The fundamental theorem of asset pricing with and without transaction costs

(preprint available at arXiv or on my website)

Christoph Kühn
Institute of Mathematics, Goethe University Frankfurt
http://www.math.uni-frankfurt.de/∼ismi/kuehn/privat.shtml

Risk & Stochastics and Financial Mathematics Seminar
London School of Economics and Political Science
February 22nd, 2024

Literature review: FTAP under proportional transaction costs

- Kabanov, Stricker (2001), Finite discrete time and $|\Omega| < \infty$:
  No arbitrage in a general "currency model" $(\Pi^{ij})_{1 \le i,j \le d}$
  (i.e., $\Pi_t^{ij}(\omega)$ is price of asset j when you pay with asset i)
  $\Leftrightarrow \exists$ **consistent price system (CPS)**, i.e., a P-martingale
  $(Z^1, \dots, Z^d) > 0$ with $1/\Pi_t^{ji} \le Z_t^j / Z_t^i \le \Pi_t^{ij}$.

Literature review: FTAP under proportional transaction costs

- Kabanov, Stricker (2001), Finite discrete time and $|\Omega| < \infty$: No arbitrage in a general "currency model" $(\Pi^{ij})_{1\leq i,j\leq d}$ (i.e., $\Pi^{ij}_t(\omega)$ is price of asset j when you pay with asset i) $\Leftrightarrow \exists$ **consistent price system (CPS)**, i.e., a P-martingale $(Z^1, \ldots, Z^d) > 0$ with $1/\Pi^{ji}_t \leq Z^j_t / Z^i_t \leq \Pi^{ij}_t$.
- Grigoriev (2005), "$\Leftrightarrow$" holds for general Ω if there is only one risky asset (besides a "bank account", i.e., $d = 2$)
- Counterexample by Schachermayer (2004) for "$\Rightarrow$" if $|\Omega| = \infty$ (with more than two assets).

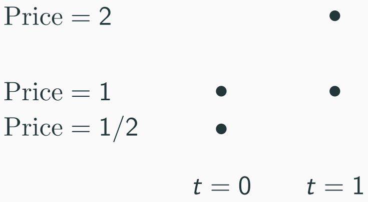
Literature review: FTAP under proportional transaction costs

- Kabanov, Stricker (2001), Finite discrete time and $|\Omega| < \infty$: No arbitrage in a general "currency model" $(\Pi^{ij})_{1 \leq i,j \leq d}$ (i.e., $\Pi^{ij}_t(\omega)$ is price of asset j when you pay with asset i) $\Leftrightarrow \exists$ **consistent price system (CPS)**, i.e., a P-martingale $(Z^1, \ldots, Z^d) > 0$ with $1/\Pi^{ji}_t \leq Z^j_t / Z^i_t \leq \Pi^{ij}_t$.
- Grigoriev (2005), "$\Leftrightarrow$" holds for general Ω if there is only one risky asset (besides a "bank account", i.e., $d = 2$)
- Counterexample by Schachermayer (2004) for "$\Rightarrow$" if $|\Omega| = \infty$ (with more than two assets).
- Schachermayer (2004), **Robust no-arbitrage** in a discrete time model $(\Pi^{ij})_{1 \leq i,j \leq d}$ $\Leftrightarrow \exists$ strictly consistent price system (SCPS), i.e., a P-martingale $(Z^1, \ldots, Z^d) > 0$ satisfying $1/\Pi^{ji}_t \leq Z^j_t / Z^i_t \leq \Pi^{ij}_t$ with "$<$" if $1/\Pi^{ji}_t < \Pi^{ij}_t$.

Basic phenomenon that can (only) occur in models with transaction costs

Example

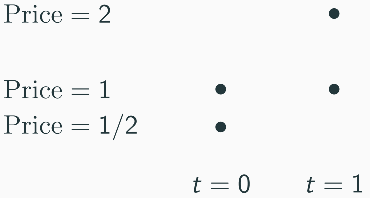
Let $T = 1$, $\Omega = \{1, 2, 3, \ldots\}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = 2^\Omega$, i.e., $\omega \in \Omega$ is revealed at time 1. Non-random bid-ask prices of asset are given by



Basic phenomenon that can (only) occur in models with transaction costs

Example

Let $T = 1$, $\Omega = \{1, 2, 3, \ldots\}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = 2^\Omega$, i.e., $\omega \in \Omega$ is revealed at time 1. Non-random bid-ask prices of asset are given by

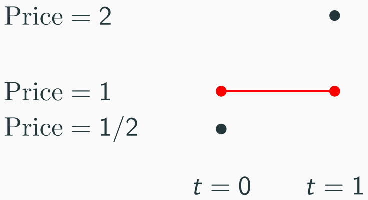


- Aim: investor wants to have ω assets and $-\omega$ monetary units.
- Problem: at price 1 asset can only be purchased in $t = 0$, but ω is not yet known.

