Optimal investment in financial markets with different liquidity effects

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Abstract

In an illiquid financial market trades of a large investor can move market prices quite significantly. In the last years different models have been suggested to capture the interdependency of the evolution of the asset price and the dynamic trading strategy of a large investor in the asset. Roughly speaking, there are two competing approaches in the literature. In Çetin, Jarrow, Protter [3], Rogers, Singh [21] and others, a transaction of a large trader only has a short-term price impact and the asset price jumps back to its previous level. By contrast, in Bank, Baum [1], Esser, Mönch [6], Frey, Stremme [7] and Platen, Schweizer [18] the market price stays at the new level after a transaction of the large investor.

We give a new microeconomic motivation of different illiquid market models and propose a continuous-time model which unifies both liquidity effects. The continuous-time self-financing condition for this model is obtained as a limit from so-called simple strategies. We analyze the large investor's utility maximization problem for utility functions which are finite on the whole real line. We prove the existence of an optimal strategy and characterize it by a marginal utility property generalizing a well-known relationship in frictionless markets.

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1 Introduction

Liquidity risk is certainly one of the most important risks faced by the finance industry. Whereas on a heuristical level it seems to be quite clear what the effect of illiquidity is about, there are different approaches to its precise mathematical modelling.

In the models of Bank and Baum [1], Jarrow [10, 11] and Platen and Schweizer [18]. the price of an illiquid asset is a function of the *total* number of shares a large investor currently holds in his portfolio. After a trade of the large investor has moved the market price it stays at the new level until new information arises or the large investor trades again. In this sense the price impact is permanent. This feature can be derived from "first principles" by regarding the price process as a sequence of temporary equilibria in a market with *one large investor* and *many small agents* (who are price takers): at each $t \in [0, T]$ a new auction takes place and small traders update their positions, i.e. all allong they compare their optimal demand for assets (depending on the asset price) with the number of assets they hold so far in their portfolios. By owning θ_t assets the large investor removes these shares from the market leading to a shortage of shares (resp. an oversupply in the case that $\theta_t < 0$). If the total amount of shares is $M \in \mathbb{R}_+$, there are $M - \theta_t$ shares available to meet the demand of the small agents. Thus the market clearing price at time t, i.e. the price at which demand equals supply, is a function of θ_t, see Section 3 for more details.

By contrast, in the models of Çetin, Jarrow, Protter [3], Çetin [2] and Rogers, Singh [21] the time-t-price of the asset is not determined by the total amount θ_t of shares, but by the large investor's excess demand $\Delta\theta_t$ resp. $d\theta_t$, i.e. the number of assets he purchases or sells at time t. This can be motivated as follows: at time t (or in a short interval $(t, t+\Delta t]$) only a small fraction of traders, which own or which are in principle interested in the asset, update their positions (in the sense that they have limit/market orders in the order book against which the orders of the large investor can be executed). At each point in time new small traders constitute the market. Therefore price impacts are nonpermanent and feedback effects are avoided. In [21] the amount of orders during a time interval $(t_1, t_2]$ is of order $t_2 - t_1$. By contrast, in [3] the large investor can realize significant transactions in arbitrary short time intervals.

Summing up, the determinant which model to take is the fraction of shares which is actually traded on the market. As the total number of shares is typically much higher than the active orders in the orderbook the models of Çetin, Jarrow, Protter [3] and Rogers, Singh [21] seem to describe the dominating liquidity effect when the large trader is not "too large". On the other hand, if an investor plans to purchase or sell positions which are not negligible w.r.t. the total number of shares he will try to acquire sellers or buyers of the asset. This could lead into the regime of the models of Bank and Baum [1] and Jarrow [10, 11] which is based on the assumption that all small agents are permanently updating their positions in the asset.

The paper is organized as follows. In Section 2 we take up the model of [1] and analyze the utility maximization problem without the assumption of a joint martingale measure for all processes $S(\vartheta, \cdot)$. The optimal strategy is characterized by some martingale property (Theorem 2.16). In Section 3 we give a new microeconomic motivation of special choices of the family $(S(\vartheta, \cdot))_{\vartheta \in \mathbb{R}}$. In Section 4 we propose a continuous-time model which unifies both liquidity effects described above. The main idea is that during some time interval only a fraction of small traders is present on the market. Then, in a future time interval these small traders are not totally replaced by other small traders (as e.g. in [21]), but some of them return which leads to a feedback effect of the asset price process and the dynamic trading strategy of the large investor. The continuous-time self-financing condition for this model is obtained as a limit from so-called simple strategies (Theorem 4.7) and the large investor's utility maximization problem is analyzed (Theorem 4.13).

Throughout the paper we fix a terminal time $T \in \mathbb{R}_+$ and a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)$ satisfying the usual conditions. To keep the exposition simple, we work with only one illiquid asset and one totally liquid asset. The liquid asset acts as a numeraire, i.e. all price processes are expressed in terms of multiples of this (tradable) numeraire security.

2 Permanent price impact

In this section we follow the approach of [1] and model an illiquid financial market by a family of semimartingales $S = (S(\vartheta, \cdot))_{\vartheta \in \mathbb{R}}$ where $S(\vartheta, \cdot)$ is the asset price process in the case that the single large investor holds a constant number of $\vartheta \in \mathbb{R}$ shares during the whole term $[0, T]$. If the large investor chooses the dynamic strategy $(\theta_t)_{t \in [0,T]}$ the market price process becomes $t \mapsto S(\theta_t, t)$ (for further motivation of this assumption see Section 3). The aim is to characterize the dynamic self-financing trading opportunities of the large investor. Throughout Sections 2 and 4 we make the following

Assumption 2.1 (Largeness is not favorable). *The "marginal" price process $S(0, \cdot)$ is a locally bounded semimartingale and*

$$S(\vartheta, t) = S(\vartheta, 0) + \int_0^t H_s^\vartheta \, dS(0, s) + \int_0^t H_s^\vartheta a_s^\vartheta \, dA_s, \quad t \in [0, T], \vartheta \in \mathbb{R}. \tag{2.1}$$

where $H : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}_+$, $(\omega, t, \vartheta) \mapsto H_t^\vartheta(\omega)$ and $a : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$, $(\omega, t, \vartheta) \mapsto a_t^\vartheta(\omega)$ are $\mathcal{P} \otimes \mathcal{B}(\mathbb{R})$-measurable functions ($\mathcal{P}$ denotes the predictable σ-algebra on $\Omega \times [0, T]$). $\vartheta \mapsto S(\vartheta, 0)$ is continuous. A is càdlàg, predictable and nondecreasing (and thus locally bounded). For any $\vartheta \in \mathbb{R}$, H^ϑ and $H^\vartheta a^\vartheta$ are $S(0, \cdot)$-integrable and A-integrable, resp. H is uniformly Lipschitz continuous in ϑ with $H^0 = 1$. a is continuous and **nonincreasing** *in ϑ with $a^0 = 0$.*

One may think of $H = 1$. In this case the different price processes only differ in predictable processes of finite variation.

Throughout the paper we follow the usual convention that $\int_{b_2}^{b_1} \ldots = -\int_{b_1}^{b_2} \ldots$ for $b_1 < b_2$.

Definition 2.2. *Let $L(x, t) := \int_0^x S(\vartheta, t) \, d\vartheta$ be the liquidation value of $x \in \mathbb{R}$ shares. The value process $V = V(\theta)$ is defined by*

$$\begin{aligned} V_t(\theta) &:= v_0 + \int_0^t L(\theta_s, ds) \\ &:= v_0 + \int_0^t \left(\int_0^{\theta_s} H_s^x \, dx \right) dS(0, s) + \int_0^t \left(\int_0^{\theta_s} H_s^x a_s^x \, dx \right) dA_s, \ t \in [0, T]. \end{aligned} \tag{2.2}$$

$v_0 \in \mathbb{R}$ is the initial endowment in bonds. θ runs through all predictable processes such that the processes

$$t \mapsto \int_0^{\theta_t} H_t^x \, dx \quad \text{and} \quad t \mapsto \int_0^{\theta_t} H_t^x a_t^x \, dx \tag{2.3}$$

are in $L(S(0,\cdot))$ and $L(A)$, resp. and the set of these strategies is denoted by $\widetilde{\Theta}$ (by the assumptions on H and a the processes in (2.3) are at least predictable if θ is).

Remark 2.3. *If θ is continuous and of finite variation with $\theta_0 = 0$, it follows by the Itô-Wentzell formula applied to the stochastic flow $(L(x,\cdot))_{x\in\mathbb{R}}$ (cf. [1]) that*

$$V_t(\theta) = v_0 + L(\theta_t, t) - \int_0^t S(\theta_s, s-) \, d\theta_s, \quad t \in [0,T]. \tag{2.4}$$

At time $t-$ the investor purchases $\theta_t - \theta_{t-dt}$ shares at price $S(\theta_t, t-)$. Thus the term $-\int_0^t S(\theta_s, s-)\, d\theta_s$ can be interpreted as the evolution of the bank account when choosing the investment strategy θ. So that (2.4) (resp. (2.2)) is a self-financing condition. These considerations only hold for continuous strategies of finite variation, but it was shown in [1] that (for continuous stock prices) the terminal wealth of semimartingale *strategies can be uniformly approximated by continuous processes of finite variation. In this section we start with (2.2) as a definition and allow for essentially any (possibly irregular) prectictable strategy. This is of course an idealization, but, for typical examples one should expect that the optimal strategies are anyhow sufficiently regular which could justify the approach ex post*. In (2.2) so called free-round-trips are possible. This means the large investor can move the market price without transaction costs. By contrast, free-round-trips are not workable in [3], [21] or in the model introduced in Section 4.*

Remark 2.4. *By Assumption 2.1 we have the following: starting from ϑ assets at some $t \in [0,T]$, the actual change of the liquidation value (2.2) by investing in further η securities during $(t, t+dt]$ is almost surely dominated by some hypothetical extra trading gains $\widetilde{\eta}(S(\vartheta, t+dt) - S(\vartheta,t))$ which one could achieve if the added assets were traded at price process $S(\vartheta,\cdot)$ (in case that $H = 1$ one can choose $\widetilde{\eta} = \eta$). This property guarantees that the set of superhedgeable terminal wealths is convex. I.e. for two admissible terminal*

*Due to this idealization we need not *formally* assume that $S(\vartheta, t)$ is nondecreasing in ϑ. This assumption is made for the model in Section 4 to exclude e.g. arbitrage opportunities for the large investor.

wealths any convex combination can be superhedged by another strategy. See Remark 2.6 for further discussions of this property.

Definition 2.5. *Let (B, C, ν) be the characteristics of the semimartingale $S(0, \cdot)$ relative to the truncation function $h(x) = x$ (this choice is possible as $S(0, \cdot)$ is locally bounded), see [9], Definition II.2.6. Possibly after redefinition of a^ϑ and A – without changing the processes in (2.1) and losing the claimed properties – we can w.l.o.g. assume that the characteristics can be desintegrated w.r.t. A. In particular, $B_t = \int_0^t b_s \, dA_s$ up to evanescence for some predictable process b. By construction semimartingale differential drifts for $S(\vartheta, \cdot)$ are given by*

$$b^\vartheta := H^\vartheta (b + a^\vartheta), \quad \vartheta \in \mathbb{R}. \tag{2.5}$$

We call the processes in (2.5) the canonical drift rates of $(S(\vartheta, \cdot))_{\vartheta \in \mathbb{R}}$. For an equivalent probability measure these quantities are defined accordingly.

Notice that the differential drift of a semimartingale is only defined up to a $(P \otimes A)$-nullset. To avoid that the nullset depends on ϑ we always take the canonical version from (2.5). Define

$$\Theta_c := \{\theta \in \widetilde{\Theta} \mid V_t(\theta) \geq -c, \ \forall t \in [0, T]\}, \quad c \in \mathbb{R}_+ \quad \text{and} \quad \Theta = \bigcup_{c \in \mathbb{R}_+} \Theta_c. \tag{2.6}$$

We want to consider the utility maximization problem

$$\sup \{E\,(u(F)) \mid F \in \mathcal{C}\}, \tag{2.7}$$

where

$$\begin{aligned} \mathcal{C} \ := \ & \big\{F \in L^0(\Omega, \mathcal{F}, P) \mid u(F) \text{ is in the } L^1(\Omega, \mathcal{F}, P)\text{-closure of} \\ & \quad \{u(V_T(\theta) - Y) \mid \theta \in \Theta, \ Y \in L^0_+(\Omega, \mathcal{F}, P)\}\big\}. \end{aligned} \tag{2.8}$$

$u : \mathbb{R} \to \mathbb{R}$ is a monotone, strictly concave utility function satisfying $\lim_{x\to\infty} u'(x) = 0$ and $\lim_{x\to-\infty} u'(x) = \infty$ (Inada conditions). In addition $\lim_{x\to\infty} \frac{u'(x)x}{u(x)} < 1$ and u is subexponential in the sense that there exist $a, b \in \mathbb{R}_+$ s.t. $u^-\left(\frac{x+y}{2}\right) \leq \sqrt{(u^-(x) + a)(u^-(y) + b)}$ for all $x, y \in \mathbb{R}$. An example is the exponential utility function. With (2.7) we follow the approach of Schachermayer [22] considering in a first instance only value processes with limited credit and then let the credit line tend to infinity. This approach seems to be reasonable as the price processes $S(\vartheta,\cdot)$ are locally bounded. In typical models for illiquid markets the absence of a credit line allows for doubling strategies leading to an arbitrage – similar to liquid models.

