Perpetual convertible bonds with credit risk

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Abstract

A convertible bond is a security that the holder can convert into a specified number of underlying shares. We enrich the standard model by introducing some *default risk* of the *issuer*. Once default has occured payments stop immediately. In the context of a reduced form model with infinite time horizon driven by a Brownian motion, analytical formulae for the no-arbitrage price of this American contingent claim are obtained and characterized in terms of solutions of free boundary problems. It turns out that the default risk changes the structure of the optimal stopping strategy essentially. Especially, the continuation region may become a disconnected subset of the state space.

Keywords: convertible bonds, exchangeable bonds, default risk, optimal stopping problems, free-boundary problems, smooth fit.

Mathematics Subject Classification (2000): 60G40, 60J50, 60G44, 91B28.

1 Introduction

The market for convertible bonds has been growing rapidly during the last years and the corresponding optimal stopping problems have attracted much attention in the literature on mathematical finance. One has to distinguish between *reduced form models* where the stock price process of the issuing firm is exogenously given by some stochastic process and *structural models* where the starting point is the *firm value* which splits in the total equity value and the total debt value. Within a firm value model the pricing problem is treated in Sîrbu, Pikovsky and Shreve [15] and Sîrbu and Shreve [16]. In contrast to earlier articles of Brennan and Schwartz [4] and Ingersoll [11, 12], [15, 16] includes the case where an earlier conversion of the bond can be optimal that necessitates to address a nontrivial free-boundary problem. In the context of a reduced form model Bielecki, Crépey, Jeanblanc and Rutkowski [2] made quite recently a comprehensive analysis of interesting features of convertible bonds. Especially they model the interplay between equity risk and credit

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Acknowledgements. We would like to thank Andreas Kyprianou for valuable discussions and comments.

risk, cf. also Remark 1.2 (iii). This is done for the nonperpetual case. Thus the pricing problem has finally to be solved by numerical methods.

In this article we work with reduced form models where such a contract without a recall option for the issuer can be expressed as a standard American contingent claim (see also Davis and Lischka [5] for a detailed introduction and a precise description of the contract). The special feature of the current article is that we enrich the standard Black and Scholes model by introducing some *default risk* of the *issuer*. Once default has occured payments stop immediately. The main purpose is to obtain analytical formulae for the no-arbitrage price of a *perpetual* convertible bond under different default intensities through characterizations in terms of free boundary problems. It turns out that the default risk changes the structural behavior of the solution essentially. Roughly speaking, in models without default bonds are converted only by the time the stock price is high, cf. [4], [9], [11], [12], [15], and [16]. The ratio behind this is that for low stock prices the holder prefers collecting the prespecified coupon payments, whereas for higher stock prices the dividends payed out exclusively to stockholders become more attractive which may cause the bondholder to convert. We model the default intensity of the issuer as a nonincreasing function of the current stock price. In this setting also a low stock price may cause the holder to convert the bond (even if the yield is low) in order to get rid of the high risk that the issuer defaults which would make the contract worthless.

The paper is organized as follows. In Subsection 1.1 we introduce the stochastic model. Stopping times depending on the default state of the issuer are reduced to stopping times without using this information. We do this in a mathematical framework differing from the standard one in credit risk modeling which is based on the progressive enlargement of the filtration without the default event, cf. e.g. Chapter 5 in [3]. We think this provides some interesting additional insights – but the resulting payoff process (1.4) is of course the same. Subsection 1.3 provides some general properties of the value function of convertible bonds with varying default intensities. In Section 2 we consider the simplifying case that there are two different default intensities depending on the current stock price. In Section 3 we consider the case that the default intensity is a power function of the current stock price (with negative exponent). In Section 4 the results of Sections 2 and 3 are represented by some plots. Parts of the unavoidable technical proofs are left to the appendix.

1.1 The model

Consider the following Black and Scholes market. We have a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_{\geq 0}\cup\{+\infty\}}, P)$, where the filtration $\mathbb{F}$ satisfies the usual conditions and $\mathcal{F} = \mathcal{F}_\infty = \sigma(\mathcal{F}_t, t \in \mathbb{R}_{\geq 0})$. The riskless asset B is given by $B_t = e^{rt}$ for all $t \geq 0$, where $r > 0$ is the interest rate. The process S models the risky stock paying dividends at rate δS_t, where $\delta \in (0, r)$. S is given by the formula

$$S_t = \exp(\sigma W_t + (r - \delta - \sigma^2/2)t), \quad t \geq 0,$$

where $\sigma > 0$ is the volatility and W a standard Brownian motion under the unique equivalent martingale measure $\mathbb{P} \sim P$. This means that the discounted cum dividend cumulative price process $(\exp(-rt)S_t + \int_0^t \exp(-ru)\delta S_u \, \mathrm{d}u)_{t\geq 0}$ is a $\mathbb{P}$-martingale. Let for each $s > 0$, the measure $\mathbb{P}_s$ be the translation of $\mathbb{P}$ such that $\mathbb{P}_s(S_0 = s) = 1$. $\mathbb{F}$ is the natural filtration generated by W.