Basic phenomenon that can (only) occur in models with transaction costs

Example

Let $T = 1$, $\Omega = \{1, 2, 3, \ldots\}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = 2^{\Omega}$, i.e., $\omega \in \Omega$ is revealed at time 1. Non-random bid-ask prices of asset are given by

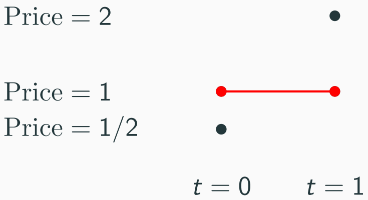


- Aim: investor wants to have ω assets and $-\omega$ monetary units.
- Problem: at price 1 asset can only be purchased in $t = 0$, but ω is not yet known.
- Solution: Buy in advance $n \in \mathbb{N}$ assets in 0, sell $(n - \omega)^+$ at the **same price** in 1 $\rightsquigarrow$ $\omega \wedge n$ assets, $-(\omega \wedge n)$ monetary units.

Basic phenomenon that can (only) occur in models with transaction costs

Example

Let $T = 1$, $\Omega = \{1, 2, 3, \ldots\}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = 2^\Omega$, i.e., $\omega \in \Omega$ is revealed at time 1. Non-random bid-ask prices of asset are given by



- Aim: investor wants to have ω assets and $-\omega$ monetary units.
- Solution: Buy in advance $n \in \mathbb{N}$ assets in 0, sell $(n-\omega)^+$ at the **same price** in 1 $\rightsquigarrow$ $\omega \wedge n$ assets, $-(\omega \wedge n)$ monetary units.
- Aim can only be achieved approximately for $n \to \infty$. **Set of attainable portfolio positions is not closed in probability.**

Way out: prospective strict no-arbitrage NA^{ps} (discrete time)

- Let $\mathcal{A}_s^t$ be the set of portfolio positions that are attainable from zero endowment by trading between s and t
- **Prospective strict no-arbitrage** NA^{ps}:

$$\mathcal{A}_0^t \cap (-\mathcal{A}_t^T) \subseteq \mathcal{A}_t^T \quad \forall t = 0, 1, \ldots, T.$$

Way out: prospective strict no-arbitrage NA^{ps} (discrete time)

- Let $\mathcal{A}_s^t$ be the set of portfolio positions that are attainable from zero endowment by trading between s and t
- **Prospective strict no-arbitrage** NA^{ps}:

$$\mathcal{A}_0^t \cap (-\mathcal{A}_t^T) \subseteq \mathcal{A}_t^T \quad \forall t = 0, 1, \ldots, T.$$

- The difference to **strict no-arbitrage** NA^s of Kabanov, Rásonyi, Stricker (2002), which says $\mathcal{A}_0^t \cap (-\mathcal{A}_t^t) \subseteq \mathcal{A}_t^t$, is: we do not distinguish between a trade at time t and a trade from which we know for sure at time t that it can be realized in the future.

Way out: prospective strict no-arbitrage NA^{ps} (discrete time)

- Let $\mathcal{A}_s^t$ be the set of portfolio positions that are attainable from zero endowment by trading between s and t
- **Prospective strict no-arbitrage** NA^{ps}:

$$\mathcal{A}_0^t \cap (-\mathcal{A}_t^T) \subseteq \mathcal{A}_t^T \quad \forall t = 0, 1, \ldots, T.$$

- The difference to **strict no-arbitrage** NA^s of Kabanov, Rásonyi, Stricker (2002), which says $\mathcal{A}_0^t \cap (-\mathcal{A}_t^t) \subseteq \mathcal{A}_t^t$, is: we do not distinguish between a trade at time t and a trade from which we know for sure at time t that it can be realized in the future.
- K. and Molitor (2019): $\mathrm{NA}^{ps} \implies \mathcal{A}_0^T$ is closed in probability. ($\implies$ $\exists$ consistent price system)
- Aim of the present talk: extend this to continuous time

Literature review: FTAP under proportional transaction costs in continuous time

- Guasoni/Lépinette/Rásonyi (2012)
  Robust no free lunch with vanishing risk (RNFLVR)
  $\Leftrightarrow \exists$ strictly consistent price system

- Guasoni, Rásonyi, and Schachermayer (2010): FTAP for a continuous mid-price process and deterministic transaction costs:
  no-arbitrage for all transaction costs > 0
  $\Leftrightarrow$ for all transaction costs > 0, $\exists$ consistent price system

Extension of prospective strict no-arbitrage to continuous time

We consider a market that consists of a bank account and one risky asset with **bid price** $\underline{S}$ and **ask price** $\overline{S}$.