Notice that by the extension of the domain to the $L^1(\Omega,\mathcal{F},P)$-closure of $\{u(V_T(\theta)-Y)\mid \theta\in\Theta,\ Y\in L^0_+(\Omega,\mathcal{F},P)\}$ the supremum in (2.7) is unchanged.

Remark 2.6. *The utility maximization problem for the model in this section was already treated in [1], but under the assumption that $(S(\vartheta,\cdot))_{\vartheta\in\mathbb{R}}$ possess a* joint *martingale measure. As mentioned there, this assumption is rather restrictive as $(S(\vartheta,\cdot))_{\vartheta\in\mathbb{R}}$ are not the price processes of assets which can be simultaneously traded on the market. To our mind the most important generalization consists in allowing for different P-drifts of $S(\vartheta,\cdot)$ whereas the P-martingale parts are at least strongly related to each other as they should be driven by the same underlying random source (e.g. the firm's dividend process, see Section 3). This is also the motivation for Assumption 2.1 which guarantees the convexity of the problem. However, the assumption that a^ϑ (and A) are predictable is mainly for convenience and could be replaced by progressively measurability together with appropriate integrability conditions on $(a^\vartheta)_{\vartheta\in\mathbb{R}}$. Essential further generalizations of the problem cannot rely on the theory in [5] ensuring the existence of a limit from the convex hull of a sequence of strategies. In addition, Theorem 2.16 and Theorem 4.13, resp., fail to hold in general.*

Assumption 2.7. *Let $S(0,\cdot)=S(0,0)+N+D$ be the canonical decomposition of the special semimartingale $S(0,\cdot)$, i.e. N is a local martingale and D is a predictable process of finite variation both starting in 0. Assume that $E\left(\int_0^T|dD_t|\right)<\infty$.*

Assumption 2.8. *Assume that there exists some $m\in\mathbb{R}_+$ such that for $\vartheta\geq m$ the process $S(\vartheta,\cdot)$ is a local supermartingale and for $\vartheta\leq -m$ a local submartingale.*

Assumption 2.9. *Assume that there exists a $m\in\mathbb{R}_+$ such that the P-differential drift b^ϑ of $S(\vartheta,\cdot)$ (cf. Definition 2.5) is larger than 1 for $\vartheta\leq -m$ and smaller than -1 for $\vartheta\geq m$, $(P\otimes A)$-a.s.*

These assumptions seem to be quite natural for illiquid markets which can be seen in Section 3. For utility maximization problems P-integrability of the stock prices is of course not necessary, but, notice that we need anyhow that $S(0,\cdot)$ is locally bounded.

The first goal is to show that (2.7) possesses a (unique) maximizer. To this end we need some auxiliary lemmas.

Lemma 2.10. *Under Assumptions 2.7 and 2.8 the family* $\{V_T^-(\theta) \in L^0(\Omega, \mathcal{F}, P) \mid \theta \in \Theta$ *with* $E\left(u(V_T(\theta))\right) \geq u(v_0) - 1\}$ *is uniformly integrable and the set* $\{V_T^+(\theta) \in L^0(\Omega, \mathcal{F}, P) \mid \theta \in \Theta$ *with* $E\left(u(V_T(\theta))\right) \geq u(v_0) - 1\}$ *is bounded in* $L^1(\Omega, \mathcal{F}, P)$.

If, in addition, Assumption 2.9 holds, then the set $\{\theta \in L^0(\Omega \times [0,T], \mathcal{P}, P \otimes A) \mid \theta \in \Theta$ *with* $E\left(u(V_T(\theta))\right) \geq u(v_0) - 1\}$ *is bounded in* $L^0(\Omega \times [0,T], \mathcal{P}, P \otimes A)$.

Notice that the boundedness of $\{\theta \in \Theta \text{ with } E\left(u(V_T(\theta))\right) \geq u(v_0) - 1\}$ does usually not hold in frictionless markets.

Proof. Step 1: By Assumptions 2.7 and 2.8 the wealth process $V = V(\theta)$, $\theta \in \Theta$, can be written as

$$V = v_0 + \widehat{B} + C,$$

with $\widehat{B}_0 = C_0 = 0$, where $\widehat{B}$ is an increasing, integrable and *strategy-independent* process. $C = C(\theta)$ is a P-local supermartingale. As each V is bounded from below by some constant c, the local supermartingale C is bounded from below by the martingale $t \mapsto -c - v_0 - E(\widehat{B}_T \mid \mathcal{F}_t)$. Thus C is a (true) supermartingale.

Step 2: Let $(V(\theta^n))_{n\in\mathbb{N}}$, $\theta^n \in \Theta$, be a sequence of wealth processes with $E(u(V_T(\theta^n))) \geq u(v_0) - 1$. Let $V(\theta^n) = v_0 + \widehat{B} + C^n$. W.l.o.g. we can assume that $u(v_0) = 0$. It is sufficient to show that the sequence $((C_T^n)^-)_{n\in\mathbb{N}}$ is uniformly integrable. For each $K \in \mathbb{R}_+$ we have

$$\begin{aligned} E\left(-u^-(v_0 + \widehat{B}_T + C_T^n)\right) &\leq u'(v_0)E\left(-(\widehat{B}_T + C_T^n)^-\right) \\ &\quad + \left[u'(-K + v_0) - u'(v_0)\right] E\left(-(\widehat{B}_T + C_T^n + K)^-\right), \quad \text{and} \end{aligned} \tag{2.9}$$

$$E\left(u^+(v_0 + \widehat{B}_T + C_T^n)\right) \leq u'(v_0)E\left((\widehat{B}_T + C_T^n)^+\right). \tag{2.10}$$

(2.9), (2.10), and the fact that C^n is a supermartingale imply that

$$\begin{aligned}
E\left(u(V_T(\theta^n))\right) &\leq u'(v_0)E(\widehat{B}_T + C_T^n) + [u'(-K+v_0) - u'(v_0)]\, E(-(\widehat{B}_T + C_T^n + K)^-) \\
&\leq u'(v_0)E(\widehat{B}_T) + [u'(-K+v_0) - u'(v_0)]\, E(-(\widehat{B}_T + C_T^n + K)^-). \quad (2.11)
\end{aligned}$$

As $u'(-K+v_0)$ tends to infinity for $K \to \infty$, by (2.11) and the assumption that $E(u(V_T(\theta^n))) \geq u(v_0) - 1$ we obtain that $\lim_{K\to\infty} \sup_{n\in\mathbb{N}} E\left((\widehat{B}_T + C_T^n + K)^-\right) = 0$ and thus

$$\lim_{K\to\infty} \sup_{n\in\mathbb{N}} E\left((C_T^n + K)^-\right) = 0. \tag{2.12}$$

(2.12) implies that $((C_T^n)^-)_{n\in\mathbb{N}}$ is uniformly integrable. For the positive parts this means anyway that

$$\sup_{n\in\mathbb{N}} E\left((C_T^n)^+\right) < \infty$$

whereas uniformly integrability need not hold.

Step 3: Assume that $\{\theta \in \Theta \mid E\left(u(V_T(\theta))\right) \geq u(v_0) - 1\}$ is not bounded in $L^0(\Omega \times [0,T], \mathcal{P}, P \otimes A)$. This means there exists an $\varepsilon > 0$ and a sequence $(\theta^n)_{n\in\mathbb{N}}$ from this strategy set s.t. $|\theta^n| > n$ on some set $D_n \in \mathcal{P}$ with $(P \otimes A)(D_n) \geq \varepsilon$. By Assumption 2.9, Lemma 2.5. in [13], and the fact that C^n is bounded from below by a martingale we obtain that

$$E(C_T^n) \leq E\left[\int_0^T \int_0^{\theta_t^n} H_t^\vartheta a_t^\vartheta \, d\vartheta \, dA_t\right] \leq E\left[\int_0^T 1_{D_n}(t)(m - |\theta_t^n|)\, dA_t\right] \leq (m-n)\varepsilon, \ \forall n \geq m$$

and thus $E\left(u(V_T(\theta^n))\right) \leq u'(v_0)\left[E(\widehat{B}_T) + (m-n)\varepsilon\right]$ which is for n large enough a contradiction to $E(u(V_T(\theta^n)) \geq u(v_0) - 1$. □

Lemma 2.11. *There exist a sequence $(\theta^n)_{n\in\mathbb{N}} \subset \Theta$ with $\lim_{n\to\infty} E(u(V_T(\theta^n))) = \sup_{\theta\in\Theta} E(u(V_T(\theta)))$, an $X \in L^0(\Omega, \mathcal{F}, P)$, and a $\theta \in L^0(\Omega \times [0,T], \mathcal{P}, P \otimes A)$ s.t. $\theta^n \to \theta$ $P \otimes A$-a.s. and $X^n \to X$ P-a.s., $n \to \infty$, for some sequence $X^n \in \operatorname{conv}(V_T(\theta^n), V_T(\theta^{n+1}), V_T(\theta^{n+2}), \ldots)$. By Assumption 2.1, $X^n \in \mathcal{C}$ and, by the concavity of u,*

$$\lim_{n\to\infty} E(u(X^n)) = \sup_{\theta\in\Theta} E(u(V_T(\theta))).$$

Proof. Let $(\theta^n)_{n\in\mathbb{N}} \subset \Theta$ be a sequence approximating the supremum in (2.7). By Lemma 2.10 and Theorem D in [5] there exists a sequence $(\widetilde{\theta}^n)_{n\in\mathbb{N}}$ with $\widetilde{\theta}^n \in \operatorname{conv}(\theta^n, \theta^{n+1}, \theta^{n+2}, \dots)$ such that $\widetilde{\theta}^n$ converges $(P\otimes A)$-a.s. to some $\widetilde{\theta} \in L^0(\Omega\times[0,T], \mathcal{P}, P\otimes A)$. By Assumption 2.1 the optimality of the sequence $(\theta^n)_{n\in\mathbb{N}}$ implies that of $(\widetilde{\theta}^n)_{n\in\mathbb{N}}$.

Similarly, by Lemma 2.10, there exists a sequence $(X^n)_{n\in\mathbb{N}}$ with $X^n \in \operatorname{conv}(V_T(\widetilde{\theta}^n), V_T(\widetilde{\theta}^{n+1}), V_T(\widetilde{\theta}^{n+2}), \dots)$ such that X^n converges P-a.s. to some $X \in L^0(\Omega, \mathcal{F}, P)$. □

Proposition 2.12. *The optimization problem (2.7) possesses a unique maximizer.*

Proof. Let $Z^1, Z^2 \in \mathcal{C}$ be two maximizers. By Assumption 2.1 the set $\{V_T(\theta) - Y \mid \theta \in \Theta,\ Y \in L^0_+(\Omega, \mathcal{F}, P)\}$ is convex and thus also its $L^1(\Omega, \mathcal{F}, P)$-closure (the latter follows as convergence is equivalent to pointwise convergence plus some uniformly integrability which translates to convex combinations by basic properties of utility functions). Thus, $\frac{Z^1+Z^2}{2} \in \mathcal{C}$. Assume that $P(Z^1 \neq Z^2) > 0$. By the strict concavity of u we arrive at $E\left(u\left(\frac{Z^1+Z^2}{2}\right)\right) > E\left(u(Z^1)\right)$ which is a contradiction to the optimality of Z^1. This implies uniqueness.

Throughout the existence proof we assume w.l.o.g. that $u(0) = 0$. By Lemma 2.11 there exists a sequence $(X^n)_{n\in\mathbb{N}} \subset \mathcal{C}$ with

$$\lim_{n\to\infty} E(u(X^n)) = \sup_{F\in\mathcal{C}} E(u(F)) \tag{2.13}$$

and $\lim_{n\to\infty} X^n = X$, P-a.s. for some $X \in L^0(\Omega, \mathcal{F}, P)$. But, we do not yet know whether X is in $\mathcal{C}$ – as we do not know whether $u(X^n)$ is a Cauchy sequence in $L^1(\Omega, \mathcal{F}, P)$. To show that X is indeed the solution of (2.7) there is to verify that both $(u^+(X^n))_{n\in\mathbb{N}}$ and $(u^-(X^n))_{n\in\mathbb{N}}$ are uniformly integrable.

