Instalment Options:
A Closed-Form Solution and the Limiting Case

Susanne Griebsch   Uwe Wystup

Frankfurt School of Finance & Management

Sonnemannstrasse 9 - 11

60314 Frankfurt am Main

GERMANY

Christoph Kühn

Frankfurt MathFinance Institute

Goethe University

Robert-Mayer-Straße 10

60054 Frankfurt am Main

GERMANY

April 10, 2008

Abstract

In Foreign Exchange Markets Compound options (options on options) are traded frequently. Instalment options generalize the concept of Compound options as they allow the holder to prolong a Vanilla Call or Put option by paying instalments of a discrete payment plan. We derive a closed-form solution to the value of such an option in the Black-Scholes model and prove that the limiting case of an Instalment option with a continuous payment plan is equivalent to a portfolio consisting of a European Vanilla option and an American Put on this Vanilla option with a time-dependent strike.

Keywords: Exotic Options
JEL classification: C15, G12
Acknowledgments. We would like to thank Michèle Vanmaele, Robert G. Tompkins, Manuel Guerra and an anonymous referee for their help and comments.

1 Introduction

An Instalment Call or Put option works similar like a Compound Call or Put respectviely, but allows the holder to pay the premium of the option in instalments spread over time. A first payment is made at inception of the trade. The buyer receives the *mother option*. On the following payment days the holder of the Instalment option can decide to prolong the contract and obtain the *daughter option*, in which case he has to pay the second instalment of the premium, or to terminate the contract by simply not paying any more. After the last instalment payment the contract turns into a plain Vanilla Call or Put option. For an Instalment Put option we illustrate two scenarios in Figure 1.

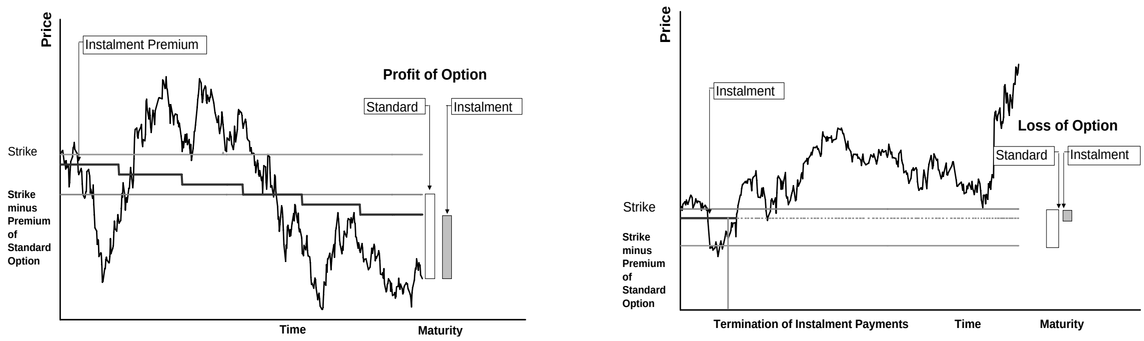

Figure 1: Comparison of two scenarios of an Instalment option. The left hand side shows a continuation of all instalment payments until expiration. The right hand side shows a scenario where the Instalment option is terminated after the first decision date.

1.1 Example

Instalment options are typically traded in Foreign Exchange markets between banks and corporates. For example, a company in the EUR-zone wants to hedge receivables from an export transaction in USD due in 12 months time. It expects a stronger EUR/weaker USD. The company wishes to be able to buy EUR at a lower spot rate if EUR becomes weaker on the one hand, but on the other hand be fully protected against a stronger EUR. The future income in USD is yet uncertain but will be under review at the end of each quarter.

In this case a possible form of protection that the company can use is to buy a EUR Instalment Call option with 4 equal quarterly premium payments as for example illustrated in Table 1.

The company pays 12,500 USD on the trade date. After one quarter, the company has the right to prolong the Instalment contract. To do this the company must pay another 12,500 USD. After 6 months, the company has the right to prolong the contract and must pay 12,500 USD in order to do so. After 9 months the same decision has to be taken. If at one of these three decision days the company does not pay, then the contract terminates. If all premium payments

Spot reference	1.1500 EUR-USD
Maturity	1 year
Notional	USD 1,000,000
Company buys	EUR Call USD Put strike 1.1500
Premium per quarter of the Instalment	USD 12,500.00
Premium of the corresponding Vanilla Call	USD 40,000.00

Table 1: Example of an Instalment Call. Four times the instalment rate sums up to USD 50,000, which is more than buying the corresponding plain Vanilla for USD 40,000.

are made, then in 9 months the contract turns into a plain Vanilla EUR Call.

Of course, besides not paying the premium, another way to terminate the contract is always to sell it in the market. So if the option is not needed, but deep in the money, the company can take profit from paying the premium to prolong the contract and then selling it.

If the EUR-USD exchange rate is above the strike at maturity, then the company can buy EUR at maturity at a rate of 1.1500.

If the EUR-USD exchange rate is below the strike at maturity the option expires worthless. However, the company would benefit from being able to buy EUR at a lower rate in the market.

Compound options can be viewed as a special case of Instalment options, and the possible variations of Compound options such as early exercise rights or deferred delivery apply analogously to Instalment options.

1.2 Reasons for Trading Compound and Instalment Options

We observe that Compound and Instalment options are always more expensive than buying the corresponding Vanilla option, sometimes substantially more expensive. So why are people buying them? One reason may be the situation that a treasurer has a budget constraint, i.e. limited funds to spend for foreign exchange risk management. With an Instalment he can then split the premium over time. This would be inefficient accounting, but a situation like this is not uncommon in practice. However, the essential motivation for a treasurer dealing with an uncertain cash-flow is the situation where he buys a Vanilla instead of an Instalment, and then is faced with a far out of the money Vanilla at time t_1, then selling the Vanilla does not give him as much as the savings between the Vanilla and the sum of the instalment payments. With an Instalment, the budget to spend for FX risk can be planned and controlled. This additional optionality comes at a cost beyond the vanilla price.