Assumption
$(\underline{S}_t)_{t\in[0,T]}$ and $(\overline{S}_t)_{t\in[0,T]}$ are adapted càdlàg processes (**not necessarily semimartingales**) with $0 \leq \underline{S} \leq \overline{S}$ and $\overline{S}_T > 0$.

We do not assume efficient friction, i.e., $\underline{S}_t(\omega) = \overline{S}_t(\omega)$ is possible.

Extension of prospective strict no-arbitrage to continuous time

We consider a market that consists of a bank account and one risky asset with **bid price** $\underline{S}$ and **ask price** $\overline{S}$.

Assumption
$(\underline{S}_t)_{t\in[0,T]}$ and $(\overline{S}_t)_{t\in[0,T]}$ are adapted càdlàg processes (**not necessarily semimartingales**) with $0 \le \underline{S} \le \overline{S}$ and $\overline{S}_T > 0$.

We do not assume efficient friction, i.e., $\underline{S}_t(\omega) = \overline{S}_t(\omega)$ is possible.

But, **continuous transitions** between zero and positive bid-ask spreads require an additional condition:

Assumption
[K. and Molitor (2022)] For every $(\omega, t) \in \Omega \times [0, T)$ with $\overline{S}_t(\omega) = \underline{S}_t(\omega)$ there exists $\varepsilon > 0$ s.t. $\overline{S}_s(\omega) = \underline{S}_s(\omega)$ for all $s \in (t, t+\varepsilon)$ or $\overline{S}_s(\omega) > \underline{S}_s(\omega)$ and $\overline{S}_{s-}(\omega) > \underline{S}_{s-}(\omega)$ for all $s \in (t, t+\varepsilon)$.

General strategies, results from K. and Molitor (2022)

- For a FTAP, general strategies are needed.
- "No unbounded profit with bounded risk" in bid-ask model with simple strategies $\Longrightarrow$ $\exists$ semimartingale S with $\underline{S} \leq S \leq \overline{S}$

General strategies, results from K. and Molitor (2022)

- For a FTAP, general strategies are needed.
- "No unbounded profit with bounded risk" in bid-ask model with simple strategies $\implies \exists$ semimartingale S with $\underline{S} \leq S \leq \overline{S}$
- Let φ^0 position in bank account, φ position in risky asset.

$\varphi^0_t := \int_0^t \varphi\, dS - \varphi_t S_t - Cost^S_t(\varphi)$ (self-financing condition for bounded predictable φ)

General strategies, results from K. and Molitor (2022)

- For a FTAP, general strategies are needed.
- "No unbounded profit with bounded risk" in bid-ask model with simple strategies $\Longrightarrow$ $\exists$ semimartingale S with $\underline{S} \leq S \leq \overline{S}$
- Let φ^0 position in bank account, φ position in risky asset.

$\varphi_t^0 := \int_0^t \varphi \, dS - \varphi_t S_t - Cost_t^S(\varphi)$ (self-financing condition for bounded predictable φ)

- φ^0 does not depend on the choice of the semimartingale S.

General strategies, results from K. and Molitor (2022)

- For a FTAP, general strategies are needed.
- "No unbounded profit with bounded risk" in bid-ask model with simple strategies $\Longrightarrow$ $\exists$ semimartingale S with $\underline{S} \leq S \leq \overline{S}$
- Let φ^0 position in bank account, φ position in risky asset.

$\varphi^0_t := \int_0^t \varphi\, dS - \varphi_t S_t - Cost^S_t(\varphi)$ (self-financing condition for bounded predictable φ)

- φ^0 does not depend on the choice of the semimartingale S.
- A general strategy (φ^0, φ) need not be of finite variation. But, for special case $\varphi = \varphi^{\uparrow} - \varphi^{\downarrow}$:
  $Cost^S_t(\varphi) = \int_0^t (\overline{S} - S)\, d\varphi^{\uparrow} + \int_0^t (S - \underline{S})\, d\varphi^{\downarrow}$
  (integrals are increasing: more trades lead to higher "costs", gains $\int_0^t \varphi\, dS$ of a bounded φ are finite, this allows to exhaust all costs in a monotone manner)

General strategies, results from K. and Molitor (2022)

- For a FTAP, general strategies are needed.
- "No unbounded profit with bounded risk" in bid-ask model with simple strategies $\implies \exists$ semimartingale S with $\underline{S} \leq S \leq \overline{S}$
- Let φ^0 position in bank account, φ position in risky asset.