Step 1: This step is similar in spirit to the proof of Theorem 5.2 in [8]. But, in [8] utility functions are finite on $\mathbb{R}_+ \setminus \{0\}$ only (and the article is about models with proportional transaction costs).

Assume that $(u^+(X^n))_{n\in\mathbb{N}}$ is *not* uniformly integrable. From Lemma 2.10 it follows that $(E(u^+(X^n)))_{n\in\mathbb{N}}$ take values in a compact interval. Therefore, under the contradiction assumption, it follows with Fatou that there is an $\alpha > 0$ such that (possibly after passing to a subsequence)

$$\lim_{n\to\infty} E(u^+(X^n)) = E(u^+(X)) + \alpha.$$

By Schachermayer [23], Lemma 3.16, for every $\varepsilon > 0$ there exist $n, m > \varepsilon^{-1}$ and a measurable set A_n such that

(i) $u^+(X^n) \geq \varepsilon^{-1}$ on A_n and $\alpha - \varepsilon \leq E(u^+(X^n)1_{A_n}) \leq \alpha + \varepsilon$

(ii) $E(u^+(X^m)1_{A_n}) \leq \varepsilon$

(iii) $E(u^+(X)) + \alpha - \varepsilon \leq E(u^+(X^k)) \leq E(u^+(X)) + \alpha + \varepsilon$ for $k = n, m$.

In addition, by Lemma 6.3 in [15] there are $\lambda > 0$, $\beta > 1$, and $x_0 \geq 0$ such that

$$u\left(\frac{x(1-\lambda)}{2}\right) \geq \frac{\beta}{2} u(x) \text{ and } u\left(-\frac{\lambda}{2}x\right) \geq \left(1 - \frac{\beta}{2}\right) u(-\lambda x) \quad \text{for all } x \geq x_0. \tag{2.14}$$

We only consider ε with $\varepsilon^{-1}\min\{\lambda, 1\} \geq x_0$ and distinguish two situations, namely the regions $M_1 := \{X^m \geq -\lambda X^n\} \cap A_n$ and $M_2 := A_n \setminus M_1$. We have by (i) and (2.14)

$$\begin{aligned} u\left(\frac{X^n + X^m}{2}\right) - \frac{1}{2}\left[u(X^n) + u(X^m)\right] &\geq u\left(\frac{1-\lambda}{2}X^n\right) - \frac{1}{2}\left[u(X^n) + u(X^m)\right] \\ &\geq \frac{\beta - 1}{2} u(X^n) - \frac{1}{2}u(X^m) \quad \text{on } M_1. \end{aligned} \tag{2.15}$$

As u' is decreasing the function $u\left(\frac{\cdot}{2}\right) - \frac{1}{2}u(\cdot)$ is decreasing on $(-\infty, 0]$. This together with (i) implies

$$\begin{aligned} &u\left(\frac{X^n + X^m}{2}\right) - \frac{1}{2}\left[u(X^n) + u(X^m)\right] \\ &\geq u\left(\frac{X^m}{2}\right) - \frac{1}{2}u(X^m) - \frac{1}{2}u(X^n) \\ &\geq u\left(\frac{-\lambda X^n}{2}\right) - \frac{1}{2}u(-\lambda X^n) - \frac{1}{2}u(X^n) \\ &\geq \frac{1}{2}(1-\beta)u(-\lambda X^n) - \frac{1}{2}u(X^n) \\ &\geq \eta u(X^n) \quad \text{on } M_2 \end{aligned} \tag{2.16}$$

for some $\eta > 0$. The last inequality is due to (2.14) and the Inada conditions by choosing ε from (i)-(iii) small enough. (2.15) and (2.16) together with (i)/(ii) yield

$$\begin{aligned} &E\left[u\left(\frac{X^n + X^m}{2}\right)1_{A_n}\right] \\ &\geq \frac{1}{2}\left(E\left[u(X^n)1_{A_n}\right] + E\left[u(X^m)1_{A_n}\right]\right) + \left(\frac{\beta-1}{2} \wedge \eta\right)(\alpha - \varepsilon) - \frac{\varepsilon}{2}. \end{aligned}$$

Furthermore we have of course that

$$u\left(\frac{X^n+X^m}{2}\right)1_{\Omega\setminus A_n} \geq \frac{1}{2}u\left(X^n\right)1_{\Omega\setminus A_n}+\frac{1}{2}u\left(X^m\right)1_{\Omega\setminus A_n} \tag{2.17}$$

which implies that

$$E\left[u\left(\frac{X^n+X^m}{2}\right)\right] \geq \frac{1}{2}\left(E\left[u\left(X^n\right)\right]+E\left[u\left(X^m\right)\right]\right)+\left(\frac{\beta-1}{2}\wedge\eta\right)(\alpha-\varepsilon)-\frac{\varepsilon}{2}.$$

As $\beta>1$, $\eta>0$, $\alpha>0$ and ε can be chosen arbitrary small this is a contradiction to the optimality of the sequence $(\theta^n)_{n\in\mathbb{N}}$. Therefore $(u^+(X^n))_{n\in\mathbb{N}}$ *is* uniformly integrable.

Step 2: Assume that $(u^-(X^n))_{n\in\mathbb{N}}$ is not uniformly integrable where $(X^n)_{n\in\mathbb{N}}$ still denotes the sequence from (2.13). Again, possibly after passing to a subsequence we can assume that

$$\lim_{n\to\infty} E(u^-(X^n)) = E(u^-(X))+\alpha'$$

for some $\alpha'>0$. As in Step 1, for all $\varepsilon>0$ there exist $n,m>\varepsilon^{-1}$ and measurable sets A_n with

(i) $u^-(X^n)\geq\varepsilon^{-1}$ on A_n and $E(u^-(X^n)1_{A_n})\geq\alpha'-\varepsilon$

(ii) $E(u^-(X^n)1_{\Omega\setminus A_n})\geq E(u^-(X))-\varepsilon$ and
$E(u^-(X^m)1_{\Omega\setminus A_n})\in[E(u^-(X))+\alpha'-\varepsilon, E(u^-(X))+\alpha'+\varepsilon]$

Using that u^- is subexponential, i.e. $u^-\left(\frac{x+y}{2}\right)\leq\sqrt{(u^-(x)+a)(u^-(y)+b)}$ for some $a,b\in\mathbb{R}_+$ and all $x,y\in\mathbb{R}$, we obtain by Cauchy-Schwarz's inequality that

$$\begin{aligned} E\left(u^-\left(\frac{X^n+X^m}{2}\right)1_{A_n}\right) &\leq \sqrt{E\left(u^-\left(X^n+a\right)1_{A_n}\right)}\sqrt{E\left(u^-\left(X^m+b\right)1_{A_n}\right)} \\ &\leq (\alpha'+\varepsilon+aP(A_n))^{1/2}(\varepsilon^{1/2}+bP(A_n))^{1/2}. \end{aligned} \tag{2.18}$$

$P(A_n)$ is of order $\mathcal{O}(\varepsilon)$ and $(u^+(V(\theta^n)))_{n\in\mathbb{N}}$ is uniformly integrable which allows us to neglect the positive parts of the utility on A_n. Furthermore, like in (2.17), averaging on $\Omega\setminus A_n$ does not decrease the utility and we have

$$E\left(u\left(\frac{X^n+X^m}{2}\right)\right)-\frac{1}{2}\left[E\left(u\left(X^n\right)\right)+E\left(u\left(X^m\right)\right)\right]\geq\frac{\alpha'}{2}+\mathcal{O}(\varepsilon)$$

which is a contradiction to the optimality of the sequence $(X^n)_{n\in\mathbb{N}}\subset\mathcal{C}$. $\square$

Under stronger assumptions we can show that the maximizer itself can be represented as the terminal value of the process (2.2) with some strategy $\widehat{\theta} \in \widetilde{\Theta}$. But, in general $\widehat{\theta} \notin \Theta$.

Proposition 2.13. *Let the assumptions of Proposition 2.12 be satisfied. $S(0,\cdot) = S(0,0) + N + B$ denotes the canonical decomposition of the special semimartingale $S(0,\cdot)$, i.e. N is a local martingale and B a predictable process of finite variation. We assume that there exists a $K \in \mathbb{R}_+$ such that for all $\theta \in \Theta$*

$$E\left(\int_0^T \left(\int_0^{\theta_t} H_t^x\, dx\right)^2 d[N,N]_t\right) + E\left(\int_0^T \left(\int_0^{\theta_t} H_t^x\, dx\right)^2 dB_t\right)$$
$$\leq K\left[1 + E\left|\int_0^T \left(\int_0^{\theta_t} H_t^x a_t^x\, dx\right) dA_t\right|\right], \tag{2.19}$$

if the rhs is finite. Then there exists a strategy $\widehat{\theta} \in \widetilde{\Theta}$ with $V_T(\widehat{\theta}) \geq X$, P-a.s. for X from Lemma 2.11.

It suggest itself to consider $\widehat{\theta}$ as an optimizer of the utility maximization problem.

Remark 2.14. *(2.19) means that the drift of $S(\vartheta,\cdot)$ has to vary at least linearly with ϑ. It is e.g. satisfied in the model $S(\vartheta,t) = S(0,t) + \vartheta t$ with appropriate integrability conditions.*

Proof. Let $(\theta^n)_{n\in\mathbb{N}} \subset \Theta$ be a maximizing sequence as in Lemma 2.11, i.e. there exists a $\theta \in \widetilde{\Theta}$ with $\theta^n \to \theta$ $(P \otimes A)$-a.s. With Fatou we obtain that

$$\limsup_{n\to\infty} \int_0^T \left(\int_0^{\theta_t^n} H_t^x a_t^x\, dx\right) dA_t \leq \int_0^T \left(\int_0^{\theta_t} H_t^x a_t^x\, dx\right) dA_t, \quad P-\text{a.s.} \tag{2.20}$$

as the integrands on the lhs are nonpositive. Especially, $\int_0^\theta H^x a^x\, dx \in L(A)$. Plugging in θ^n for θ in (2.19), the expectations on the rhs are uniformly bounded due to (2.12). Thus, by passing to convex combinations we can assume that the sequence $\int_0^\cdot \left(\int_0^{\theta_t} H_t^x\, dx\right) dN_t$ converges to some semimartingale Y in $\mathcal{H}^2$ and therewith in the semimartingale topology, cf. e.g. [19], Chapter V. By Lemma 2.15 we have $Y = \int_0^\cdot K_t\, dS(0,t)$ where $K = \int_0^{\theta_\cdot} H_\cdot^x\, dx$ with θ from (2.20). Together with the uniformly integrability w.r.t. dB_t due to (2.19) this yields

$$\int_0^T \left(\int_0^{\theta_t^n} H_t^x\, dx\right) dS(0,t) \to \int_0^T \left(\int_0^{\theta_t} H_t^x\, dx\right) dS(0,t), \text{ in probability} \tag{2.21}$$

for $n \to \infty$. Putting (2.20) and (2.21) together yields the assertion. □

Lemma 2.15. *Let X be a semimartingale with differential characteristics (b, c, F, A), $(H^n)_{n\in\mathbb{N}} \subset L(X)$ with $H^n \to H$, $(P \otimes A)$-a.s. for some H and*

$$H^n \bullet X \to Y, \quad n \to \infty, \quad \textit{in the semimartingale topology}, \tag{2.22}$$

for some semimartingale Y. Then $H \in L(X)$ and $H \bullet X = Y$ up to evanescence.

Proof. The following identities for stochastic processes are understood up to evanescence. By definition of the stochastic integral it is sufficient to show that

$$(1_{\{|H|\leq m\}} H) \bullet X = 1_{\{|H|\leq m\}} \bullet Y, \quad \forall m \in \mathbb{N}. \tag{2.23}$$

For $m \in \mathbb{N}$, $\varepsilon \in (0, 1)$ define a *continuous* "truncation" function $h_{m,\varepsilon} : \mathbb{R} \to \mathbb{R}$ by

$$h_{m,\varepsilon}(x) := \begin{cases} 1 & : |x| \leq m \\ 1 - \frac{|x|-m}{\varepsilon} & : m < |x| \leq m + \varepsilon \\ 0 & : m + \varepsilon < |x| \end{cases}$$

By (2.22) we have that

$$h_{m,\varepsilon}(H^n) \bullet (H^n \bullet X) - h_{m,\varepsilon}(H^n) \bullet Y$$

converges to zero uniformly in probability. Furthermore, by $H^n \to H$, $(P \otimes A)$-a.s. and the continuity of the stochastic integral (cf. e.g. [9] Theorem I.4.31), we have that $(h_{m,\varepsilon}(H^n)H^n) \bullet X$ converges to $(h_{m,\varepsilon}(H)H) \bullet X$ and $h_{m,\varepsilon}(H^n) \bullet Y$ to $h_{m,\varepsilon}(H) \bullet Y$, uniformly in probability. Putting this together we obtain that $(h_{m,\varepsilon}(H)H) \bullet X = h_{m,\varepsilon}(H) \bullet Y$. Again by the continuity of the stochastic integral, letting $\varepsilon \to 0$ yields (2.23). □

Theorem 2.16. *Assume that the requirements of Proposition 2.13 are satisfied. Let θ be an optimizer with terminal wealth X in the sense of Proposition 2.13 and assume that*

$$\limsup_{x\to-\infty} \frac{u(x-a)}{u(x)} < \infty, \quad \forall a \in \mathbb{R}_+. \tag{2.24}$$

Then, $E(u'(X)) < \infty$. Define the measure Q by

$$\frac{dQ}{dP} = \frac{u'(X)}{E(u'(X))}. \tag{2.25}$$

For every $\vartheta \in \mathbb{R}$ the canonical Q-drift rate of $S(\vartheta,\cdot)$ vanishes on the predictable set $\{\theta=\vartheta\}$ and the process

$$t \mapsto M_t := \int_0^t S(\theta_s,\, ds) := \int_0^t H_s^{\theta_s}\, dS(0,s) + \int_0^t H_s^{\theta_s} a_s^{\theta_s}\, dA_s, \quad t \in [0,T]. \tag{2.26}$$

is a $Q-\sigma$-local martingale.