From a trader's viewpoint, an instalment is a bet on the future change of the term structure of volatility. For instance, if the forward volatility (12) is higher than a trader's belief of the later materializing volatility, then he would go short an instalment. Some volatility arbitrage focussed hedge funds are trying to identify situations like this.

1.3 Literature on Instalment Options

There is not much literature available on the valuation of Instalment options. Here we mention the papers we know about this topic which were published in the past years.
In the paper of Davis, Tompkins and Schachermayer [6] no-arbitrage bounds on the price of Instalment options are derived, which are used to set up static hedges and to compare them to dynamic hedging strategies. Ben-Ameur, Breton and François [2] develop a dynamic programming procedure to compute the value of Instalment options and investigate the properties of Instalment options through theoretical and numerical analysis. Recently Ciurlia and Roko [3] construct a dynamic hedging portfolio and derive a Black-Scholes partial differential equation for the initial value of an American continuous Instalment option. In [11] Kimura and Kikuchi develop a Laplace transform based valuation of Instalment options. The valuation and risk management of Instalment options is related to Bermudan options as in both cases there is a discrete time scale with time points requiring decisions. For details on Bermudan contracts see, e.g., Baviera and Giada [1], Henrard [9] and Pietersz and Pelsser [12].

In the next section we discuss the valuation of Instalment options in the Black-Scholes model in closed-form. In Section 3 we examine the limiting case of an Instalment option, where instalment rates are paid continuously over the lifetime of the option. We will see, that this limiting case can be expressed *model-independently* as a portfolio of other options. In Section 4 we analyze the performance and convergence of our results numerically. Concluding remarks are given in Section 5.

2 Valuation in the Black-Scholes Model

The goal of this section is to obtain a closed-form formula for the n-variate Instalment option in the Black-Scholes model. For the cases $n = 1$ and $n = 2$ the Black-Scholes formula and Geske's Compound option formula (see [8]) are already well known.

We consider an exchange rate process S_t, whose evolution is modeled by a geometric Brownian motion

$$\frac{dS_t}{S_t} = (r_d - r_f)dt + \sigma dW_t, \tag{1}$$

where W is a standard Brownian motion, the volatility is denoted by σ and the domestic and foreign interest rates are denoted by r_d and r_f respectively.
This means

$$S_T = S_0 \exp\left(\left(r_d - r_f - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z\right), \tag{2}$$

where S_0 is the current exchange rate, Z is a standard normal random variable and T the time to maturity of the option.

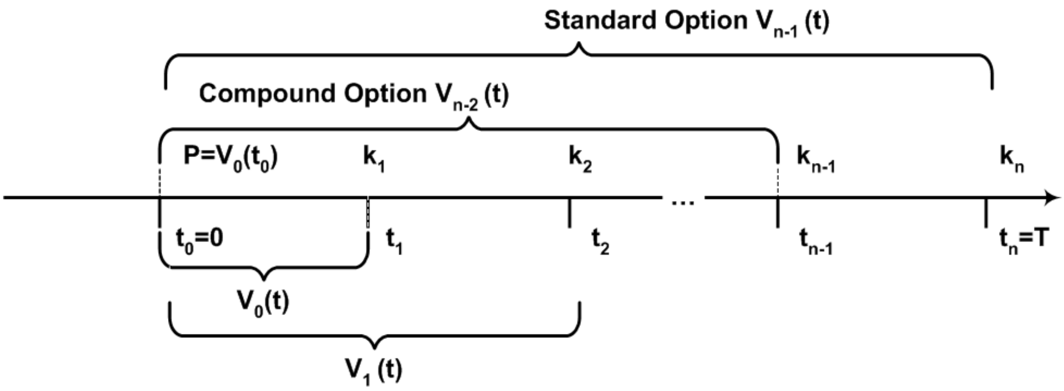

Figure 2: Lifetime of the options with value V_i

As illustrated in Figure 2 we let $t_0 = 0$ be the Instalment option inception date and $t_0 < t_1 < t_2 < \ldots < t_n = T$ a schedule of decision dates in the contract on which the option holder has to decide whether to continue to pay the premiums $k_1, k_2, \ldots, k_{n-1}$ to keep the option alive. These premiums can be chosen to be all equal or to have different values. However, the first premium V_0 of the Instalment option is determined dependent on the other premiums. To compute the value of the Instalment option, which is the up front payment V_0 at t_0 to enter the contract, we begin with the option payoff at maturity T

$$V_n(s) \triangleq [\phi_n(s - k_n)]^+ \triangleq \max\left(\phi_n(s - k_n), 0\right), \tag{3}$$

where $s = S_T$ is the price of the underlying currency at T, k_n the strike price and as usual $\phi_n = +1$ for the underlying standard Call option, $\phi_n = -1$ for a Put option. V_n is the value of the underlying option at time t_n, whose value at time t_{n-1} is given by its discounted expectation. And in turn we can again define a payoff function on this value, which would correspond to the payoff of a Compound option.