$\varphi^0_t := \int_0^t \varphi \, dS - \varphi_t S_t - Cost^S_t(\varphi)$ (self-financing condition for bounded predictable φ)

- φ^0 does not depend on the choice of the semimartingale S.
- A general strategy (φ^0, φ) need not be of finite variation.
  But, for special case $\varphi = \varphi^\uparrow - \varphi^\downarrow$:
  $Cost^S_t(\varphi) = \int_0^t (\overline{S} - S) \, d\varphi^\uparrow + \int_0^t (S - \underline{S}) \, d\varphi^\downarrow$
  (integrals are increasing: more trades lead to higher "costs", gains $\int_0^t \varphi \, dS$ of a bounded φ are finite, this allows to exhaust all costs in a monotone manner)
- Extension to unbounded φ.

Admissibility (numéraire-free)

In continuous time, strategies like "favorable" doubling strategies have to be ruled out.

Useful mathematical objects are **actual bid/ask prices**:

$$\underline{X}_t = \operatorname{essinf}_{\mathcal{F}_t} \sup_{u \in [t,T]} \underline{S}_u$$
$$\overline{X}_t = \operatorname{esssup}_{\mathcal{F}_t} \inf_{u \in [t,T]} \overline{S}_u$$

Of course, $\underline{X} \geq \underline{S}$ and $\overline{X} \leq \overline{S}$

Admissibility (numéraire-free)

- Let $M \in \mathbb{R}_+$. A self-financing strategy (φ^0, φ) is called **M-admissible** iff $\varphi^0 + M + (\varphi + M)^+ \underline{X} - (\varphi + M)^- \overline{X} \geq 0$.
- Stating the admissibility condition in terms of the actual bid/ask prices $\underline{X}$ and $\overline{X}$ is equivalent to **freezing** a portfolio position as in Guasoni, Lépinette, Rásonyi (2012), but our condition is "numéraire-free".

Admissibility (numéraire-free)

- Let $M \in \mathbb{R}_+$. A self-financing strategy (φ^0, φ) is called **M-admissible** iff $\varphi^0 + M + (\varphi + M)^+ \underline{X} - (\varphi + M)^- \overline{X} \geq 0$.
- Stating the admissibility condition in terms of the actual bid/ask prices $\underline{X}$ and $\overline{X}$ is equivalent to **freezing** a portfolio position as in Guasoni, Lépinette, Rásonyi (2012), but our condition is "numéraire-free".
- The market model satisfies the numéraire-free no-arbitrage condition $(\mathrm{NA}^{nf}) :\Leftrightarrow \forall M \in \mathbb{R}_+, \nexists$ M-admissible strategy (φ^0, φ) with $\varphi^0_0 = \varphi_0 = 0$, $P(\varphi^0_T \geq 0, \varphi_T \geq 0) = 1$, and $P(\{\varphi^0_T > 0\} \cup \{\varphi_T > 0\}) > 0$.

M-admissibility: $\varphi^0 + M + (\varphi + M)^+ \underline{X} - (\varphi + M)^- \overline{X} \geq 0$

Lemma (Guasoni/ Léppinette/ Rásonyi (2012) and K. (2023++) for the current setting)
Assume that the model satisfies NA^{nf}*. Let* $M, M' \in \mathbb{R}_+$ *and let* (φ^0, φ) *be a M-admissible strategy with* $P(\varphi^0_T \geq -M', \varphi_T \geq -M') = 1$*. Then,* (φ^0, φ) *is* M'*-admissible.*

- Analogue to the frictionless case, the lemma plays an important role in the proof of a FTAP: uniformly small losses at T are only possible if losses are uniformly small for all $t \in [0, T]$.

Lemma (Guasoni/ Léppinette/ Rásonyi (2012) and K. (2023++) for the current setting)
Assume that the model satisfies NA^{nf}. *Let* $M, M' \in \mathbb{R}_+$ *and let* (φ^0, φ) *be a* M*-admissible strategy with* $P(\varphi^0_T \geq -M', \varphi_T \geq -M') = 1$. *Then,* (φ^0, φ) *is* M'*-admissible.*

- Analogue to the frictionless case, the lemma plays an important role in the proof of a FTAP: uniformly small losses at T are only possible if losses are uniformly small for all $t \in [0, T]$.
- **Interpretation:** Assume you own 1 stock at some $t_0 \in [0, T]$ and want to maximize the worst-case reward. Just freeze the stock and sell it at price $\underline{X}_{t_0}$, dynamic strategies cannot do better.