(2.24) is of course satisfied for the exponential utility function.

Proof. Step 1: Let $\theta \in \widetilde{\Theta}$ be an optimal solution in the sense of Proposition 2.13. There exists a sequence $(D_n)_{n\in\mathbb{N}} \subset \mathcal{P}$ with $D_n \uparrow \Omega \times [0,T]$, $(P \otimes A)$-a.s. s.t. on D_n, θ, $H^{\theta-2}$, $H^{\theta+2}$ $a^{\theta-2}$, $a^{\theta+2}$, $S(0,\cdot)$, and A are bounded by n (note that $\vartheta \mapsto a^{\vartheta}$ is monotone and the processes $S(0,\cdot)$ and A are even locally bounded). This implies that $M^n := 1_{D_n} \bullet M$ is bounded and the expectations below are well-defined and finite. Assuming that M^n is not a Q-martingale, consider first the case that

$$E_Q\left(1_A\left(M^n_{t_2} - M^n_{t_1}\right)\right) > 0 \tag{2.27}$$

for some $0 \le t_1 < t_2 \le T$, $A \in \mathcal{F}_{t_1}$. For $\varepsilon \in (0,1]$ define $\psi^{\varepsilon} := \varepsilon 1_{(A\times(t_1,t_2])\cap D_n}$. We have $\int_0^{\theta+\psi^{\varepsilon}} H^x\, dx \in L(S(0,\cdot))$ and $\int_0^{\theta+\psi^{\varepsilon}} H^x a^x\, dx \in L(A)$. Let $\tau^{\varepsilon} := \inf\{t \in [0,T] \mid |V_t(\theta+\psi^{\varepsilon}) - V_t(\theta)| \ge 1\} \wedge T$. By the uniformly Lipschitz continuity of $x \mapsto H^x$ and the continuity of the stochastic integral (in the sense of [9], I.4.31) it follows that $P(\tau^{\varepsilon} = T) \to 1$ for $\varepsilon \to 0$. By $dS(0,t) = \frac{1}{H_t^{\theta_t}} dM_t - a_t^{\theta_t} dA_t$ we arrive at

$$\begin{aligned}
&\tfrac{1}{\varepsilon} E_Q\left[V_{\tau^{\varepsilon}}(\theta+\psi^{\varepsilon}) - V_{\tau^{\varepsilon}}(\theta)\right] \\
&= \tfrac{1}{\varepsilon} E_Q\left[1_A \int_{t_1\wedge\tau^{\varepsilon}}^{t_2\wedge\tau^{\varepsilon}} 1_{D_n} \int_{\theta_s}^{\theta_s+\varepsilon} S(x,ds)\, dx\right] \\
&= E_Q\left[1_A \int_{t_1\wedge\tau^{\varepsilon}}^{t_2\wedge\tau^{\varepsilon}} 1_{D_n} \frac{\int_{\theta_s}^{\theta_s+\varepsilon} H_s^x\, dx}{\varepsilon H_s^{\theta_s}}\, dM_s\right] \\
&+ E_Q\left[1_A \int_{t_1\wedge\tau^{\varepsilon}}^{t_2\wedge\tau^{\varepsilon}} 1_{D_n} \tfrac{1}{\varepsilon}\left(\int_{\theta_s}^{\theta_s+\varepsilon} H_s^x a_s^x\, dx - a_s^{\theta_s}\int_{\theta_s}^{\theta_s+\varepsilon} H_s^x\, dx\right) dA_s\right] =: I + II.
\end{aligned} \tag{2.28}$$

Again by continuity, I is positive for $\varepsilon > 0$ small enough. The integrand in II is nonpositive, nondecreasing in ε, and tends pointwise to zero for $\varepsilon \to 0$. By construction of D_n the integral is finite for $\varepsilon = 1$. Therefore dominated convergence yields that under assumption (2.27)

$$E_Q(V_T(\theta+\widetilde{\psi}^{\varepsilon}) - V_T(\theta)) > 0 \quad \text{for } \varepsilon > 0 \text{ small enough, where } \widetilde{\psi}^{\varepsilon} := \psi^{\varepsilon} 1_{[\![0,\tau^{\varepsilon}]\!]}. \tag{2.29}$$

Step 2: Remember that in general $\theta \notin \Theta$, i.e. θ does not necessarily generate a liquidation value process which is bounded from below. Thus it is not clear that $\theta + \widetilde{\psi}^\varepsilon$ is an admissible strategy in any sense (note that trading gains are not linear in the strategy). Let us show that $\left(X + V_T(\theta + \widetilde{\psi}^\varepsilon) - V_T(\theta)\right) \in \mathcal{C}$. By assumption there exists a sequence $(\theta^m)_{m\in\mathbb{N}}$ with $\theta^m \in \Theta_m$ s.t. $\theta^m \to \theta$, $(P\otimes A)$-a.s. and

$$u(V_T(\theta^m) - Y_m) \to u(X) \text{ in } L^1(\Omega, \mathcal{F}, P) \text{ for some sequence } (Y_m)_{m\in\mathbb{N}} \subset L^0_+(\Omega, \mathcal{F}, P) \tag{2.30}$$

For $\widetilde{\psi}^\varepsilon$, $\varepsilon \in (0,1]$, consider the approximating sequence

$$\varphi^{\varepsilon,m} := \varepsilon 1_{(A\times(t_1,t_2])\cap D_n \cap [\![0,\tau^\varepsilon]\!]\cap\{\theta^m\in[\theta-1,\theta+1]\}}.$$

We have $\varphi^{\varepsilon,m} \to \widetilde{\psi}^\varepsilon$ $(P\otimes A)$-a.s. for $m\to\infty$ (and $|\varphi^{\varepsilon,m}| \le 1$, $\forall m\in\mathbb{N}$). Thus by the continuity of the stochastic integral we obtain that the sequence $V(\theta^m + \varphi^{\varepsilon,m}) - V(\theta^m)$ converges to $V(\theta+\widetilde{\psi}^\varepsilon) - V(\theta)$ *uniformly in probability*, see [9], I.4.31 and the definition of V in (2.2). Consequently, the sequence $R_m := \inf\{t\in[0,T] \mid |V_t(\theta^m + \varphi^{\varepsilon,m}) - V_t(\theta^m)| \ge |V_t(\theta+\widetilde{\psi}^\varepsilon) - V_t(\theta)| + 1\} \wedge T$ is localizing. As the limiting process $V(\theta+\widetilde{\psi}^\varepsilon) - V(\theta)$ is bounded, $|V_{\cdot\wedge R_m}(\theta^m + \varphi^{\varepsilon,m}) - V_{\cdot\wedge R_m}(\theta^m)|$ is bounded by some $a\in\mathbb{R}_+$ which depends on n but not on m (the jumps of $V(\theta^m+\varphi^{\varepsilon,m}) - V(\theta^m)$ are bounded by construction). We have $\theta^m + 1_{[\![0,R_m]\!]}\varphi^{\varepsilon,m} \in \Theta_{m+a}$. By (2.24), (2.30) implies that $u(V_T(\theta^m + \varphi^{\varepsilon,m}1_{[\![0,R_m]\!]}) - Y_m)$ converges to $u(X + V_T(\theta+\widetilde{\psi}^\varepsilon) - V_T(\theta))$ in $L^1(\Omega,\mathcal{F},P)$ for $m\to\infty$.

Step 3: Obviously (2.29) is equivalent to

$$E\left[u'(X)(V_T(\theta+\widetilde{\psi}^\varepsilon) - V_T(\theta))\right] > 0. \tag{2.31}$$

As argued at the beginning of the proof of Proposition 2.12, $\mathcal{C}$ is convex. This allows us to apply a marginal utility criterion for optimality, see Lemma 2.2 in [12], and obtain that (2.31) is a contradiction to the optimality of θ ($\mathcal{C}$ does not fit in the requirements of Lemma 2.2, but for the proof it is sufficient that $\mathcal{C}$ is convex). The case that $E_Q\left(1_A\left(M^n_{t_2} - M^n_{t_1}\right)\right) < 0$ is treated analogously by substituting $-\varepsilon$ for ε. □

3 Microeconomic motivation

In this section special choices of S within the model of Bank and Baum [1] are motivated. In a first step we consider only buy-and-hold strategies of the large investor and derive the resulting price processes from "first principles". Let us assume that there is an inelastic supply of $M \in \mathbb{R}_+$ shares. In addition there are finitely many *reference traders* maximizing expected utility from terminal wealth. We assume that the reference traders have no random endowment and their risk aversions are constant, i.e. they possess exponential utility functions. Thus they can be accumulated to a *representative agent* with utility function $u(x) = -\exp(-\alpha x)$ for some $\alpha > 0^\dagger$. The firm pays a one-time dividend of $H \in L^0(\Omega, \mathcal{F}_T, P)$ units per share at $T^\ddagger$. To ensure existence of all price processes we assume that

$$E(\exp(aH)) < \infty \text{ for all } a \in \mathbb{R}. \tag{3.1}$$

Then, S is uniquely determined by the following

Proposition 3.1. *For any $\vartheta \in \mathbb{R}$ there is up to evanescence a unique semimartingale $S(\vartheta, \cdot)$ satisfying the following conditions*

(1) $S(\vartheta, T) = H, \quad P$-a.s.

(2) Given the price process $S(\vartheta, \cdot)$, $\varphi = M - \vartheta$ is an optimal strategy for the representative agent.

$S(\vartheta, \cdot)$ is a càdlàg version of

$$S(\vartheta, t) = E_{Q^\vartheta}(H|\mathcal{F}_t), \quad t \in [0, T], \ \textit{where}\ \frac{dQ^\vartheta}{dP} = \frac{\exp(-\alpha(M - \vartheta)H)}{E(\exp(-\alpha(M - \vartheta)H))}. \tag{3.2}$$

Remark 3.2. *(2) is the (dynamic) market clearing condition if the large investor holds constantly ϑ shares and the representative agent acts as a price taker with rational expectations.*

Remark 3.3. *If we replace (3.1) by the condition that $H \geq 0$, price processes may be defined only for $\vartheta < M$ and we possibly have to restrict ourselves to $(-\infty, M)$-valued trading strategies for the large investor.*

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†Let $\alpha_i > 0$ be the risk aversion of the ith agent. Then, α is given by $\frac{1}{\alpha} = \sum_i \frac{1}{\alpha_i}$.

‡As agents are maximizing expected utility from terminal wealth (and there is no intermediate consumption) a whole dividend process $D = (D_t)_{t \in [0,T]}$ would enter in the solution of the problem only by its terminal value D_T. For an infinite time horizon the effects would be morally the same.

Remark 3.4. *The equilibrium approach behind Proposition 3.1 corresponds to Lucas' tree economy, see [17], but it differs from the standard equilibrium approach in a pure exchange economy. Roughly speaking in the pure exchange equilibrium approach agents are from the first endowed with a random income process. Assets act as an instrument to exchange claims between agents but they are not linked to the dividend payments of some firm (as in our example). Consequently, a large investor running short the supply of shares to* $M - \vartheta$ *(by holding* ϑ *assets constantly through time) would not change the yield return of the asset. Namely, the outcoming price process would change only by a factor. However, the neutrality of the price impact breaks down if the large investor takes a time-depending strategy, see the article of Frey and Stremme [7] on feedback effects from dynamic hedging. In contrast to [7] in our approach the asset is linked to the dividend payment* H *by some firm. In Proposition 3.1 we obtain different rates of return for different* $\vartheta \in \mathbb{R}$.