Generally, at time t_i the option holder can either terminate the contract or pay k_i to continue. Therefore by the risk-neutral pricing theory, the time-t_i-value is given by the backward recursion

$$V_i(s) = \left[e^{-r_d(t_{i+1}-t_i)} I\!E[V_{i+1}(S_{t_{i+1}}) \mid S_{t_i} = s] - k_i\right]^+ \quad \text{for } i = 1, \ldots, n-1, \tag{4}$$

where $V_n(s)$ is given by (3). Following this principal the unique arbitrage-free initial premium of the Instalment option – given $k_1, \ldots, k_n$ – is

$$k_0 \quad \triangleq \quad V_0(s) = e^{-r_d(t_1-t_0)} I\!E[V_1(S_{t_1}) \mid S_{t_0} = s]. \tag{5}$$

In practice, we normally want to have

$$k_0 = k_1 = \cdots = k_{n-1}. \tag{6}$$

We notice that one way to determine the value of this Instalment option is to evaluate the nested expectations in Equation (5) through multiple numerical integration of the payoff functions via backward iteration. Another numerical procedure by Ben-Ameur, Breton and François is presented in [2]. In this paper the recursive structure of the value in Equation (5), which is illustrated in Figure 2, is used to develop a dynamic programming procedure to price Instalment options. Thirdly, it is possible to compute the value in closed-form, which is one of the results of this paper.

2.1 The Curnow and Dunnett Integral Reduction Technique

For the derivation of the closed-form pricing formula of an Instalment option, we see from Equations (2) and (4) that we need to compute integrals of the form

$$\int_{\mathbb{R}} [\text{option value}(y) - \text{strike}]^+ \, n(y) dy$$

with respect to the standard normal density $n(\cdot)$. This essentially means to compute integrals of the form

$$\int_{-\infty}^{h} N_i(f(y)) n(y) \, dy,$$

where $N_i(\cdot)$ is the i-dimensional cumulative normal distribution function, f some vector-valued function and h some boundary. The following result provides this relationship.

Denote the n-dimensional multivariate normal distribution function with upper limits $h_1, \ldots, h_n$ and correlation matrix $R_n \stackrel{\Delta}{=} (\rho_{ij})_{i,j=1,\ldots,n}$ by $N_n(h_1, \ldots, h_n; R_n)$, and the univariate standard normal density function by $n(\cdot)$. Let the correlation matrix be non-singular and $\rho_{11} = 1$. Under these conditions Curnow and Dunnett [4] derive the following *reduction formula*

$$N_n(h_1, \cdots, h_n; R_n) = \int_{-\infty}^{h_1} N_{n-1} \left(\frac{h_2 - \rho_{21} y}{(1 - \rho_{21}^2)^{1/2}}, \cdots, \frac{h_n - \rho_{n1} y}{(1 - \rho_{n1}^2)^{1/2}}; R_{n-1}^* \right) n(y) dy,$$

where the $n-1$-dimensional correlation matrix R^* is given by

$$\begin{aligned} R_{n-1}^* &\stackrel{\Delta}{=} (\rho_{ij}^*)_{i,j=2,\ldots,n}, \\ \rho_{ij}^* &\stackrel{\Delta}{=} \frac{\rho_{ij} - \rho_{i1}\rho_{j1}}{(1 - \rho_{i1}^2)^{1/2}(1 - \rho_{j1}^2)^{1/2}}. \end{aligned} \tag{7}$$

2.2 A Closed-Form Solution for the Value of an Instalment Option

An application of the above result of Curnow and Dunnett yields the derived closed-form pricing formula for Instalment options given in Theorem 2.1. Before stating the result, we will make an observation about its structure.

The formula in Theorem 2.1 below has a similar structure as the Black-Scholes formula for Basket options, namely $S_0 N_n(\cdot) - k_n N_n(\cdot)$ minus the later premium payments $k_i N_i(\cdot)$ $(i = 1, \ldots, n-1)$. This structure is a result of the integration of the Vanilla option payoff,

$$\int_{\mathbb{R}} [S_T(y) - \text{strike}]^+ \, n(y) dy$$

which is again integrated after substracting the next instalment,

$$\int_{\mathbb{R}} [\text{Vanilla option value}(y) - \text{strike}]^+ \, n(y) \, dy$$

which in turn is integrated with the following instalment and so forth. By this iteration the Vanilla payoff is integrated with respect to the normal density function n times and the i-th payment is integrated i times for $i = 1, \ldots, n-1$.

Theorem 2.1 *Let $\vec{k} = (k_1, \ldots, k_n)$ be the strike price vector, $\vec{t} = (t_1, \ldots, t_n)$ the vector of the exercise dates of an n-variate Instalment option and $\vec{\phi} = (\phi_1, \ldots, \phi_n)$ the vector of the Put/Call-indicators of these n options.*