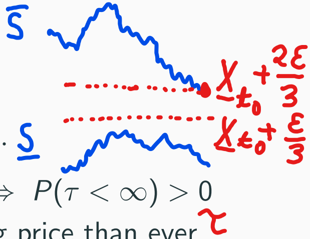
Lemma (Guasoni/ Léppinette/ Rásonyi (2012) and K. (2023++) for the current setting)
Assume that the model satisfies NA^{nf}. *Let* $M, M' \in \mathbb{R}_+$ *and let* (φ^0, φ) *be a* M*-admissible strategy with* $P(\varphi^0_T \geq -M', \varphi_T \geq -M') = 1$. *Then,* (φ^0, φ) *is* M'*-admissible.*

Idea of proof: To show: $\forall t_0 \in (0, T)$ $\forall$ admissible (φ^0, φ) with $\varphi^0_{t_0} = 0$, $\varphi_{t_0} = 1$, and $\varphi_T = 0$, we have $\operatorname{essinf}_{\mathcal{F}_{t_0}} \varphi^0_T \leq \underline{X}_{t_0}$.

Lemma (Guasoni/ Léppinette/ Rásonyi (2012) and K. (2023++) for the current setting)
Assume that the model satisfies NA^{nf}. *Let* $M, M' \in \mathbb{R}_+$ *and let* (φ^0, φ) *be a* M*-admissible strategy with* $P(\varphi^0_T \geq -M', \varphi_T \geq -M') = 1$. *Then,* (φ^0, φ) *is* M'*-admissible.*

Idea of proof: To show: $\forall t_0 \in (0, T)$ $\forall$ admissible (φ^0, φ) with $\varphi^0_{t_0} = 0$, $\varphi_{t_0} = 1$, and $\varphi_T = 0$, we have $\operatorname{essinf}_{\mathcal{F}_{t_0}} \varphi^0_T \leq \underline{X}_{t_0}$.
W.l.o.g. let $\mathcal{F}_{t_0}$ be P-trivial. Assume **by contradiction** that $\exists\ \varepsilon > 0$ s.t. $P(\varphi^0_T \geq \underline{X}_{t_0} + \varepsilon) = 1$.

$$\tau := \inf\{t \geq t_0 : (\overline{S}_t \leq \underline{X}_{t_0} + 2\varepsilon/3 \text{ or } \varphi_t \leq 0) \text{ and } \underline{S}_u \leq \underline{X}_{t_0} + \varepsilon/3\ \forall u \in (t_0, t)\}.$$



$P(\varphi_T = 0,\ \sup_{u \in [t_0, T]} \underline{S}_u \leq \underline{X}_{t_0} + \varepsilon/3) > 0 \implies P(\tau < \infty) > 0$
Bad prices before $\tau < \infty$ and at τ better buying price than ever before. Switch from strategy 0 to strategy φ at $\tau \rightsquigarrow$ arbitrage. ↯

The **cost value process** introduced by Bayraktar and Yu (2018) is defined as the cost to enter a portfolio position:

$$V^{\text{cost}}(\varphi) := \varphi^0 + \varphi^+ \overline{X} - \varphi^- \underline{X} \quad \text{for } (\varphi^0, \varphi) \text{ self-financing}$$

The **cost value process** introduced by Bayraktar and Yu (2018) is defined as the cost to enter a portfolio position:

$$V^{\text{cost}}(\varphi) := \varphi^0 + \varphi^+ \overline{X} - \varphi^- \underline{X} \quad \text{for } (\varphi^0, \varphi) \text{ self-financing}$$

Definition (Merge NUPBR with prospective version of NA^s)

The market model satisfies the **prospective strict no unbounded profit with bounded risk ($\mathrm{NUPBR}^{\mathsf{ps}}$)** condition iff

$$\left\{ \sup_{t \in [0,T]} V_t^{\text{cost}}(\varphi) \ : \ (\varphi^0, \varphi) \text{ 1-admissible} \right\} \text{ is bounded in } L^0,$$

and for every 1-admissible sequence $(\varphi^{0,n}, \varphi^n)_{n \in \mathbb{N}}$ such that φ^n are bounded and $(\varphi_T^{0,n}, \varphi_T^n) \to (C^0, C)$ a.s., where (C^0, C) is a maximal element, there exist forward convex combinations $(\lambda_{n,k})_{n \in \mathbb{N},\ k=0,\dots,k_n}$, $k_n \in \mathbb{N}$, i.e., $\lambda_{n,k} \in \mathbb{R}_+$ & $\sum_{k=0}^{k_n} \lambda_{n,k} = 1$, s.t.