At T *the asset is totally liquid, i.e. the price does not depend on* ϑ *anymore. As there exists by assumption a perfectly liquid market for riskless bonds, illiquidity disappears with randomness.*

3.1 Dynamic strategies of the large investor

Given the family of stochastic processes $S = (S(\vartheta, \cdot))_{\vartheta \in \mathbb{R}_+}$ from (3.2) and a dynamic strategy θ of the large investor it suggest itself to consider $t \mapsto S(\theta_t, t)$ as the resulting market price process. There arise the questions

(1) *Is* $M - \theta_t$ *the optimal demand of the representative agent given the price process* $t \mapsto S(\theta_t, t)$ *? (rational expectations for small investors)*

(2) *Is there already enough structure in the model to obtain a result like Theorem 2.16 ?*

(1) does not hold – not even in very pleasing models. Namely, given that $t \mapsto S(\theta_t, t)$ is the stochastic price process where e.g. $\theta_t = t$ (i.e. the large investor builds up a position in the asset), the strategy $M - t$ is probably not optimal. In fact, as $t \mapsto S(\theta_t, t)$ tends to increase it would be better to purchase assets slightly before the large investor does so to benefit from rising prices.

However, as the model is based on asymmetric information property (1) may anyway be deemed as not very desirable. Therefore, we formulate a different condition reflecting rational expectations .

Definition 3.5. *The model in Proposition 3.1 satisfies the rational expectations condition for small traders, if for any predictable process* $(\theta_t)_{t\in[0,T]}$*, the strategy* $M-\theta$ *is optimal for the representative small investor given the price process*

$$t \mapsto \int_0^t S(\theta_s, ds), \;\; t\in[0,T]. \tag{3.3}$$

If this only holds for deterministic processes $(\theta_t)_{t\in[0,T]}$ *we say that the model satisfies the weak rational expectations condition for small traders.*

(3.3) can be interpreted as the sum of the price movements *not* caused by the large investor. By reason of Example 3.6 and the following remarks we make throughout Sections 2 and 4 the restrictive Assumption 2.1. In that case we do not need the nonlinear stochastic integral in its full generality, but only in the form (2.26) resp. (2.2). Therefore we do not introduce the general integral formally in continuous time, but refer to [16]. In discrete time (like in Example 3.6) the process (3.3) is given by $\sum_{i=1}^{\cdot} (S(\theta_i,i) - S(\theta_i,i-1))$ with the standard notation for liquid markets in discrete time that θ_i is the number of shares hold between $i-1$ and i.

In general there are even no weak rational expectations in the sense of Definition 3.5. We want to demonstrate this by the following two-period example.

Example 3.6. *The dividend payment at* $T=2$ *is given by*

$$H = X_1 + K(X_1+1)X_2 \tag{3.4}$$

where the random variables X_1*,* X_2 *are i.i.d. with* $P(X_i=-1)=P(X_i=1)=1/2$*,* $i=1,2$*, and* $K\in\mathbb{R}_+$ *is large.* $\mathcal{F}_1=\sigma(X_1)$ *and* $\mathcal{F}_2=\sigma(X_1,X_2)$[§]*. Recall that by construction of* S *the representative agent faced with the price process* $S(\vartheta,\cdot)$ *behaves optimally by holding constantly* $M-\vartheta$ *shares in his portfolio. Let us consider the following auxiliary problem which may appear at first glance rather artificial but which turns out to be of some importance for the large investor's portfolio optimization problem.*

[§] (3.4) can be regarded as a very simple (complete) stochastic volatility model.

Auxiliary optimization problem. *Assume that trader A can invest both in the process $S(\vartheta_1,\cdot)$ and in the process $S(\vartheta_2,\cdot)$ with $\vartheta_1 < \vartheta_2$ in such a way that the numbers of shares are nonnegative and add up to 1 (in addition there is a riskless bond with price process 1). This means the frictionless market is given by the risky processes $(S(\vartheta_1,\cdot), S(\vartheta_2,\cdot))$ where the set of admissible strategies consists of all predictable processes φ taking values in the set $\{(\lambda, 1-\lambda) \in \mathbb{R}^2 \mid \lambda \in [0,1]\}$.*

If only buy-and-hold strategies were allowed, then it would certainly be optimal to invest everything in the cheaper asset $S(\vartheta_1,\cdot)$ as the achievable trading gains are given by

$$\begin{aligned}
&\lambda[S(\vartheta_1,2) - S(\vartheta_1,0)] + (1-\lambda)[S(\vartheta_2,2) - S(\vartheta_2,0)] \\
&\quad = \lambda[H - S(\vartheta_1,0)] + (1-\lambda)[H - S(\vartheta_2,0)] = H + \lambda[S(\vartheta_2,0) - S(\vartheta_1,0)] - S(\vartheta_2,0)
\end{aligned}$$

which takes its pointwise maximum at $\lambda = 1$ as $S(\vartheta_1,0) \leq S(\vartheta_2,0)$. One could expect that the strategy $\varphi = (1,0)$ is also dynamically optimal. But, this is not true in general: take $\vartheta_1 < \vartheta_2 = M$ and $u^A(x) = -\exp(-\alpha(M-\vartheta_1)x)$ for A's utility function[¶]*. We have $S(M,t) = E(H|\mathcal{F}_t)$, i.e. $S(M,1) = X_1$ and $S(M,0) = 0$. As $S(\vartheta_1,1) < S(M,1)$ and $S(\vartheta_1,2) = S(M,2)$ for the investment in the second period the optimum has to be $\varphi_2 = (1,0)$. However, it can easily be seen that $\varphi_1 = (1,0)$ cannot be optimal. Namely, starting from $\varphi = (1,0)$ there is some incentive to invest a small amount in $S(M,1) - S(M,0) = X_1$ and reduce at the same time the investment in $S(\vartheta_1,1) - S(\vartheta_1,0)$. Namely, by construction of $S(\vartheta_1,\cdot)$ we have the marginal utility criterion*

$$\begin{aligned}
&E\left[(u^A)'(H - S(\vartheta_1,0))\,(S(M,1) - S(M,0) - S(\vartheta_1,1) + S(\vartheta_1,0))\right] \\
&= \widetilde{K}E\left[\exp(-\alpha(M-\vartheta_1)H)\,(S(M,1) - S(M,0) - S(\vartheta_1,1) + S(\vartheta_1,0))\right] \\
&= E\left[\exp(-\alpha(M-\vartheta_1)H)X_1\right] > 0,
\end{aligned} \tag{3.5}$$

where $\widetilde{K}$ is some positive constant and the inequality holds for K large enough. (3.5) tells us that even though the process $S(\vartheta_1,\cdot)$ has a higher P-drift than $S(M,\cdot)$ it can be worthwhile to exchange gains from $S(\vartheta_1,\cdot)$ for gains from $S(M,\cdot)$. The reason for

[¶]This means that the unrestricted optimum in the market consisting only of one risky asset with price process $S(\vartheta_1,\cdot)$ is the constant strategy 1.

this is that the martingale parts of the processes are polar in the first period and the trading gain $S(M,1)-S(M,0)$ can be used as a hedging instrument against the random endowment $S(\vartheta_1,2)-S(\vartheta_1,1)$.

In some sense the large trader can invest in special linear combinations of processes $S(\vartheta,\cdot)$, namely in $\int_0^t\int_0^{\theta_s} S(\vartheta,ds)\,d\vartheta$, see (2.2). If the martingale parts of different $S(\vartheta,\cdot)$ are not connected to each other the convexity Assumption 2.1 is violated and Theorem 2.16 cannot hold in general. If the martingale parts compensate each other, there can even arise an arbitrage opportunity for the large investor.

Remark 3.7. *Let us show that not even weak rational expectations in the sense of Definition 3.5 hold for (3.4). Take the large trader's strategy $(\theta_t)_{t=1,2}=(M,\vartheta_1)$. We have to show that the corresponding strategy of the representative agent, $(0,M-\vartheta_1)$, is not optimal given the trading space spanned by the gains $S(M,1)-S(M,0)$ and $S(\vartheta_1,2)-S(\vartheta_1,1)$ which is the process (3.3). The reason for this is the same as before: although $E(S(M,1)-S(M,0))=0$ it is worthwile to invest in the asset in the first period as a hedging instrument for the trading gains $S(\vartheta_1,2)-S(\vartheta_1,1)$ from the second period. As $u(x)=-\exp(-\alpha x)$ we obtain the marginal utility criterion*

$$
\begin{aligned}
&E[u'((M-\vartheta_1)(S(\vartheta_1,2)-S(\vartheta_1,1)))1_{\{X_1=1\}}]\\
&=-\alpha E[u((M-\vartheta_1)(S(\vartheta_1,2)-S(\vartheta_1,1)))1_{\{X_1=1\}}]\\
&<-\alpha E[u(0)1_{\{X_1=1\}}]\\
&=-\alpha E[u((M-\vartheta_1)(S(\vartheta_1,2)-S(\vartheta_1,1)))1_{\{X_1=-1\}}]\\
&=E[u'((M-\vartheta_1)(S(\vartheta_1,2)-S(\vartheta_1,1)))1_{\{X_1=-1\}}]
\end{aligned}
\tag{3.6}
$$

where the inequality is by construction of $S(\vartheta_1,\cdot)$ and the second equality holds as $S(\vartheta_1,2)=S(\vartheta_1,1)$ in the event $\{X_1=-1\}$ and $P(X_1=-1)=P(X_1=1)$. It follows that the representative agent can increase his expected utility by shorting $S(M,1)-S(M,0)=X_1$. Put differently, the representative agent's strategy fails to be optimal if he recognizes that the large trader takes a dynamic strategy (even if it is deterministic) rather than assuming at any time that the strategy remains constant up to maturity.

The effects described above – based on polar martingale parts – disappear if the dividend payment is generated by a process L with independent increments. In this case $dS(\vartheta,t)$ is determined by dL_t only.

Assumption 3.8. *The one-time dividend-payment is given by $H = L_T$ (satisfying (3.1)), where $L = (L_t)_{t\in[0,T]}$ is a semimartingale with independent (not necessarily stationary) increments w.r.t. $((\mathcal{F}_t)_{t\in[0,T]}, P)$. Let (B^L, C^L, ν^L) be the (deterministic) semimartingale characteristics relative to the truncation function $h(x) = x$ (this choice is possible as by the assumptions above L must be a special semimartingale). By Proposition II.2.9 in [9] there exists some $A^L \in \mathcal{A}^+_{\mathrm{loc}}$ such that the characteristics can be written in differential form*

$$B^L = b^L \bullet A^L, \quad C^L = c^L \bullet A^L, \nu^L = A^L \otimes F^L,$$

where b^L and c^L are $\mathbb{R}$-valued resp. $\mathbb{R}_+$-valued predictable processes and F^L is a transition kernel from $(\Omega \times [0,T], \mathcal{P})$ into $(\mathbb{R}, \mathcal{B})$.

Proposition 3.9. *Under Assumption 3.8 the price processes from (3.2) reduce to*

$$S(\vartheta, t) = L_t + \lambda^\vartheta_T - \lambda^\vartheta_t, \quad t \in [0,T] \tag{3.7}$$

where

$$\begin{aligned}
\lambda^\vartheta_t &:= \int_0^t (-a^\vartheta_s)\, dA^L_s \\
&:= \int_0^t \big(b^L_s - \alpha(M-\vartheta)c^L_s \\
&+ \int_{\mathbb{R}} \left(\frac{x \exp(-\alpha(M-\vartheta)x)}{1 + \int_y (\exp(-\alpha(M-\vartheta)y) - 1)\, \nu^L(\{s\} \times dy)} - x \right) F^L_s(dx) \Big)\, dA^L_s. \qquad (3.8)
\end{aligned}$$

The drift rate a^ϑ is $P \otimes A$-a.s. decreasing in $\vartheta \in \mathbb{R}$. For any deterministic *process $\theta = (\theta_t)_{t\in[0,T]}$ s.t. a^θ is A^L-integrable the strategy $\varphi := M - \theta$ maximizes the expected utility of the representative agent given the risky price process (3.3), i.e. the representative agent has weak rational expectations in the sense of Definition 3.5.*

As the increments of L are independent and the representative agent has an exponential utility function, $dS(\vartheta,t)$ can be derived from dL_t and there is no hedging component.

Therefore the processes (3.7) only differ in their drifts and Assumption 2.1 is satisfied with $H = 1$.

Proof. (3.7) follows immediately from Esscher's change of measure (see [14]). As $S(\vartheta, \cdot)$ has independent increments, the process in (3.3) given by $L_t - L_0 + \int_0^t a_s^{\theta_s}\, dA_s^L$ also possesses independent increments if θ is deterministic. In this case the portfolio optimization problem for the representative agent becomes myopic which implies by construction the optimality of the strategy $M - \theta$. $\square$

Proposition 3.10. *If the large investor's utility function is given by $u(x) = -\exp(-\widetilde{\alpha}x)$, $\widetilde{\alpha} > 0$, the constant strategy $\theta = \frac{M\alpha}{\alpha+\widetilde{\alpha}}$ is optimal.*

4 The general case

In this section we introduce a model where the price impact caused by a transaction of a large trader has both a short-term and a permanent component. The main idea is similar to Rogers and Singh [21] where, however, the price impact is solely short-term. Namely, we assume that during some time interval $(t_1, t_2]$ only a fraction of small traders is present on the market – in the sense that they are trying to update their positions in the asset by emitting buy or sell orders. Then, in a future time interval these small traders are not totally replaced by other small traders (as in [21]), but *some* of them return which leads to a feedback effect of the asset price process and the dynamic trading strategy of the large investor.