In this market we consider a perpetual convertible bond, that is an American contingent claim with infinite horizon which gives the holder the right to convert the contract at a (stopping) time of his choosing in a predetermined number $\gamma \in \mathbb{R}_{>0}$ of stocks, while receiving coupon payments at rate $c > 0$ up to this (possibly never occuring) time. If default occurs before the conversion time of the holder, the contract is terminated and the holder is left with only the coupon payments he has collected up to default. For simplicity (and as it would not be an interesting feature in combination with default risk) we do not allow for recalling, i.e. the issuer may not terminate the contract.

For including default in the mathematical model we extend the probability space above to $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_{>0})$ containing a random variable $\mathbf{e} \in \mathbb{R}_{>0}$ which is both under P and under $\mathbb{P}$ independent of S and exponentially distributed with parameter 1. We allow for the default intensity of the issuer to depend on the current value of the stock, namely it is given by the process $(\chi(S_t))_{t\geq 0}$ for some suitable non-negative Borel-measurable function χ. That is to say, defining the process φ by

$$\varphi_t = \int_0^t \chi(S_u) \, \mathrm{d}u, \quad t \geq 0, \tag{1.1}$$

the time of default is defined as

$$\varphi^{-1}(\omega, \mathbf{e}) := \inf\{t \geq 0 \,|\, \varphi_t(\omega) \geq \mathbf{e}\},$$

which is the generalized left-continuous inverse of φ (with the usual convention that $\inf \emptyset = \infty$). Note that this corresponds to the first jump time of a Cox process with intensity process $(\chi(S_t))_{t\geq 0}$. Throughout this article we will only consider non-negative intensity functions $\chi : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ for which (1.1) defines a finitely valued non-decreasing continuous process.

The payoff process X corresponding to such defaultable convertible bond is thus given by

$$X_t(\omega, \mathbf{e}) := \mathbf{1}_{\{\varphi_t(\omega) < \mathbf{e}\}} \left(e^{-rt}\gamma S_t(\omega) + \int_0^t ce^{-ru} \, \mathrm{d}u \right) + \mathbf{1}_{\{\varphi_t(\omega) \geq \mathbf{e}\}} \int_0^{\varphi^{-1}(\omega,\mathbf{e})} ce^{-ru} \, \mathrm{d}u$$

for all $t \geq 0$ and $X_\infty(\omega, \mathbf{e}) := \int_0^{\varphi^{-1}(\omega,\mathbf{e})} ce^{-ru} \, \mathrm{d}u$.

Definition 1.1. *A stopping time w.r.t. the enlarged information is an $(\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_{>0}) - \mathcal{B}(\mathbb{R}_{\geq 0} \cup \{+\infty\}))$-measurable mapping $\tau : \Omega \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ with $\{\omega \in \Omega \mid \tau(\omega, u) \leq t\} \in \mathcal{F}_t$ for all $t \in \mathbb{R}_{\geq 0}$, $u \in \mathbb{R}_{>0}$ such that for all $\omega \in \Omega$, $u \in \mathbb{R}_{>0}$ the implication*

$$\tau(\omega, u) < \varphi^{-1}(\omega, u) \quad \Longrightarrow \quad \forall u' > \varphi_{\tau(\omega,u)}(\omega) : \tau(\omega, u') = \tau(\omega, u) \tag{1.2}$$

holds. The set of these stopping times is denoted by $\widetilde{\mathcal{T}}$.

Remarks 1.2. *(i) The lhs of (1.2) means that there is pre-default stopping. As the default event should be non-predictable we assume that this stopping takes place irrespective of when exactly default occurs after $\tau(\omega, u)$, i.e. for all u' with $\varphi^{-1}(\omega, u') > \tau(\omega, u)$ we should have $\tau(\omega, u') = \tau(\omega, u)$.*

(ii) By augmenting the model with the default event, the market becomes incomplete. On the enlarged probability space the set of martingale measures is no longer a singleton. The measure $\mathbb{P}$ introduced above is the so-called minimal martingale measure *of Föllmer and Schweizer [8]. This measure has the nice property that it respects orthogonality in the sense that the "untradable" random variable* $\mathbf{e}$ *remains independent of S and possesses the same distribution as under P.*

(iii) In our model default of the issuer is not identified with default of the firm. This includes so-called exchangeable bonds *where the issuer is not the firm itself but typically one of its major shareholders. Thus the default intensity $\chi(S_t)$ does not enter into the no-arbitrage drift condition. Note that this differs e.g. from the model in [2]. An exchangeable bond may be converted into* existing *shares and not into new shares. This destroys the advantages a firm value model possesses in comparison to a reduced form model.*

Since X stays constant after default and by the non-predictability of $\mathbf{e}$ from Definition 1.1, it is enough to consider $\mathbb{F}$-stopping times and average over $\mathbf{e}$.