$$\sup_{n \in \mathbb{N}} \sup_{t \in [0,T]} V_t^{\text{cost}}\Big(\sum_{k=0}^{k_n} \lambda_{n,k} \varphi^{n+k}\Big) < \infty \quad \text{a.s.}$$

Complicated situation: Spread is continuously moving away from zero

Assumption 2.18 (simplified, see K. (2023++) for exact version)
Let τ be a starting time of an excursion of the spread $\overline{X} - \underline{X}$ away from zero (by assumption a stopping time!). Then, there exists a stopping time σ with $\sigma > \tau$ on $\{\overline{X}_\tau = \underline{X}_\tau\}$ s.t. the fictitious frictionless market:

$$\begin{aligned} \text{asset price for long positions:} \quad & (\overline{X}^\sigma - \overline{X}^\tau) \\ \text{asset price for short positions:} \quad & (\underline{X}^\sigma - \underline{X}^\tau) \end{aligned}$$

satisfies NFLVR.

Complicated situation: Spread is continuously moving away from zero

Assumption 2.18 (simplified, see K. (2023++) for exact version)
Let τ be a starting time of an excursion of the spread $\overline{X} - \underline{X}$ away from zero (by assumption a stopping time!). Then, there exists a stopping time σ with $\sigma > \tau$ on $\{\overline{X}_\tau = \underline{X}_\tau\}$ s.t. the fictitious frictionless market:

$$\text{asset price for long positions:} \quad (\overline{X}^\sigma - \overline{X}^\tau)$$
$$\text{asset price for short positions:} \quad (\underline{X}^\sigma - \underline{X}^\tau)$$

satisfies NFLVR.

Interpretation. Local tightening of the $\mathrm{NUPBR}^{\mathsf{ps}}$ condition: The process V^{cost} values a purchased share at the higher ask price as long as it is in the portfolio. In the fictitious frictionless market, the position can even be liquidated at the ask price.

Definition
A two-dimensional **consistent price system (CPS)** is a pair (S, Q) s.t. Q is a probability measure equivalent to P and S is a Q-martingale with $\underline{X} \leq S \leq \overline{X}$ (and thus a fortiori $\underline{S} \leq S \leq \overline{S}$).

Theorem (Version of FTAP)
If the market model satisfies NA^{nf}, NUPBR^{ps}, *and Assumption 2.18, then there exists a CPS. Conversely, if* (S, Q) *is a CPS, then the bid-ask model with bid price* S *and ask price* S *satisfies* NA^{nf}, NUPBR^{ps}, *and Assumption 2.18.*

- **In the proof, we show that the set of attainable terminal portfolios (from zero endowment) is Fatou-closed.**

Definition

A two-dimensional **consistent price system (CPS)** is a pair (S, Q) s.t. Q is a probability measure equivalent to P and S is a Q-martingale with $\underline{X} \leq S \leq \overline{X}$ (and thus a fortiori $\underline{S} \leq S \leq \overline{S}$).

Theorem (Version of FTAP)

If the market model satisfies NA^{nf}, NUPBR^{ps}, *and Assumption 2.18, then there exists a CPS. Conversely, if* (S, Q) *is a CPS, then the bid-ask model with bid price* S *and ask price* S *satisfies* NA^{nf}, NUPBR^{ps}, *and Assumption 2.18.*

- **In the proof, we show that the set of attainable terminal portfolios (from zero endowment) is Fatou-closed.**
- The name "FTAP" is justified because the sufficient conditions to obtain a CPS are fulfilled in the special case of a frictionless market with NFLVR ("allowable" version).
- The conditions are weaker than those in the literature.

Idea of proof

- The proof is very technical and uses many standard methods in continuous time finance. But, the key idea is new.
- Under **Robust no free lunch with vanishing risk**, one argues that the trading volume of 1-admissible strategies cannot explode ("trading costs" cannot be compensated for with certainty by "trading gains")

Idea of proof that $\{(\varphi^0_T, \varphi_T) : (\varphi^0, \varphi) \text{ admissible}\}$ is Fatou-closed

- Consider a sequence of 1-admissible strategies, i.e.,

  $$\varphi^{0,n} + 1 + (\varphi^n + 1)^+ \underline{X} - (\varphi^n + 1)^- \overline{X} \geq 0 \quad \forall n \in \mathbb{N},$$

  with $(\varphi^{0,n}_T, \varphi^n_T) \to (C^0, C)$ a.s., where $(C^0, C) \in L^0(\mathbb{R}^2)$ is a maximal element.