Formally we model this activity by a subadditiv set function on $\mathcal{B}([0,T])$.

Definition 4.1. *Let $\mu : \mathcal{B}([0,T]) \to [0,1]$ be a set function with $\mu(\emptyset) = 0$, $\mu([0,T]) \leq 1$ and*

$$\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2) - \mu(A_1)\mu(A_2), \quad \forall A_1, A_2 \in \mathcal{B}([0,T]) \text{ with } A_1 \cap A_2 = \emptyset. \tag{4.1}$$

Assume that μ is continuous, i.e. $A_n \downarrow \emptyset \implies \mu(A_n) \downarrow 0$, and there is finite activity in the sense that

$$\sup_{n\in\mathbb{N},\ A_1,\dots,A_n\in\mathcal{B}([0,T]),\ A_k \text{ disjoint}} \sum_{k=1}^n \mu(A_k) < \infty. \tag{4.2}$$

Let $\widetilde{\mu}$ be the smallest measure dominating μ. By (4.2) we have that $\widetilde{\mu}([0,T]) < \infty$ and

$$\mu(A) = 1 - \prod_{t \in A}(1 - \widetilde{\mu}(\{t\}) \exp\left(-\widetilde{\mu}(A) + \sum_{t \in A} \widetilde{\mu}(\{t\})\right). \tag{4.3}$$

Remark 4.2. *μ is monotone and subadditiv. $\mu(A)$ can be interpreted as the fraction of small traders which are active "at least once" during $A \in \mathcal{B}([0,T])$. The universe of small traders is normalized by 1, but we only assume that $\mu([0,T]) \leq 1$, i.e. there can be small traders (shareholders) which never appear at the market during the interval $[0,T]$.*

Condition (4.1) can be interpreted as memoryless. *For two disjoint sets A_1 and A_2 the fraction of active traders in $A_1 \cup A_2$ is given by $\mu(A_1) + [1-\mu(A_1)]\mu(A_2)$ – i.e. the fraction of traders being active in both sets is $\mu(A_1)\mu(A_2)$. For the modeling of dynamic trading it is crucial whether the large investor is all along faced with* new *small traders constituting the market (as it is the idea behind [21]) or whether he meets some of the traders again. In the latter case the positions of the "current" small traders (and therewith their current excess demand for risky assets) are affected by the large trader's past transactions. Therefore a quite complex dependency on former trading arises. In principle the whole history of each small trader has to be modeled which is of course far away from tractability. The memoryless condition (4.1) simplifies things a lot. It is a middle course: some small traders have already traded with the large investor in the past and some not.*

Example 4.3. *Take $\mu(A) = 1 - \exp(-c\lambda(A))$, $\forall A \in \mathcal{B}[0,T]$, $c \in \mathbb{R}_+$ where λ is the Lebesgue-measure. This implies $\widetilde{\mu}(A) = c\lambda(A)$. In this example only rebalancings which are continuous w.r.t. the Lebesgue-measure can be realized by the large trader, i.e. $|d\theta_t| \ll dt$.*

Another example is $\mu'(A) = 1 - \prod_{t_i \in A}(1-\alpha_i)$ for some $\alpha_i \in (0,1]$, $0 \leq t_1 < t_2 < \ldots \leq T$. We have $\widetilde{\mu}' = \sum_{T_i \in A} \alpha_i$.

μ corresponds to "continuous trading" and μ' to auctions. Of course combinations of both examples are also covered.

Based on this activity function μ we will build-up a continuous-time model with dynamic trading of the large investor.

As usual in mathematical finance the starting point of dynamic trading should be so-called *simple strategies* allowing for finitely many regroupings of investments. But, typically a simple strategy in the usual sense ("buy one stock at time t_1 and sell it at time t_2") is not feasible as $\mu(\{t_1\})$ and $\mu(\{t_2\})$ can be zero. This means that it is impossible to purchase or sell a finite amount of assets at a single point in time. To circumvent this problem we consider in a first step a fining sequence of grids $(\sigma_n)_{n\in\mathbb{N}}$ with $\sigma_n = (T_0^n, \ldots, T_{k_n}^n)$, where $k_n \in \mathbb{N}$ and $0 = T_0^n < \ldots < T_{k_n}^n = T$ are stopping times. Trading (auctions) only takes place at the grid points and all orders arising during the period $(T_{k-1}^n, T_k^n]$ are executed at the endpoint T_k^n-. We start from the very beginning with a (given) time-continuous strategy $\theta = (\theta_t)_{t\in[0,T]}$ of the large investor, but the portfolio is only rebalanced at each T_k^n-, $k = 1, \ldots, k_n$, by buying (resp. selling) $\theta_{T_k^n} - \theta_{T_{k-1}^n}$ shares. The *market clearing* condition at time T_k^n- becomes

$$\underbrace{\theta_{T_k^n} - \theta_{T_{k-1}^n}}_{\text{demand of large trader}} + \underbrace{\mu((T_{k-1}^n, T_k^n]) \left[M - S^{-1}(\cdot, T_k^n-)(p) - (M - \theta_{T_{k-1}^n}) \right]}_{\text{demand of small traders}} \stackrel{!}{=} 0, \quad (4.4)$$

where M is the total number of shares and p is the price for one share which is to determine by (4.4). $S^{-1}(\cdot, T_k^n-)(p)$ denotes the inverse of S w.r.t. the first component evaluated at p. Namely, by construction of μ and $(S(\vartheta, \cdot))_{\vartheta\in\mathbb{R}}$ (cf. Sections 2 and 3), and some "independence assumption" of the risk aversion of a small agent and his presence on the market the random variable $\mu((T_{k-1}^n, T_k^n])(M - S^{-1}(\cdot, T_k^n-)(p))$ is considered to be the aggregated optimal investment of the active small traders given the price p and $\mu((T_{k-1}^n, T_k^n])(M - \theta_{T_{k-1}^n})$ is the total number of shares in their portfolios at T_{k-1}^n. Solving (4.4) for p yields

$$p = S\left(\theta_{T_{k-1}^n} + \frac{\theta_{T_k^n} - \theta_{T_{k-1}^n}}{\mu((T_{k-1}^n, T_k^n])}, T_k^n-\right) \quad (4.5)$$

and the discrete-time self-financing condition becomes

$$\beta_{T_k^n}^n - \beta_{T_{k-1}^n}^n = -\left(\theta_{T_k^n} - \theta_{T_{k-1}^n}\right) S\left(\theta_{T_{k-1}^n} + \frac{\theta_{T_k^n} - \theta_{T_{k-1}^n}}{\mu((T_{k-1}^n, T_k^n])}, T_k^n-\right), \quad k = 1, \ldots, k_n. \quad (4.6)$$

where $\beta_{T_k^n}^n$ denotes the number of (liquid) riskless bonds (with price 1) in the portfolio at time T_k^n. For the use of the left limit cf. Remark 4.5. Let β^n be the càdlàg extension of the process in (4.6). As θ is given, the self-financing dynamics of the bank account characterizes the dynamic trading in the asset. Assume that θ is predictable and absolutely continuous w.r.t. $\widetilde{\mu}$. Then, there exists a predictable density which we denote by y, i.e.

$$\theta_t = \int_0^t y_s \, d\widetilde{\mu}_s, \quad t \in [0,T]. \tag{4.7}$$

We will show convergence of β^n in model (4.6) to the corresponding process in the following continuous-time model.

Definition 4.4. *A strategy $(\beta_t, \theta_t)_{t\geq 0}$ is self-financing with initial capital β_0 iff*

$$\beta_t = \beta_0 - \int_0^t y_s S(\theta_{s-} + y_s, s-) \, d\widetilde{\mu}_s, \quad t \in [0,T]. \tag{4.8}$$

Remark 4.5. *(4.4) and (4.8) are in line with the standard theory for liquid markets where θ_t can be interpreted as the number of shares the investor holds between $t-$ and t (i.e. the investment in the jump ΔS_t). This means $y_t \widetilde{\mu}(\{t\})$ is the number of shares purchased at $t-$ at the price $S(\theta_{t-} + y_t, t-)$.*

Remark 4.6. *One may look at the model of [1], see also Section 2 of the present paper, as a special case or at least as a limiting case of the model introduced in this section (when the fraction of "active" traders tends to 1). For this, let $\mu(A) = 1 \ \forall A \neq \emptyset$. Then, the price at T_k^n- is merely a function of θ_k^n (rather than of the pair $(\theta_{T_{k-1}^n}, \theta_{T_k^n} - \theta_{T_{k-1}^n})$). However, (4.2) would be violated and there would not exist a finite $\widetilde{\mu}$. Therefore the structure is quite different and we prefer to treat the two models separately.*

Definition 4.4 can be justified by the following.

Theorem 4.7. *Let θ be a strategy of the form (4.7) with a density y satisfying*

$$\int_0^T y_s^2 \, d\widetilde{\mu}_s < \infty, \quad P\text{-a.s.} \tag{4.9}$$

and $\int_0^T |y_s| \, dA_s < \infty$, P-a.s. Let $(\sigma_n)_{n\in\mathbb{N}} = (T_0^n, T_1^n, \ldots, T_{k_n}^n)_{n\in\mathbb{N}}$ be a fining sequence of grids with $\sup_{k=1,\ldots,k_n} |T_k^n - T_{k-1}^n|$ converging to zero in probability. Assume that each $t \in [0,T]$ with $\mu(\{t\}) > 0$ is finally contained in $(\sigma_n)_{n\in\mathbb{N}}$. In addition, assume that H is bounded and there exists a localizing sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ s.t. for each $n \in \mathbb{N}$, $\vartheta \mapsto S(\vartheta, \cdot) 1_{[\![0,\tau_n[\![}$ and $\vartheta \mapsto a^{\vartheta} 1_{[\![0,\tau_n[\![}$ are uniformly Lipschitz continuous. Then, the sequence of processes $(\beta_t^n)_{t\in[0,T]}$ from (4.6) converges for $n \to \infty$ to the process $\beta = (\beta_t)_{t\in[0,T]}$ from (4.8) uniformly in probability.

Remark 4.8. *The assumed Lipschitz continuity is for instance in the model from Proposition 3.1 not always satisfied. It holds if the dividend process L is solely driven by a diffusion and bounded jumps at finitely many predictable times. But, it is violated if jumps occur at Poisson times. This can be seen from (3.8). Alternatively to the Lipschitz condition one may assume that the dividend payoff H in Proposition 3.1 is nonnegative and the strategy of the large investor takes values in $(-\infty, K]$ for some $K \in \mathbb{R}_+$ (this means short-holdings of the counterparty – the representative agent – are limited).*

With another structure of $(S(\vartheta,\cdot))_{\vartheta\in\mathbb{R}}$ other assumptions than (4.9) are needed. As reasonable illiquid market models are quite diverse one should replace (4.9) by appropriate conditions for the specific model under consideration (in the case that the Lipschitz continuity does not hold there).

