode if the worst-case risk of the strategies is bounded.”

**Theorem (Dimitrov and K. (2026+))**

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### **Theorem (Dimitrov and K. (2026+))**

*No-arbitrage and  $\{V^\alpha(x, N) : N \text{ strategy}\} - L_+^0(\mathcal{F}_T)$  is closed in probability*

$\implies \sup_N \mathbb{E}[U(V^\alpha(x, N))] \text{ is attained (if it is finite)}$

**Key Lemma** Let  $d$  be the number of stocks. In a one-period (frictionless) model satisfying no-arbitrage (NA), there exists a strategy  $\hat{\beta} \in \mathbb{R}_+^d$  (resp.  $\mathbb{R}^d$ ) s.t.  $\forall \beta \in \mathbb{R}_+^d$  (resp.  $\mathbb{R}^d$ ):

$$(2^{d+1} - 1) \left[ 1 + r_1 + \left\langle \hat{\beta}, S_1 - (1 + r_1)S_0 \right\rangle \right] \\ \geq 1 + r_1 + \langle \beta, S_1 - (1 + r_1)S_0 \rangle \text{ a.s.} \quad \text{if RHS} \geq 0 \text{ a.s.}$$

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- Directly extendable to multiperiod model: you need initial capital  $(2^{d+1} - 1)^T$  to dominate all strategies with initial capital 1 simultaneously (of course, we need conditional versions)
- Since  $V^\alpha(x, N) \leq V^0(x, N)$  for all strategies  $N$ , but by concavity etc.,

$$u^\alpha(x) < \infty \implies u^0(x) < \infty \implies u^0((2^{d+1} - 1)^T x) < \infty,$$

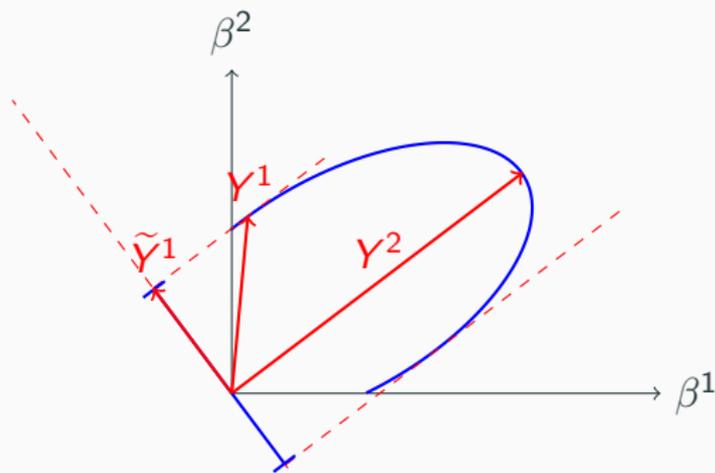
we have an **integrable majorant**.

## Idea of proof of simultaneous superhedging

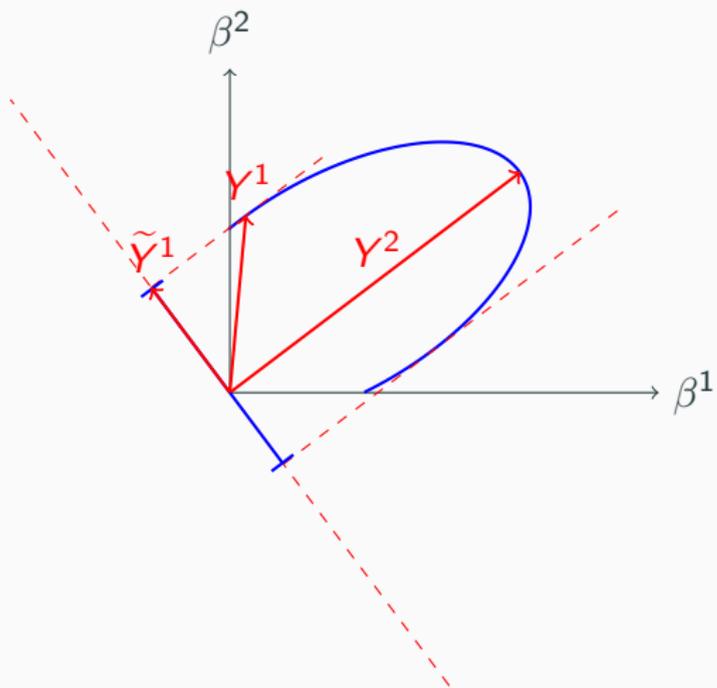
**Idea of proof:** Consider the compact-valued(!) random set

$$A := \{\beta \in \mathcal{P}_0 : 1 + r_1 + \langle \beta, S_1 - (1 + r_1)S_0 \rangle \geq 0 \text{ a.s.}\}$$

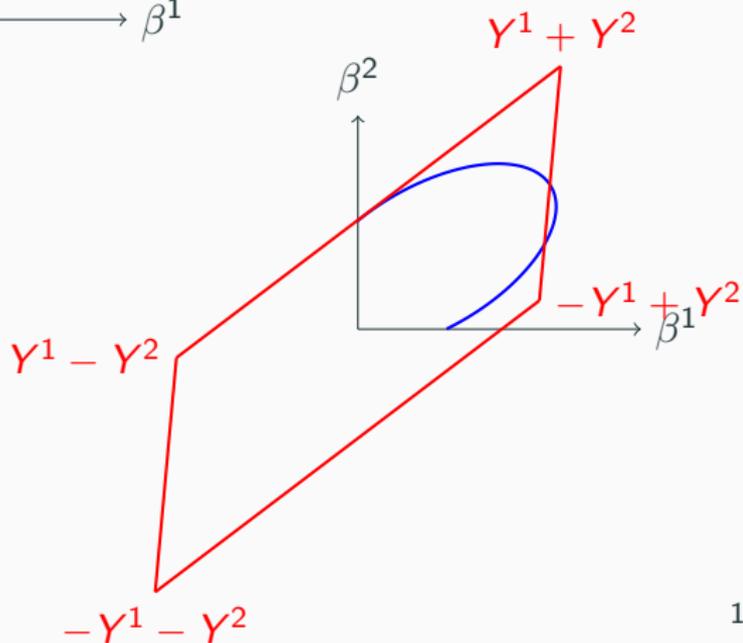
$\mathcal{P}_0 \subseteq \mathbb{R}_+^d$ : “purely non-reversible” strategies