  To show: $\exists (\varphi^0, \varphi)$ admissible s.t. $(\varphi^0_T, \varphi_T) = (C^0, C)$.

Idea of proof that $\{(\varphi^0_T, \varphi_T) : (\varphi^0, \varphi) \text{ admissible}\}$ is Fatou-closed

- Consider a sequence of 1-admissible strategies, i.e.,

$$\varphi^{0,n} + 1 + (\varphi^n + 1)^+ \underline{X} - (\varphi^n + 1)^- \overline{X} \geq 0 \quad \forall n \in \mathbb{N},$$

with $(\varphi^{0,n}_T, \varphi^n_T) \to (C^0, C)$ a.s., where $(C^0, C) \in L^0(\mathbb{R}^2)$ is a maximal element.

To show: $\exists (\varphi^0, \varphi)$ admissible s.t. $(\varphi^0_T, \varphi_T) = (C^0, C)$.

- By $\mathrm{NUPBR}^{\mathbf{ps}}$ and after passing to forward convex combinations, we get estimate for the maximal **cost to enter a portfolio position** $(\varphi^{0,n}, \varphi^n)$ at a time $t \in [0, T]$:

$$\sup_{n \in \mathbb{N}} \sup_{t \in [0,T]} \left(\varphi^{0,n}_t + (\varphi^n)^+_t \overline{X}_t - (\varphi^n)^-_t \underline{X}_t \right) < \infty \quad \text{a.s.}$$

Idea of proof that $\{(\varphi^0_T, \varphi_T) : (\varphi^0, \varphi) \text{ admissible}\}$ is Fatou-closed

- Consider a sequence of 1-admissible strategies, i.e.,

$$\varphi^{0,n} + 1 + (\varphi^n + 1)^+ \underline{X} - (\varphi^n + 1)^- \overline{X} \geq 0 \quad \forall n \in \mathbb{N},$$

with $(\varphi^{0,n}_T, \varphi^n_T) \to (C^0, C)$ a.s., where $(C^0, C) \in L^0(\mathbb{R}^2)$ is a maximal element.

To show: $\exists (\varphi^0, \varphi)$ admissible s.t. $(\varphi^0_T, \varphi_T) = (C^0, C)$.

- By $\mathrm{NUPBR}^{\mathbf{ps}}$ and after passing to forward convex combinations, we get estimate for the maximal **cost to enter a portfolio position** $(\varphi^{0,n}, \varphi^n)$ at a time $t \in [0, T]$:

$$\sup_{n \in \mathbb{N}} \sup_{t \in [0,T]} \left(\varphi^{0,n}_t + (\varphi^n)^+_t \overline{X}_t - (\varphi^n)^-_t \underline{X}_t \right) < \infty \quad \text{a.s.}$$

- Consider a fixed (ω, t) with $\overline{X}_t(\omega) - \underline{X}_t(\omega) > 0$. We get an estimate for $|\varphi^n_t(\omega)|$ which does not depend on $n \in \mathbb{N}$.

Idea of proof that $\{(\varphi^0_T, \varphi_T) : (\varphi^0, \varphi) \text{ admissible}\}$ is Fatou-closed

- For simplicity, assume that $\overline{X}_t(\omega) - \underline{X}_t(\omega) \gg 0$ uniformly in t. Then, $|\varphi^n_t(\omega)|$ can be estimated uniformly in n and t. Thus,

$$\begin{aligned} C^0 + CS_T &\approx \varphi^{0,n}_T + \varphi^n_T S_T \\ &= \underbrace{\int_0^T \varphi^n \, dS}_{L^0-\text{bounded}} - \underbrace{Cost^S_T(\varphi^n)}_{\Longrightarrow L^0-\text{bounded}}, \end{aligned}$$

where S is a semimartingale with $\underline{X} \leq S \leq \overline{X}$.