The value function of an n-variate Instalment option is given by

$$
\begin{aligned}
V_0(S_0,\vec{k},\vec{t},\vec{\phi}) &= e^{-r_f t_n} S_0 \phi_1 \cdot \ldots \cdot \phi_n \\
&\quad \times N_n\left[\frac{\ln\frac{S_0}{S_1^*}+\mu^{(+)}t_1}{\sigma\sqrt{t_1}},\frac{\ln\frac{S_0}{S_2^*}+\mu^{(+)}t_2}{\sigma\sqrt{t_2}},\ldots,\frac{\ln\frac{S_0}{S_n^*}+\mu^{(+)}t_n}{\sigma\sqrt{t_n}};R_n\right] \\
&- e^{-r_d t_n} k_n \phi_1 \cdot \ldots \cdot \phi_n \\
&\quad \times N_n\left[\frac{\ln\frac{S_0}{S_1^*}+\mu^{(-)}t_1}{\sigma\sqrt{t_1}},\frac{\ln\frac{S_0}{S_2^*}+\mu^{(-)}t_2}{\sigma\sqrt{t_2}},\ldots,\frac{\ln\frac{S_0}{S_n^*}+\mu^{(-)}t_n}{\sigma\sqrt{t_n}};R_n\right] \\
&- e^{-r_d t_{n-1}} k_{n-1} \phi_1 \cdot \ldots \cdot \phi_{n-1} \\
&\quad \times N_{n-1}\left[\frac{\ln\frac{S_0}{S_1^*}+\mu^{(-)}t_1}{\sigma\sqrt{t_1}},\frac{\ln\frac{S_0}{S_2^*}+\mu^{(-)}t_2}{\sigma\sqrt{t_2}},\ldots,\frac{\ln\frac{S_0}{S_{n-1}^*}+\mu^{(-)}t_{n-1}}{\sigma\sqrt{t_{n-1}}};R_{n-1}\right] \\
&\quad \vdots \\
&- e^{-r_d t_2} k_2 \phi_1 \phi_2 N_2\left[\frac{\ln\frac{S_0}{S_1^*}+\mu^{(-)}t_1}{\sigma\sqrt{t_1}},\frac{\ln\frac{S_0}{S_2^*}+\mu^{(-)}t_2}{\sigma\sqrt{t_2}};\rho_{12}\right] \\
&- e^{-r_d t_1} k_1 \phi_1 N\left[\frac{\ln\frac{S_0}{S_1^*}+\mu^{(-)}t_1}{\sigma\sqrt{t_1}}\right] \qquad (8) \\
&= e^{-r_f t_n} S_0 \prod_{i=1}^{n} \Phi_i N_n\left[\left(\frac{\ln\frac{S_0}{S_m^*}+\mu^{(+)}t_m}{\sigma\sqrt{t_m}}\right)_{1,\ldots,n}\right] \\
&\quad -\sum_{i=1}^{n} e^{-r_d t_i} k_i \prod_{j=1}^{i} \Phi_j N_i\left[\left(\frac{\ln\frac{S_0}{S_m^*}+\mu^{(-)}t_m}{\sigma\sqrt{t_m}}\right)_{1,\ldots,i}\right], \qquad (9)
\end{aligned}
$$

where $\mu^{(\pm)}$ is defined as $r_d - r_f \pm \frac{1}{2}\sigma^2$.

The correlation coefficients in R_i of the i-variate normal distribution function can be expressed through the exercise times t_i,

$$\rho_{ij} = \sqrt{t_i/t_j} \quad \text{for} \quad i,j = 1,\ldots,n \quad \text{and} \quad i < j. \tag{10}$$

S_i^* $(i = 1,\ldots,n)$ denotes the price of the underlying at time t_i for which the price of the underlying option is equal to k_i,

$$V_i(S_i^*) \stackrel{!}{=} k_i.$$

Remark 2.1 *S_i^* $(i = 1,\ldots,n)$ is determined as the largest resp. smallest spot price S_t for which the initial price of the corresponding renewed i-th-Instalment option $(i = 1,\ldots,n)$ is equal to* *zero. In the case of calls S_i^* is the largest underlying price at which the renewed Instalment option becomes worthless. This problem can be solved by a root finding procedure, e.g. Newton-Raphson. For a Vanilla Call the root S_{n-1}^* always exists and is unique as the Black-Scholes price of a Vanilla Call is a bijection in the starting price of the underlying. Even for a simple Vanilla Put the root S_{n-1}^* may not exist, because the price of a Put is bounded above. In general the existence of the S_i^* can't be guaranteed, but has to be checked on an individual basis. If one of the S_i^* does not exist, then the pricing formula cannot be applied. It means that the strikes are chosen too large. In such a case the strike k_i has to be lowered. In particular, if $\phi_n = -1$ we need to ensure that $\sum_{i=0}^{n-1} k_i < k_n$. This means, that because the price of a vanilla put is bounded above by the strike price, the sum of the future payments must not exceed the upper bound. In practice, arbitrary mixes of calls and puts do not occur. The standard case is a series of calls on a final vanilla product.*

Remark 2.2 *The correlation coefficients ρ_{ij} of these normal distribution functions contained in the formula arise from the overlapping increments of the Brownian motion, which models the price process of the underlying S_t, at the particular exercise times t_i and t_j.*

Proof. The proof is established with Equation (7). [1] □

Obviously Equation (8) readily extends to a term structure of interest rates and volatility. Therefore we will now deal with how to compute the necessary forward volatilities.

2.3 Forward Volatility

The *daughter option* of the Compound option requires knowing the volatility for its lifetime, which starts on the exercise date t_1 of the *mother option* and ends on the maturity date t_2 of the daughter option. This volatility is not known at inception of the trade, so the only proxy traders can take is the *forward volatility* $\sigma_f(t_1, t_2)$ for this time interval. In the Black-Scholes model the *consistency equation* for the forward volatility is given by

$$\sigma^2(t_1)(t_1 - t_0) + \sigma_f^2(t_1, t_2)(t_2 - t_1) = \sigma^2(t_2)(t_2 - t_0), \tag{11}$$

where $t_0 < t_1 < t_2$ and $\sigma(t)$ denotes the at-the-money volatility up to time t. We extract the forward volatility via

$$\sigma_f(t_1, t_2) = \sqrt{\frac{\sigma^2(t_2)(t_2 - t_0) - \sigma^2(t_1)(t_1 - t_0)}{t_2 - t_1}}. \tag{12}$$

2.4 Forward Volatility Smile

The more realistic way to look at this unknown forward volatility is that the fairly liquid market of Vanilla Compound options could be taken to back out the forward volatilities since this is the only unknown. These should in turn be consistent with other forward volatility sensible

[1] A variation of Formula (8) has been independently derived by Thomassen and Wouve in [15].

products like forward start options, window barrier options or faders.