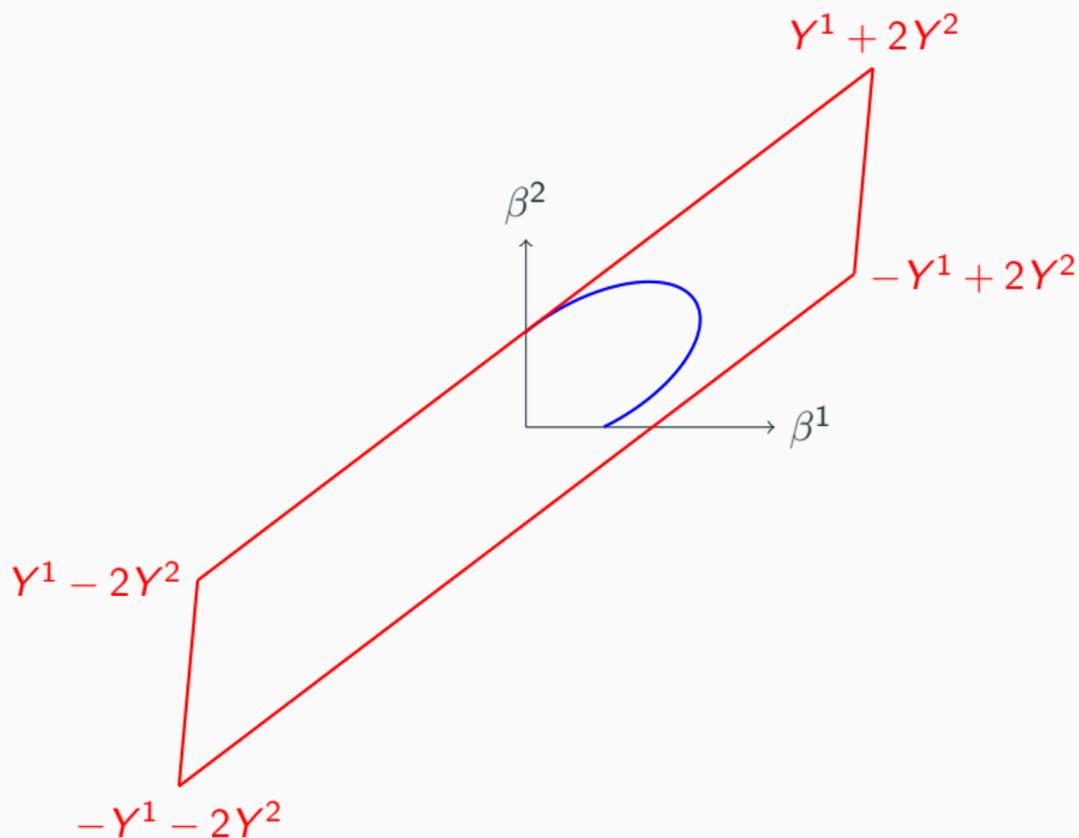
Show:  $\exists (Y^i)_{i=1, \dots, d} \in A \forall \beta \in A \exists \lambda_i \in [-2^{i-1}, 2^{i-1}] \beta = \sum_{i=1}^d \lambda_i Y^i$   
 $Y^2$  maximizes the Euclidean norm on  $A$  and  $\tilde{Y}^1$  that on the orthogonal projection of  $A$  along  $Y^2$ .  $Y^1 \in A$  is projected on  $\tilde{Y}^1$ .



One needs  $\pm Y^1 \pm 2Y^2$



One needs  $[-1, 1]Y^1 + [-2, 2]Y^2$  to overlap  $A$ .



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$\mathcal{P}_0 \subseteq \mathbb{R}_+^d$ : “purely non-reversible” strategies ( $A$  random set for  $t > 0$ )

He have shown:  $\exists (Y^i)_{i=1, \dots, d} \in A$  s.t.

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Then, for  $\lambda_i \in [0, 2^{i-1}]$ :

$$\lambda_i \langle Y^i, S_1 - (1 + r_1)S_0 \rangle \leq 2^{i-1} [1 + r_1 + \langle Y^i, S_1 - (1 + r_1)S_0 \rangle]$$

For  $\lambda_i \in [-2^{i-1}, 0)$ :

$$\lambda_i \langle Y^i, S_1 - (1 + r_1)S_0 \rangle \leq 2^{i-1} [2(1 + r_1) + \langle Y^i, S_1 - (1 + r_1)S_0 \rangle]$$

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We conclude:

$$\begin{aligned} & 1 + r_1 + \left\langle \sum_{i=1}^d \lambda_i Y^i, S_1 - (1 + r_1)S_0 \right\rangle \\ & \leq 1 + r_1 + 2 \sum_{i=1}^d 2^{i-1} (1 + r_1) + \left\langle \sum_{i=1}^d 2^{i-1} Y^i, S_1 - (1 + r_1)S_0 \right\rangle \end{aligned}$$

**Thank you for your attention!**

## Selected references

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