Idea of proof that $\{(\varphi^0_T, \varphi_T) : (\varphi^0, \varphi) \text{ admissible}\}$ is Fatou-closed

- For simplicity, assume that $\overline{X}_t(\omega) - \underline{X}_t(\omega) \gg 0$ uniformly in t. Then, $|\varphi^n_t(\omega)|$ can be estimated uniformly in n and t. Thus,

$$
\begin{aligned}
C^0 + CS_T &\approx \varphi^{0,n}_T + \varphi^n_T S_T \\
&= \underbrace{\int_0^T \varphi^n \, dS}_{L^0-\text{bounded}} - \underbrace{Cost^S_T(\varphi^n)}_{\Longrightarrow L^0-\text{bounded}},
\end{aligned}
$$

  where S is a semimartingale with $\underline{X} \leq S \leq \overline{X}$.
- This implies that $(\varphi^n)_{n \in \mathbb{N}}$ is of (uniformly) bounded variation. Schachermayer's stochastic version of Helly's theorem yields a limiting strategy.

Idea of proof that $\{(\varphi^0_T, \varphi_T) : (\varphi^0, \varphi) \text{ admissible}\}$ is Fatou-closed

- For simplicity, assume that $\overline{X}_t(\omega) - \underline{X}_t(\omega) \gg 0$ uniformly in t. Then, $|\varphi^n_t(\omega)|$ can be estimated uniformly in n and t. Thus,

$$
\begin{aligned}
C^0 + CS_T &\approx \varphi^{0,n}_T + \varphi^n_T S_T \\
&= \underbrace{\int_0^T \varphi^n \, dS}_{L^0-\text{bounded}} - \underbrace{Cost^S_T(\varphi^n)}_{\Longrightarrow L^0-\text{bounded}} ,
\end{aligned}
$$

  where S is a semimartingale with $\underline{X} \leq S \leq \overline{X}$.
- This implies that $(\varphi^n)_{n\in\mathbb{N}}$ is of (uniformly) bounded variation. Schachermayer's stochastic version of Helly's theorem yields a limiting strategy.
- Delbaen-Schachermayer argument: for $\overline{X}_t(\omega) - \underline{X}_t(\omega) = 0$, one has $V^{\text{cost}}_t(\varphi^n) \approx V^{\text{cost}}_t(\varphi^m)$ for n, m large, otherwise switch strategies with $(\varphi^{0,n}_T, \varphi^n_T) \approx (\varphi^{0,m}_T, \varphi^m_T)$. ↯

Idea of proof that $\{(\varphi^0_T, \varphi_T) : (\varphi^0, \varphi)$ admissible$\}$ is Fatou-closed

- For simplicity, assume that $\overline{X}_t(\omega) - \underline{X}_t(\omega) \gg 0$ uniformly in t. Then, $|\varphi^n_t(\omega)|$ can be estimated uniformly in n and t. Thus,

$$\begin{aligned} C^0 + CS_T &\approx \varphi^{0,n}_T + \varphi^n_T S_T \\ &= \underbrace{\int_0^T \varphi^n \, dS}_{L^0-\text{bounded}} - \underbrace{Cost^S_T(\varphi^n)}_{\Longrightarrow L^0-\text{bounded}} , \end{aligned}$$

  where S is a semimartingale with $\underline{X} \leq S \leq \overline{X}$.
- This implies that $(\varphi^n)_{n\in\mathbb{N}}$ is of (uniformly) bounded variation. Schachermayer's stochastic version of Helly's theorem yields a limiting strategy.
- Delbaen-Schachermayer argument: for $\overline{X}_t(\omega) - \underline{X}_t(\omega) = 0$, one has $V^{\mathrm{cost}}_t(\varphi^n) \approx V^{\mathrm{cost}}_t(\varphi^m)$ for n, m large, otherwise switch strategies with $(\varphi^{0,n}_T, \varphi^n_T) \approx (\varphi^{0,m}_T, \varphi^m_T)$. $\lightning$
- By Assumption 2.18, for positive but small spreads gains vanish

Thank you for your attention!

Selected references

- Delbaen, F., Schachermayer, W.: A general version of the fundamental theorem of asset pricing. Math. Ann. **300**, 463–520 (1994)
- Guasoni, P., Lépinette, E., Rásonyi, M.: The fundamental theorem of asset pricing under transaction costs. Finance & Stoch. **16**, 741–777 (2012)
- Kabanov, Y., Rásonyi, M., Stricker, C.: No-arbitrage criteria for financial markets with efficient friction. Finance & Stoch. **6**, 371–382 (2002)
- Kühn, C., Molitor, A.: Prospective strict no-arbitrage and the fundamental theorem of asset pricing under transaction costs. Finance & Stoch. **23**, 1049–1077 (2019)
- Kühn, C., Molitor, A.: Semimartingale price systems in models with transaction costs beyond efficient friction. Finance & Stoch. **26**, 927–982 (2022)