In a market with smile the payoff of a Compound option can be approximated by a linear combination of Vanillas, whose market prices are known. For the payoff of the Compound option itself we can take the forward volatility as in Equation (12) for the at-the-money value and the smile of today as a proxy. More details on this can be found, e.g. in Schilling [14]. The actual forward volatility, however, is a trader's view and can only be taken from market prices. More details on how to include weekend and holiday effects into the forward volatility computation can be found in Wystup [16].

3 Instalment Options with a Continuous Payment Plan

We will now examine what happens if we make the difference between the instalment payment dates t_i smaller. This will also cause the prolongation payments k_i to become smaller. In the limiting case the holder of the continuous instalment plan keeps paying at a rate p per time unit until she decides to terminate the contract. It is intuitively clear that the above procedure converges as the sum of the strikes increases and is bounded above by the price of the underlying (a call option will never cost more than the underlying). In the limiting case it appears also intuitively obvious that the instalment plan is equivalent to the corresponding Vanilla plus a right to return it any time at a pre-specified rate, which is equivalent to the somehow discounted cumulative prolongation payment which one would have to pay for the time after termination. We will now formalize this intuitive idea.

Let $g = (g_t)_{t\in[0,T]}$ be the stochastic process describing the discounted net payoff of an Instalment option expressed as multiples of the domestic currency. If the holder stops paying the premium at time t, the difference between the option payoff and premium payments (all discounted to time 0) amounts to

$$g(t) \quad = \quad \begin{cases} e^{-r_d T}(S_T - K)^+ \mathbf{1}_{(t=T)} - \frac{p}{r_d}(1 - e^{-r_d t}) & \text{if } r_d \neq 0 \\ (S_T - K)^+ \mathbf{1}_{(t=T)} - pt & \text{if } r_d = 0 \end{cases}, \tag{13}$$

where K is the strike. Given the premium rate p, the Instalment option can be taken as an American contingent claim with a payoff which may become negative. Thus, the unique no-arbitrage premium P_0 to be paid at time 0 (supplementary to the rate p) is given by

$$P_0 = \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}_Q(g_\tau), \tag{14}$$

where Q denotes the risk-neutral measure and $\mathcal{T}_{0,T}$ denotes the set of stopping times with values in $[0, T]$. Ideally, p is chosen as the *minimal* rate such that

$$P_0 = 0. \tag{15}$$

Note that P_0 from Equation (14) can never become negative as it is always possible to stop payments immediately. Thus, besides (15), we need a minimality assumption to obtain a unique rate. We want to compare the Instalment option with the American contingent claim $f = (f_t)_{t\in[0,T]}$ given by

$$f_t = e^{-r_d t}(K_t - C_E(T-t, S_t))^+, \quad t \in [0,T], \tag{16}$$

where $K_t = \frac{p}{r_d}\left(1 - e^{-r_d(T-t)}\right)$ for $r_d \neq 0$ and $K_t = p(T-t)$ when $r_d = 0$. C_E is the value of a standard European Call. Equation (16) represents the payoff of an American Put on a European Call where the variable strike K_t of the Put equals the part of the instalments *not* to be paid if the holder decides to terminate the contract at time t. Define by $\tilde{f} = (\tilde{f}_t)_{t\in[0,T]}$ a similar American contingent claim with

$$\tilde{f}(t) = e^{-r_d t}\left[(K_t - C_E(T-t, S_t))^+ + C_E(T-t, S_t)\right], \quad t \in [0,T]. \tag{17}$$

As the process $t \mapsto e^{-r_d t} C_E(T-t, S_t)$ is a Q-martingale we obtain that

$$\sup_{\tau\in\mathcal{T}_{0,T}} \mathbb{E}_Q(\tilde{f}_\tau) = C_E(T, s_0) + \sup_{\tau\in\mathcal{T}_{0,T}} \mathbb{E}_Q(f_\tau). \tag{18}$$

Theorem 3.1 *An Instalment Call option with continuous payments is the sum of a European Call plus an American Put on this European Call, i.e.*

$$\underbrace{P_0 + p\int_0^T e^{-r_d s}\, ds}_{\textit{total premium payments}} = C_E(T, s_0) + \sup_{\tau\in\mathcal{T}_{0,T}} \mathbb{E}_Q(f_\tau),$$

where P_0 is the Instalment option price and $\sup_{\tau\in\mathcal{T}_{0,T}} \mathbb{E}_Q(f_\tau)$ is the price of an American put with a time-dependent strike.

Proof. Define a new claim $\widetilde{g} = (\widetilde{g}_t)_{t\in[0,T]}$ differing from g only by a constant, namely $\widetilde{g}(t) = g(t) + p\int_0^T e^{-r_d s}\, ds$. In view of (18) we have to show that

$$\sup_{\tau\in\mathcal{T}_{0,T}} \mathbb{E}_Q[\widetilde{g}(\tau)] = \sup_{\tau\in\mathcal{T}_{0,T}} \mathbb{E}_Q[\widetilde{f}(\tau)]. \tag{19}$$

The inequality with $\leq$ in (19) is obvious as we have $\widetilde{g} \leq \widetilde{f}$ pointwise. Let us show the other direction. Denote by $V = (V_t)_{t\in[0,T]}$ the Snell envelope of the potentially larger process $\widetilde{f}$, i.e. V is a càdlàg process (right continuous paths with left limits) with

$$V_t = \text{ess.sup}_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}_Q[\widetilde{f}(\tau) \mid \mathcal{F}_t], \quad P\text{-a.s.}, \quad t \in [0,T],$$

where $(\mathcal{F}_t)_{t\in[0,T]}$ is the canonical filtration of the process S. Define by $h = h(u,s)$ the value of the Call plus the Put on the Call, if the initial price of the underlying is $s \in \mathbb{R}_+$ and time to maturity of the contract is $u \in \mathbb{R}_+$, i.e.