Proof. Step 1: For $s \in [0,T]$, $n \in \mathbb{N}$ take k with $T^n_{k-1} < s \le T^n_k$. We have

$$\frac{\int_{T^n_{k-1}}^{T^n_k} y_u \, d\widetilde{\mu}_u}{\widetilde{\mu}((T^n_{k-1}, T^n_k])} \to y_s, \text{ for } \widetilde{\mu}\text{-almost all } s \in [0,T], \quad n \to \infty. \tag{4.10}$$

Namely, for fixed $\omega \in \Omega$ the lhs of (4.10) can be written as $E_{\widetilde{\mu}}(Y \mid \mathcal{G}_n)$ where $Y : [0,T] \to \mathbb{R}$ is a Borel-measurable function and $(\mathcal{G}_n)_{n\in\mathbb{N}}$ an increasing sequence of σ-algebras with $\sigma\left(\bigcup_{n\in\mathbb{N}} \mathcal{G}_n\right) = \mathcal{B}([0,T])$. Then, $E_{\widetilde{\mu}}(Y \mid \mathcal{G}_n) \to Y$, $\widetilde{\mu} - a.s.$, $n \to \infty$ (see e.g. Corollary II.2.4 in [20]) which is (4.10). (4.9) corresponds to $E_{\widetilde{\mu}}(Y^2) < \infty$ which implies uniformly $\widetilde{\mu}$-integrability of the sequence $(Y E_{\widetilde{\mu}}(Y \mid \mathcal{G}_n))_{n\in\mathbb{N}}$. Therefore, we have $E_{\widetilde{\mu}}|Y(Y - E_{\widetilde{\mu}}(Y \mid \mathcal{G}_n))| \to 0$, $n \to \infty$, i.e.

$$\sum_{k=1}^{k_n} \int_{T^n_{k-1}}^{T^n_k} \left| y_s \left(y_s - \frac{\int_{T^n_{k-1}}^{T^n_k} y_u \, d\widetilde{\mu}_u}{\widetilde{\mu}((T^n_{k-1}, T^n_k])} \right) \right| d\widetilde{\mu}_s \to 0, \quad n \to \infty, \quad P\text{-a.s.} \tag{4.11}$$

Step 2: By the Lipschitz continuity of $S(\cdot, T^n_k-)$ we have for all $n \in \mathbb{N}$, $k = 1, \ldots, k_n$

$$\left| \int_{T^n_{k-1}}^{T^n_k} y_s S\left(\theta_{s-} + y_s, T^n_k-\right) d\widetilde{\mu}_s - \int_{T^n_{k-1}}^{T^n_k} y_s S\left(\theta_{T^n_{k-1}} + \frac{\int_{T^n_{k-1}}^{T^n_k} y_u \, d\widetilde{\mu}_u}{\widetilde{\mu}((T^n_{k-1}, T^n_k])}, T^n_k-\right) d\widetilde{\mu}_s \right|$$
$$\le M \left(\int_{T^n_{k-1}}^{T^n_k} |y_s| \left| y_s - \int_{T^n_{k-1}}^{T^n_k} \frac{\int_{T^n_{k-1}}^{T^n_k} y_u \, d\widetilde{\mu}_u}{\widetilde{\mu}((T^n_{k-1}, T^n_k])} \right| d\widetilde{\mu}_s + \int_{T^n_{k-1}}^{T^n_k} |y_s| \left| \theta_{s-} - \theta_{T^n_{k-1}} \right| d\widetilde{\mu}_s \right)$$

where $M \in \mathbb{R}_+$ is independent of k and n. By (4.11) and the right-continuity of θ this implies that

$$\sum_{k=1}^{k_n} \left| \int_{T_{k-1}^n}^{T_k^n} y_s S\left(\theta_{s-} + y_s, T_k^n-\right) d\widetilde{\mu}_s \right. \tag{4.12}$$
$$\left. - \int_{T_{k-1}^n}^{T_k^n} y_s S\left(\theta_{T_{k-1}^n} + \frac{\int_{T_{k-1}^n}^{T_k^n} y_u \, d\widetilde{\mu}_u}{\widetilde{\mu}((T_{k-1}^n, T_k^n])}, T_k^n-\right) d\widetilde{\mu}_s \right| \to 0, \ n \to \infty \text{ in probability.}$$

From (4.3) it can be seen that $\max_{k=1,\dots,k_n} |\widetilde{\mu}((T_{k-1}^n, T_k^n])/\mu((T_{k-1}^n, T_k^n]) - 1|$ converges to zero uniformly in probability (w.l.o.g. $\mu((T_{k-1}^n, T_k^n]) > 0$, for all k). By the uniformly Lipschitz continuity of $\vartheta \mapsto S(\vartheta, t)$ this implies

$$\sum_{l=1}^{k} \left(\theta_{T_l^n} - \theta_{T_{l-1}^n}\right) S\left(\theta_{T_{l-1}^n} + \frac{\theta_{T_l^n} - \theta_{T_{l-1}^n}}{\mu((T_{l-1}^n, T_l^n])}, T_l^n-\right) \tag{4.13}$$
$$= \sum_{l=1}^{k} \int_{T_{l-1}^n}^{T_l^n} y_s \, d\widetilde{\mu}_s S\left(\theta_{T_{l-1}^n} + \frac{\int_{T_{l-1}^n}^{T_l^n} y_u \, d\widetilde{\mu}_u}{\widetilde{\mu}((T_{l-1}^n, T_l^n])}, T_l^n-\right) + o(1), \text{ uniformly in } k = 1, \dots, k_n$$

for $n \to \infty$.

Step 3: Let us show that

$$\sup_{k=1,\dots,k_n} \sum_{l=1}^{k} \int_{T_{l-1}^n}^{T_l^n} y_s \left(S\left(\theta_{s-} + y_s, T_l^n-\right) - S\left(\theta_{s-} + y_s, s-\right)\right) d\widetilde{\mu}_s \tag{4.14}$$

tends to zero in probability for $n \to \infty$. Define the family $(M(\vartheta, \cdot))_{\vartheta \in \mathbb{R}}$ of martingales by $M(\vartheta, t) := \int_0^t H_s^\vartheta \, dS^M(0, s)$, where S^M is the martingale part of the special semimartingale $S(0, \cdot)$. Set $B(\vartheta, t) := \int_0^t H_s^\vartheta \, dS^B(0, s) + \int_0^t H_s^\vartheta a_s^\vartheta \, dA_s$, where S^B is the predictable finite variation part of $S(0, \cdot)$. Aiming at convergence uniformly in probability we can w.l.o.g. assume that S^M, S^B, A, and θ are bounded (instead of only locally bounded, notice that θ is predictable and possesses càdlàg paths). By $0 \le H \le K \in \mathbb{R}_+$ we have

$$E\left(\sum_{l=1}^{k} \int_{T_{l-1}^n}^{T_l^n} y_s (M(\theta_{s-} + y_s, T_l^n-) - M(\theta_{s-} + y_s, s-)) \, d\widetilde{\mu}_s\right)^2$$
$$\le K^2 E\left(\sum_{l=1}^{k} \int_{T_{l-1}^n}^{T_l^n-} (\theta_s - \theta_{T_{l-1}^n})^2 \, d[S^M]_s\right)$$
$$\le K^2 E\left(\max_{k=0,\dots,k_n} \sup_{s \in [T_{l-1}^n, T_l^n)} (\theta_{T_{l-}^n} - \theta_s)^2 \sum_{l=1}^{k_n} \left([S^M]_{T_l^n-} - [S^M]_{T_{l-1}^n}\right)\right) \tag{4.15}$$

and

$$
\begin{aligned}
&E\left(\sum_{k=1}^{k_n}\left|\int_{T_{k-1}^n}^{T_k^n} y_s(B(\theta_{s-}+y_s,T_k^n-)-B(\theta_{s-}+y_s,s-))\,d\widetilde{\mu}_s\right|\right)\\
&\le K'E\left(\sum_{k=1}^{k_n}\int_{T_{k-1}^n}^{T_k^n-}|\theta_s-\theta_{T_{l-1}^n}|(\,d\mathrm{Var}(S^B)_s+(1+|y_s|)dA_s)\right)\\
&\le K'E\left(\max_{k=0,\dots,k_n}\sup_{s\in[T_{l-1}^n,T_l^n)}|\theta_{T_{l-}^n}-\theta_s|\sum_{k=1}^{k_n}\Big(\mathrm{Var}(S^B)_{T_k^n-}-\mathrm{Var}(S^B)_{T_{k-1}^n}+A_{T_k^n-}-A_{T_{k-1}^n}\right.\\
&\qquad\left.+\int_{T_{k-1}^n}^{T_k^n-}|y_s|\,dA_s\Big)\right).
\end{aligned}
\tag{4.16}
$$

Both (4.15) and (4.16) tend to zero when $n\to\infty$. By Doob's inequality (for square integrable martingales) also the pathwise maxima of the sums up to $k=1,\dots,k_n$ in (4.15) converge to zero and we arrive at (4.14). (4.12), (4.13), and (4.14) imply that

$$\max_{k=0,\dots,k_n}|\beta^n_{T_k^n}-\beta_{T_k^n}|\to 0,\ \ n\to\infty,\ \text{in probability.}$$

By (4.9) and the Lipschitz continuity, we have that $\int_0^T|y_sS(\theta_{s-}+y_s,s-)|\,d\widetilde{\mu}^d_s<\infty$ where $\widetilde{\mu}^d$ is the discontinuous part of the increasing process $\widetilde{\mu}$. As each of the at most countable many discontinuity points of $\widetilde{\mu}$ is finally contained in the fining sequence of grids we can make the distance $\sup_{t\in[0,T]}|\beta_t-\beta_{T_{k-1}^n}|$, where $T_{k-1}^n\le t<T_k^n$, arbitrary small. As β^n is by construction a right-continuous step function this implies the assertion. □

For the rest of the article we make the following

Assumption 4.9. *$\vartheta\mapsto S(\vartheta,\cdot)$ is nondecreasing and continuously differentiable.*

It is quite natural that $S(\vartheta,t)$ is nondecreasing in ϑ. Otherwise the large investor possesses an arbitrage opportunity which consists in purchasing assets and sell them back immediately (or vice versa). However, Assumption 4.9 is not needed in Section 2 which is due to the idealized definition of the dynamics of the liquidation value in (2.2).

Definition 4.10. *Define the (liquidation) value process $V=(V_t)_{t\in[0,T]}$ by*

$$V_t=\beta_t+\int_0^{\theta_t}S(x,t)\,dx,\quad t\in[0,T]. \tag{4.17}$$

An application of the Itô-Wentzell formula to the stochastic flow $(\int_0^y S(x,t)\,dx)_{y\in\mathbb{R}}$ and a càdlàg process θ of finite variation with $\theta_0 = 0$ yields

$$\begin{aligned}\int_0^{\theta_t} S(x,t)\,dx &= \int_0^t \int_0^{\theta_s} S(x,ds)\,dx + \int_0^t S(\theta_s,s-)\,d\theta_s \\ &\quad + \sum_{0<s\le t}\left\{\int_{\theta_{s-}}^{\theta_s} S(x,s-)\,dx - S(\theta_s,s-)\Delta\theta_s\right\}\end{aligned}$$

where in the framework of this article

$$\int_0^t \int_0^{\theta_s} S(x,ds)\,dx := \int_0^t \left(\int_0^{\theta_s} H_s^x\,dx\right)\,dS(0,s) + \int_0^t \left(\int_0^{\theta_s} H_s^x a_s^x\,dx\right)\,dA_s$$

cf. (2.2). Thus (4.8) is equivalent to

$$\begin{aligned}V_t(\theta) &= \beta_0 + \int_0^t \int_0^{\theta_s} S(x,ds)\,dx + \int_0^t y_s\left(S(\theta_s,s-) - S(\theta_{s-}+y_s,s-)\right)\,d\widetilde{\mu}_s \\ &\quad + \sum_{0<s\le t}\left\{\int_{\theta_{s-}}^{\theta_s} S(x,s-)\,dx - S(\theta_s,s-)\Delta\theta_s\right\},\ t\in[0,T]. \end{aligned} \tag{4.18}$$

The process $L :=$

$$\int_0^\cdot y_s\left(S(\theta_s,s-) - S(\theta_{s-}+y_s,s-)\right)\,d\widetilde{\mu}_s + \sum_{0<s\le\cdot}\left\{\int_{\theta_{s-}}^{\theta_s} S(x,s-)\,dx - S(\theta_s,s-)\Delta\theta_s\right\} \tag{4.19}$$

can be interpreted as transaction costs. As $\theta_s = \theta_{s-} + y_s\Delta\widetilde{\mu}_s = \theta_{s-} + y_s\mu(\{s\})$, $0 \le \mu(\{s\}) \le 1$, and by Assumption 4.9, (4.19) is a decreasing process.