$$h(u,s) = \sup_{\tau\in\mathcal{T}_{0,u}} \mathbb{E}_s\left[e^{-r_d\tau}\left[\left[\frac{p}{r_d}(1 - e^{-r_d(u-\tau)}) - C_E(u-\tau, \widetilde{S}_\tau)\right]^+ + C_E(u-\tau, \widetilde{S}_\tau)\right]\right],$$

where $\widetilde{S}$ is again a geometric Brownian motion with the same probabilistic characteristics as S. Using the Markov property of S we can apply Theorem 3.4 in El Karoui/Lepeltier/Millet (1992) and obtain

$$V_t = \text{ess.sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_Q \left[\widetilde{f}(\tau) \mid \mathcal{F}_t \right] = \text{ess.sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_Q \left[\widetilde{f}(\tau) \mid S_t \right] = e^{-r_d t} h(T-t, S_t).$$

As $\widetilde{f}$ has continuous paths the optimal exercise time is given by

$$\begin{aligned} \widehat{\tau} &= \inf\{t \in [0,T] \mid V_t = \widetilde{f}(t)\} \\ &= \inf\{t \in [0,T] \mid e^{-r_d t} h(T-t, S_t) = \widetilde{f}(t)\}. \end{aligned} \tag{20}$$

Keeping this in mind, we want to show that

$$h(u,s) > C_E(u,s) \quad \text{for all } u > 0, s > 0. \tag{21}$$

As the process $t \mapsto e^{-r_d t} C_E(T-t, S_t)$ is a martingale we can pull it out of the optimal stopping problem and obtain

$$h(u,s) = C_E(u,s) + \sup_{\tau \in \mathcal{T}_{0,u}} \mathbb{E}_s \left[e^{-r_d \tau} \left[\frac{p}{r_d}(1 - e^{-r_d(u-\tau)}) - C_E(u-\tau, \widetilde{S}_\tau) \right]^+ \right],$$

and thus $h(u,s) > C_E(u,s)$, for all $u > 0, s > 0$, as the underlying Call C_E can always get into the money with positive probability as long as $u > 0$. Therefore, we obtain for $t \in [0,T)$ and $s \in (0,\infty)$ the following implication

$$h(T-t,s) = (K_t - C_E(T-t,s))^+ + C_E(T-t,s) \Rightarrow K_t > C_E(T-t,s). \tag{22}$$

This means that by (20) $\widetilde{f}$ is only exercised prematurely when $K_t > C_E(T-t, S_t)$. But, in this case we have $\widetilde{f}(t) = \widetilde{g}(t)$. As at maturity the payoffs of $\widetilde{f}$ and $\widetilde{g}$ coincide anyway, we have for the optimal exercise time $\widehat{\tau}$ of the process $\widetilde{f}$

$$\widetilde{f}(\widehat{\tau}) = \widetilde{g}(\widehat{\tau}), \quad P\text{-a.s.}$$

Therefore we arrive at (19) and the assertion of the theorem follows. □

Remark 3.1 *We could use the same argument to prove that an Instalment put option is the sum of a European Put plus an American Put on this European Put.*

4 Numerical Results

4.1 Implementational Aspects

In the appendix we give a sample implementation for the discrete case of an Instalment option in both

- Mathematica, which solves the nested integration for the value recursively as mentioned in Section 2, and
- R, which computes the value using Equation (8) in Theorem 2.1.

Both implementations are used to investigate the performance and convergence of Instalment option values.

4.2 Performance

To compare the various methods to determine the value of Instalment options – to calculate the initial premium at time 0 dependent on the remaining strikes – we compare values of a specific trivariate Instalment option. We implement on the same machine

1. a binomial tree method in C++,
2. the closed-form formula in the statistical language R (see [13]),
3. the dynamic programming algorithm of Breton et al. [2],
4. a numerical integration using Gauß quadrature methods with $50,000$ supporting points, and
5. a recursive algorithm implemented in Mathematica for the calculation of the value of an n-variate Instalment option.

In Table 2 the results and computation times of all these five methods are shown for two representative examples of a 3-variate-Instalment option. The computational times are given in seconds and lie close together for most of the applied techniques.

Numerical Method	**Value of V_3**		**CPU Time**
Binomial trees for $n = 4000$	1.69053	0.0137335	1109
Closed-form formula for $n = 3$	1.69092	0.0137339	< 1
Algorithm based on [2] with $p = 4000$	1.69084	0.0137332	168
Numerical integration (50000-point Gauß-Legendre)	1.69087	0.0137339	176
Numerical integration with Mathematica	1.69091	0.0137299	47

Table 2: Performance comparison of Instalment valuation algorithms. We use $S_0 = 100$, $k_1 = 100$, $k_{2,3} = 3$, $\sigma = 20\%$, $r_d = 10\%$, $r_f = 15\%$, $T = 1$, $\Delta t = 1/3$, $\phi_{1,2,3} = 1$ and $S_0 = 1.15$, $k_1 = 1.15$, $k_{2,3} = 0.02$, $\sigma = 10\%$, $r_d = 1\%$, $r_f = 2\%$, $T = 1$,$t = 1/3$, $\phi_{1,2,3} = 1$.

Our experiences with the application of these methods show that

- The results using binomial tree methods oscillate heavily even with a large depth of the tree. Our examples show variations in the fourth digit of the value by using binomial trees with a depth of the tree from up to 7000.
- The trivariate formula is the fastest of all compared methods. Its accuracy and computation time essentially depend on the quality of the root finding procedure and on the calculation of the multivariate normal distribution function.

- The techniques, which are based on numerical integration as well as the dynamic programming approach of Breton et al. [2] lie in the middle field of all observed computation times.

4.3 Convergence

We illustrate the convergence of the overall Instalment premium to the limiting case in Figure 3.

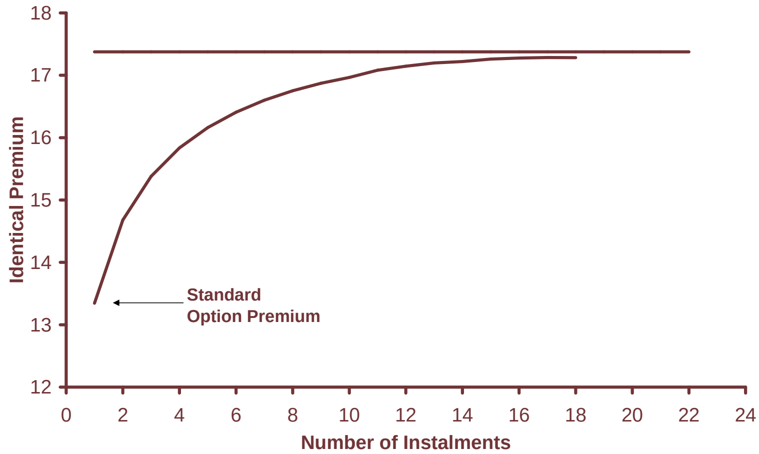

Figure 3: Convergence of uniform premium in discrete case to continuous premium. We have used the data $S_0 = 100$, $K = 95$, $\sigma = 0.2$, $r_d = 0.05$, $r_f = 0$, $T = 1$.

Here we investigate our result in Theorem 3.1 for a practical example, where a number of identical premiums of their corresponding n-variate-Instalment option for $n = 1, ..., 18$ is evaluated. The identical premium of a 1-variate-Instalment option is obviously the value of a standard Call option at time 0. All other identical premiums are calculated by a root finding procedure with respect to the strike price of the function

$$V_0(k) - k = 0,$$

which is the value of the particular Instalment option at time 0 minus the strike price. Here we use the closed-form Equation (8). It is implemented in the statistical language R (see [13]) as it contains the multivariate normal distribution function. The source code is listed in the appendix.
This calculation requires a high degree of accuracy and therefore takes a long time to compute. The identical premium for a 18-variate-Instalment option is 17.28. The limit U is calculated following Theorem 3.1 using the value of a European Call plus an American Put on this Call.

The Black-Scholes formula is used to determine the value of the European Call, and for the calculation of the American product of the portfolio we use binomial tree methods. The limit U lies approximately at 17.51 for the parameter set in Figure 3 and is approached here from below.

5 Summary

We have presented a closed-form solution for Instalment Call and Put options in the Black-Scholes model, discussed its application and numerical implementation. We proved the equivalence of the limiting case of a continuous instalment plan with a portfolio of the corresponding Vanilla and an American Put on that claim with a time dependent strike.

Further research could be done to explore closed-form valuation of Instalment options in models beyond Black-Scholes, such as stochastic volatility models or behavior in interest rate models. The case of Compound options ($n = 2$) has been examined in stochastic volatility models by Fouque and Han [5].
Another approach would be to analyze Instalment options with a more generalized payoff function at maturity, so that the Instalment plan's final option is a more exotic product than a Vanilla.

A Mathematica Code

A.1 The Package `instalment.m`

```
BeginPackage["Options`Instalment`"]

Instalment::usage = "Instalment[S,K,T,vol,rd,rf,phi,N] \n
Black-Scholes value for European Instalment options\n
S: spot\n
K: strike list of individual options\n
T: time differences to maturity in years between individual options\n
beginning with Vanilla option maturity\n
vol:  volatility\n
rd:  domestic risk free rate:  discounting is done as Exp[-T[[i]]*rd] \n
rf:  foreign risk free rate:  discounting is done as Exp[-T[[i]]*rf]\n
phi:  list of +1 for Calls, -1 for Puts\n
N: number of options in Instalment option"

Begin["`Private`"]

ncum[x_] := 1/2*(Erf[x/Sqrt[2]] + 1); (*cumulative standard normal*)
ndf[x_] := Evaluate[D[ncum[x], x]]; (*standard normal density*)
```

```
Vanilla[x_, K_, vol_, r_, rf_, T_, fi_]:=Block[dp, dm,
    dp = (Log[x/K] + (r- rf + 0.5*vol*vol)*T)/(vol*Sqrt[T]);
    dm = (Log[x/K] + (r - rf - 0.5*vol*vol)*T)/(vol*Sqrt[T]);
    fi*(Exp[-rf*T]*x*ncum[fi*dp] - Exp[-r*T]*K*ncum[fi*dm])];

Instalment[S_, K_, T_, vol_, rd_, rf_, phi_, N_] := Block[mu,
          mu = rd - rf - 0.5*vol*vol;
    If[N == 1, Vanilla[S, K[[1]], vol, rd, rf, T[[1]], phi[[1]]],
          Exp[-T[[N]]*rd]*
    NIntegrate[
          Max[0, phi[[N]]*(Instalment[S*Exp[vol*Sqrt[T[[N]]]*z + mu* T[[N]]],
             K, T, vol, rd, rf, phi, N - 1] -
          K[[N]])]*ndf[z], z, -10, 10]]];

End[]
EndPackage[]
```

A.2 The Testing Environment instalment_testenv.nb

```
spot = 100
vol = 0.2
tau = {1/3, 1/3, 1/3}
rd = 0.10
rf = 0.15
strike = {100, 3, 3}
phi = {1, 1, 1}

Instalment[spot, strike, tau, vol, rd, rf, phi, 3]

1.69085
```