4.1 Merton's problem

The large investor endowed with $\beta_0 \in \mathbb{R}$ bonds can trade in the stock market according to the self-financing condition (4.8). Similar to Section 2 we define

$$\Theta_c := \{\theta \in \widetilde{\Theta} \mid V_t(\theta) \ge -c,\ \forall t\in[0,T]\},\quad c\in\mathbb{R}_+ \quad\text{and}\quad \Theta = \bigcup_{c\in\mathbb{R}_+}\Theta_c.$$

where V is now the value process from (4.18). At maturity T the position in the illiquid stock has to be liquidated, i.e. $\theta_T = 0$. The aim is to maximize expected utility from terminal wealth which is measured in units of bonds, i.e.

$$E\left(u(\beta_T)\right)\ \max_y!\quad \theta = \int_0^\cdot y_s\,d\widetilde{\mu}_s,\ \theta\in\Theta,\ \theta_T = 0,\quad (\beta,\theta)\text{ self-financing},\quad \beta_0\in\mathbb{R}\text{ is given.}$$

Remark 4.11. *Notice that the value of the process (4.18) during* $(0,T)$ *enters in the optimization problem only marginally via* Θ *(the set of admissible strategies) and it is not part of the optimization criterion itself. Namely, at* T *the stock position is liquidated and (despite of some technical subtleness arising in continuous time models) the evaluation of intermediate positions in the stock has no influence on the optimization problem.*

V *corresponds to the liquidation value process in Section 2. However, the portfolio "value"* V_t *in (4.17) cannot be realized by closing the asset position instantly as rebalancings of the portfolio have to be continuous w.r.t. the measure* $\widetilde{\mu}$. V_t *ignores a part of (future) transaction/liquidation costs for the* θ_t *shares, essentially the non-permanent price impact caused by liquidation. But, these unaccounted future "transaction costs" only concern the assets sold at that moment and no mark-to-market of the other shares is needed.* V *could be interpreted as the smallest overestimation of current positions – taking into account that future transactions should not enter into today's evaluation. This makes* V *to a suitable quantity for the mathematical analysis.*

Assumption 4.12. *For each* $\vartheta \in \mathbb{R}$, $S(\vartheta, \cdot)$ *is bounded.*

Theorem 4.13. *Assume that* $\widetilde{\mu}(A) > 0$ *for all* $A \in \mathcal{B}([0,T])$ *with positive Lebesgue measure and Assumption 4.12 holds. Assume that there exists an optimal solution* y *that is bounded and whose paths have right limits which we denote by* y_+. *In addition, let* $\vartheta \mapsto S(\vartheta, \cdot)$ *be uniformly Lipschitz continuous. Then, the process*

$$\begin{aligned} M_t &:= \int_0^t S(\theta_s, ds) + S(\theta_t + y_{t+}, t) - S(\theta_t, t) + y_{t+}\partial_1 S(\theta_t + y_{t+}, t) \qquad (4.20) \\ &:= \int_0^t H_s^{\theta_s}\, dS(0,s) + \int_0^t H_s^{\theta_s} a_s^{\theta_s}\, dA_s + S(\theta_t + y_{t+}, t) - S(\theta_t, t) + y_{t+}\partial_1 S(\theta_t + y_{t+}, t) \end{aligned}$$

restricted to $[0,T)$ *is a* Q*- martingale.*

Remark 4.14. *Of course, in general there need not exist an optimal strategy whose* $\widetilde{\mu}$*-density possesses right limits.*

For Theorem 2.16 we do not need these boundedness assumptions as investment can be restricted to stochastic intervals $[\![0, T_n]\!]$ *or predictable sets in general. By contrast the model in this section is much less flexible as some time is needed to reduce positions. If price processes were only locally bounded resp. transaction costs could explode, it would not* *be guaranteed that positions can be liquidated fast enough in order to arrive at a terminal wealth bounded from below.*

As $y_T \widetilde{\mu}(\{T\})$ is the number of assets purchased (resp. sold) at $T-$ the investor can only arrive at $\theta_T = 0$ if the asset position is already liquidated at $T-$. As there is no way to invest in the jump ΔM_T, nothing can be said about it. To avoid this problem one can of course substitute T by $T+1$ with $S(\vartheta, \cdot)$ being constant on $[T, T+1]$.

Remark 4.15. *The term $l_t := y_{t+}\partial_1 S(\theta_t + y_{t+}, t) + S(\theta_t + y_{t+}, t) - S(\theta_t, t)$ can be interpreted as marginal transaction costs (also called liquidity costs) for buying one extra unit of assets during $(t, t+\Delta t]$ and $\int_0^t S(\theta_s, ds)$ are the trading gains in the interval $[0,t]$. l_t is nonnegative if $y_t > 0$ and nonpositive otherwise. For small perturbations of the density y the term $\sum_{0<s\le t}\left\{\int_{\theta_{s-}}^{\theta_s} S(x, s-)\,dx - S(\theta_s, s-)\Delta\theta_s\right\}$ varies less than linearly and thus it does not enter in l.*

Proof. The line of argument is quite similar to the proof of Theorem 2.16. But, it is simpler as by the assumptions made we need not care about integrability.

Let $\theta = \int_0^\cdot y_t\, d\widetilde{\mu}_t$ be the optimal solution. Assume that M is not a Q-martingale. Consider first the case that

$$\begin{aligned} E_Q\Bigg[1_A\Bigg(&\int_{t_1}^{t_2} S(\theta_s, ds) + y_{t_2}\partial_1 S(\theta_{t_2} + y_{t_2}, t_2) - y_{t_1}\partial S(\theta_{t_1} + y_{t_1}, t_1) \\ &+ S(\theta_{t_2} + y_{t_2}, t_2) - S(\theta_{t_2}, t_2) - S(\theta_{t_1} + y_{t_1}, t_1) + S(\theta_{t_1}, t_1))] > 0 \end{aligned} \tag{4.21}$$

for some $0 \le t_1 < t_2 < T$, $A \in \mathcal{F}_{t_1}$. For $\varepsilon > 0$ define $\theta^\varepsilon := \int_0^\cdot y_s^\varepsilon\, d\widetilde{\mu}_s$ where

$$y^\varepsilon := y + \varepsilon 1_{A\times(t_1, t_1+\delta(\varepsilon))} - \varepsilon 1_{A\times(t_2, t_2+\delta'(\varepsilon))}$$

with $\widetilde{\mu}((t_1, t_1+\delta(\varepsilon))) = \widetilde{\mu}((t_2, t_2+\delta'(\varepsilon))) = \varepsilon$ (assuming that ε is small enough s.t. $t_1 + \delta(\varepsilon) < t_2$ and $t_2 + \delta'(\varepsilon) < T$). As the Lebesgue measure is absolutely continuous w.r.t. $\widetilde{\mu}$, $\delta(\varepsilon) \to 0$ and $\delta'(\varepsilon) \to 0$ for $\varepsilon \to 0$. W.l.o.g. we can assume that $\Delta\widetilde{\mu} = 0$ as otherwise we replace $1_{A\times(t_1, t_1+\delta(\varepsilon))}$ by $1_{(A\times(t_1, t_1+\delta(\varepsilon)))\cap\{\Delta\widetilde{\mu}=0\}}$ (and for t_2 accordingly). We have for all $s \in (t_1, t_1+\delta(\varepsilon))$

$$\begin{aligned} &(y+\varepsilon)\left(S(\theta_s^\varepsilon, s-) - S(\theta_{s-}^\varepsilon + y_s + \varepsilon, s-)\right) - y_s\left(S(\theta_s, s-) - S(\theta_{s-} + y_s, s-)\right) \\ &= \varepsilon\left(S(\theta_s, s-) - S(\theta_{s-} + y_s, s-)\right) + y_s\left(S(\theta_{s-} + y_s, s-) - S(\theta_{s-} + y_s + \varepsilon, s-)\right) \\ &\quad + \text{error}. \end{aligned}$$

where

$$\text{error} = (y+\varepsilon)\left(S(\theta^\varepsilon_s, s-) - S(\theta_s, s-) + S(\theta_{s-}+y_s+\varepsilon, s-) - S(\theta^\varepsilon_{s-}+y_s+\varepsilon, s-)\right) \\ +\varepsilon\left(S(\theta_{s-}+y_s, s-) - S(\theta_{s-}+y_s+\varepsilon, s-)\right). \tag{4.22}$$

For $s \in (t_2, t_2+\delta'(\varepsilon))$ we have to substitute $-\varepsilon$ for ε. As $|\theta^\varepsilon_s - \theta_s| \le \varepsilon^2$ and by the Lipschitz continuity of $x \mapsto S(x,t)$ the error term (4.22) is of order $o(\varepsilon)$ (uniformly in s). As $(S(x,\cdot))_{x\in\mathbb{R}}$ have right-continuous paths (uniformly in x on compacta), we obtain

$$\begin{aligned}
&E_Q(V_T(\theta^\varepsilon) - V_T(\theta)) \\
&= E_Q\left[\int_{t_1}^{t_2}\int_{\theta_s}^{\theta_s+\varepsilon^2} S(y, ds)\,dy\right] + E_Q\left[\int_{t_1}^{t_1+\delta(\varepsilon)} \varepsilon\left(S(\theta_s, s-) - S(\theta_{s-}+y_s, s-)\right)\right. \\
&+y_s\left(S(\theta_{s-}+y_s, s-) - S(\theta_{s-}+y_s+\varepsilon, s-)\right) d\widetilde{\mu}_s + \int_{t_2}^{t_2+\delta'(\varepsilon)} (-\varepsilon)\left(S(\theta_s, s-)\right. \\
&\left.-S(\theta_{s-}+y_s, s-)) + y_s\left(S(\theta_{s-}+y_s, s-) - S(\theta_{s-}+y_s-\varepsilon, s-)\right) d\widetilde{\mu}_s\right] + o(\varepsilon^2) \\
&= E_Q\left[\varepsilon^2\int_{t_1}^{t_2} S(\theta_s, ds)\right] \\
&+\varepsilon^2 E_Q\left[S(\theta_{t_1}, t_1) - S(\theta_{t_1}+y_{t_1+}, t_1) + y_{t_1+}\partial_1 S(\theta_{t_1}+y_{t_1+}, t_1)\right] \\
&-\varepsilon^2 E_Q\left[S(\theta_{t_2}, t_2) - S(\theta_{t_2}+y_{t_2+}, t_2) + y_{t_2+}\partial_1 S(\theta_{t_2}+y_{t_2+}, t_2)\right] + o(\varepsilon^2).
\end{aligned}$$

This means that

$$E_Q(V_T(\theta^\varepsilon) - V_T(\theta)) = \varepsilon^2 E_Q(1_A(M_{t_2} - M_{t_1})) + o(\varepsilon^2), \quad \varepsilon \to 0.$$

As $V(\theta)$ is bounded from below and $V(\theta^\varepsilon) - V(\theta)$ converges uniformly to zero, we obtain by the marginal utility criterion a contradiction to the optimality of θ (without using any convexity assumption). Again, the case that $E_Q\left(1_A(M_{t_2} - M_{t_1})\right) < 0$ instead of (4.21) is treated analogously by substituting $-\varepsilon$ for ε. □

References

[1] P. Bank and D. Baum. Hedging and portfolio optimization in financial markets with a large trader. *Mathematical Finance*, 14:1–18, 2004.

[2] U. Çetin. Optimal portfolios and utility based pricing in markets with limited depth. *Preprint*, 2005.

[3] U. Çetin, R.A. Jarrow, and P. Protter. Liquidity risk and arbitrage pricing theory. *Finance & Stochastics*, 8:311–341, 2004.

[4] U. Çetin and L.C.G. Rogers. Modelling liquidity effects in discrete time. *Mathematical Finance, forthcoming*, 2005.

[5] F. Delbaen and W. Schachermayer. A compactness principle for bounded sequences of martingales with applications. *Proceedings of the Seminar of Stochastic Analysis, Random Fields and Applications, Progress in Probability*, 45:137–173, 1999.

[6] A. Esser and B. Mönch. Modeling feedback effects with stochastic liquidity. *Preprint*, 2002.

[7] R. Frey and A. Stremme. Market volatility and feedback effects from dynamic hedging. *Mathematical Finance*, 7(4):351–374, 1999.

[8] P. Guasoni. Optimal investment with transaction costs and without semimartingales. *The Annals of Applied Probability*, 12(4):1227–1246, 2002.

[9] J. Jacod and A.N. Shiryaev. *Limit Theorems for Stochastic Processes*. Springer-Verlag, Berlin, second edition, 2003.

[10] R. Jarrow. Market manipulation, bubbles, corners and short squeezes. *Journal of Financial and Quantitative Analysis*, 27:311–336, 1992.

[11] R. Jarrow. Derivative security markets, market manipulation and option pricing. *Journal of Financial and Quantitative Analysis*, 29:241–261, 1994.

[12] J. Kallsen and C. Kühn. Pricing derivatives of American and game type in incomplete markets. *Finance & Stochastics*, 8:261–284, 2004.

[13] J. Kallsen and A. Shiryaev. Time change representations of stochastic integrals. *Theory of Probability and its Applications,*, 46:522–528, 2001.

[14] J. Kallsen and A. Shiryaev. The cumulant process and Esscher's change of measure. *Finance & Stochastics,*, 6:397–428, 2004.

[15] D. Kramkov and W. Schachermayer. The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *The Annals of Applied Probability*, 9:904–950, 1999.

[16] H. Kunita. *Stochastic Flows and Stochastic Differential Equations*. Cambridge University Press, Cambridge, 1990.

[17] R.E. Lucas. Asset prices in an exchange economy. *Econometrica*, 46:1429–1445, 1978.

[18] E. Platen and M. Schweizer. On feedback effects from hedging derivatives. *Mathematical Finance*, 8:67–84, 1998.

[19] P. Protter. *Stochastic Integration and Differential Equations*. Springer, Berlin, second edition, 2004.

[20] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion*. Springer, Berlin, third edition, 1999.

[21] C. Rogers and S. Singh. Modelling liquidity and its effects on price. *Preprint*, 2006.

[22] W. Schachermayer. Optimal investment in incomplete markets when wealth may become negative. *The Annals of Applied Probability*, 11:694–734, 2001.

[23] W. Schachermayer. *Portfolio Optimization in Incomplete Financial Markets*. Notes of the Scuola Normale Superiore Cattedra Galileiana, Pisa, ISBN 8876421416. 2004.