B R Code

B.1 The R Functions

```
installments <- function(spot,strikes,times,phis,rd,rf,sigma,interval)
{
   n <- length(times)
   s <- 1:(n-1)
   roots <- rep(0,n)
   roots[n] <- strikes[n]

   for(i in s)
   {
```

```
        tau <- rep(0,i)
        for(j in (1:i))
            tau[j] <- times[n-j+1]-times[n-i]

        f<-function(x){recur(x,strikes[(n+1-i):n],rev(tau),phis[(n+1-i):n],
                        rd,rf,sigma,roots[(n-i+1):n])-strikes[i]}
            roots[n-i] <- uniroot(f,interval)[1]
    }

    result <- recur(spot,strikes,times,phis,rd,rf,sigma,roots)
    return(result)
}

recur <- function(spot,k,t,phis,rd,rf,sigma,roots){

        library(mnormt)

        n <- length(t)
        s <- 1:n
        args1 <- rep(0,n)
        args2 <- rep(0,n)
        multi <- rep(0,n)
        rho <- matrix(rep(0,n^2),nrow=n,ncol=n)

        for(i in s)
        {
            for(j in i:n)
            {
                rho[i,j] <- sqrt(t[i]/ t[j])
                rho[j,i] <- rho[i,j]
            }
        }

        muplus <- rd - rf + 0.5*sigma^2
        muminus <- rd - rf - 0.5*sigma^2

        for(i in s)
        {
            args1[i]<-(log(spot/roots[[i]])+muplus*t[i])/(sigma*sqrt(t[i]))
            args2[i]<-(log(spot/roots[[i]])+muminus*t[i])/(sigma*sqrt(t[i]))
        }

        for(i in s)
        {
```

```
        if(i==1)
            multi[i]<-prod(phis[1:i])
            *pmnorm(x=args2[1:i],mean=0,varcov=1,abseps = 1e-10)[1]
        else
            multi[i]<-prod(phis[1:i])
            *pmnorm(x=args2[1:i],mean=rep(0,i),varcov=rho[1:i,1:i])[1]
    }

    if(n==1)
        part1<-exp(-rf*t[n])*spot*prod(phis)
                *pmnorm(x=args1,mean=0,varcov=1,abseps = 1e-10)[1]
    else
        part1<-exp(-rf*t[n])*spot*prod(phis)
                *pmnorm(x=args1,mean=rep(0,i),varcov=rho[1:i,1:i])[1]

    part2 <- sum(exp(-rd*t)*k*multi)
    return(part1-part2)
}
```

B.2 The R Testing Environment

```
interval <- c(0,150)
strikes<-c(3,3,100)
times<-c(1/3,2/3,1.0)
phis <- c(1,1,1)
rd = 0.1
rf = 0.15
sigma = 0.2
spot= 100
installments(spot,strikes,times,phis,rd,rf,sigma,interval)

interval <- c(0,10)
strikes<-c(0.02,0.02,1.15)
times<-c(1/3,2/3,1.0)
phis <- c(1,1,1)
rd = 0.01
rf = 0.02
sigma = 0.1
spot= 1.15
installments(spot,strikes,times,phis,rd,rf,sigma,interval)
```

References

[1] Baviera, R. and Giada, L. (2006). *Perturbative Approach to Bermundan Option Pricing.* Working Paper 941318, SSRN.

[2] BEN-HAMEUR, H., BRETON, M. and FRANÇOIS, P. (2006). A Dynamic Programming Approach to Price Installment Options. *European Journal of Operational Research*, **169** 667-676.

[3] CIURLIA, P. and ROKO, I. (2005). Valuation of American Continuous-Installment Options. *Computational Economics* **1-2**, 143-165.

[4] CURNOW, R.N. and DUNNETT, C.W. (1962). The Numerical Evaluation of Certain Multivariate Normal Integrals. *Ann. Math. Statist.* **33**, 571-579.

[5] FOUQUE, J. and HAN, C. (2005). Evaluation of Compound Options using Perturbation Approximation. *Journal of Computational Finance* **9**.

[6] DAVIS, M., SCHACHERMAYER, W. and TOMPKINS, R. (2001). Pricing, No-arbitrage Bounds and Robust Hedging of Installment Options. *Quantitative Finance* **1**, 597-610.

[7] BROWN, B., LOVATO, J. and RUSSELL, K. (2004). CDFLIB - C++ - library, `http://www.csit.fsu.edu/~burkardt/cpp_src/dcdflib/dcdflib.html`

[8] GESKE R. (1979). The Valuation of Compound Options. *Journal of Financial Economics*, **7**, 63-81.

[9] HENRARD, M. (2005). *Bermudan Swaptions in a Hull-White One-Factor Model: Analytical and Numerical Approaches.* Ewp-fin 0505023, Economics Working Paper Archive.

[10] EL KAROUI, N., LEPELTIER, J.P. and MILLET, A. (1992). A Probabilistic Approach of the Reduite. *Probability and Mathematical Statistics* **13**, 97–121.

[11] KIMURA, T. and KIKUCHI, K. (2007). *Valuing Continuous Installment Options via Laplace Transforms.* Working Paper, Hokkaido University.

[12] PIETERSZ, R. and PELSSER, A.A. (2003). *Risk Managing Bermudan Swaptions in the LIBOR BGM Model.* Working Paper 383580, SSRN.

[13] R PROJECT, `http://www.r-project.org`.

[14] SCHILLING, H, (2002). Compound Options. In *Foreign Exchange Risk.* Risk Publications. London.

[15] THOMASSEN, L. and VAN WOUVE, M. (2002). *A Sensitivity Analysis for the N-fold Compound Option.* Research Paper, Faculty of Applied Economics, University of Antwerpen.

[16] WYSTUP, U. (2006). *FX Options and Structured Products.* Wiley Finance.