Proposition 1.3. *Let $\mathcal{T}_{a,b}$ denote the set of $[a, b]$-valued $\mathbb{F}$-stopping times. We have for all $s \in \mathbb{R}_{>0}$*

$$\sup_{\tau \in \widetilde{\mathcal{T}}} \mathbb{E}_s [X_\tau] = \sup_{\mathcal{T}_{0,\infty}} \mathbb{E}_s [L_\tau], \tag{1.3}$$

where the $\mathbb{F}$-adapted continuous process $(L_t)_{t \in \mathbb{R}_{\geq 0} \cup \{+\infty\}}$ is given by

$$L_t := e^{-rt-\varphi_t} \gamma S_t + \int_0^t c e^{-ru-\varphi_u} \, du, \qquad t \in \mathbb{R}_{\geq 0} \tag{1.4}$$

and $L_\infty := \int_0^\infty c e^{-ru-\varphi_u} \, du$.

Remark 1.4. *The proof is based on representation (1.8) which says that any stopping time w.r.t. the enlarged information can be expressed by $\mathbb{F}$- stopping times. This is an analogous result to Dellacherie, Maisonneuve, and Meyer [6], page 186, for the standard mathematical framework based on the progressive enlargement of the filtration without the default event, cf. Chapter 5 in [3].*

Proof. *Step 1.* Given a $\sigma \in \mathcal{T}_{0,\infty}$ we obviously have that $\tau(\omega, \mathbf{e}) := \sigma(\omega)$, $\forall \mathbf{e} \in \mathbb{R}_{>0}$, is an element of $\widetilde{\mathcal{T}}$ and we can calculate

$$\begin{aligned}
\mathbb{E}_s\left[X_{\tau(\omega,\mathbf{e})}(\omega,\mathbf{e})\right] &= \mathbb{E}_s\Bigg[\mathbf{1}_{\{\varphi_{\sigma(\omega)}(\omega)<\mathbf{e}\}}\Bigg(e^{-r\sigma(\omega)}\gamma S_{\sigma(\omega)}(\omega) \\
&\qquad + \int_0^{\sigma(\omega)} ce^{-ru}\,\mathrm{d}u\Bigg) + \mathbf{1}_{\{\varphi_{\sigma(\omega)}(\omega)\geq\mathbf{e}\}}\int_0^{\varphi^{-1}(\omega,\mathbf{e})} ce^{-ru}\,\mathrm{d}u\Bigg] \\
&= \mathbb{E}_s\Bigg[e^{-\varphi_{\sigma(\omega)}(\omega)}\Bigg(e^{-r\sigma(\omega)}\gamma S_{\sigma(\omega)}(\omega) + \int_0^{\sigma(\omega)} ce^{-ru}\,\mathrm{d}u\Bigg) \\
&\qquad + \int_0^{\varphi_{\sigma(\omega)}(\omega)} e^{-\xi}\int_0^{\varphi^{-1}(\omega,\xi)} ce^{-ru}\,\mathrm{d}u\,\mathrm{d}\xi\Bigg],
\end{aligned} \tag{1.5}$$

where the second equality uses that $\mathbf{e}$ is independent of $\mathcal{F}$ and exponentially distributed with parameter 1. By interchanging the order of integration and using that $u < \varphi^{-1}(\omega, \xi) \Leftrightarrow \varphi(\omega, u) < \xi$ we obtain for any $\omega \in \Omega$

$$\begin{aligned}
\int_0^{\varphi_{\sigma(\omega)}(\omega)} e^{-\xi}\int_0^{\varphi^{-1}(\omega,\xi)} ce^{-ru}\,\mathrm{d}u\,\mathrm{d}\xi &= \int_0^{\sigma(\omega)} ce^{-ru}\int_{\varphi_u(\omega)}^{\varphi_{\sigma(\omega)}(\omega)} e^{-\xi}\,\mathrm{d}\xi\,\mathrm{d}u \\
&= \int_0^{\sigma(\omega)} ce^{-ru-\varphi_u(\omega)}\,\mathrm{d}u - e^{-\varphi_{\sigma(\omega)}(\omega)}\int_0^{\sigma(\omega)} ce^{-ru}\,\mathrm{d}u.
\end{aligned}$$

Thus the rhs of (1.5) coincides with $\mathbb{E}_s\left[L_{\sigma(\omega)}(\omega)\right]$ which implies that $\sup_{\tau\in\widetilde{\mathcal{T}}}\mathbb{E}_s[X_\tau] \geq \sup_{\mathcal{T}_{0,\infty}}\mathbb{E}_s[L_\tau]$.

Step 2. To establish the opposite direction, take a $\tau \in \widetilde{\mathcal{T}}$ and let

$$\sigma(\omega) := \inf\{t \in \mathbb{Q}_{>0} \mid \tau(\omega, u) \leq t, \text{ for some } u \in \mathbb{Q}_{>0} \text{ with } \varphi_t(\omega) < u\}, \tag{1.6}$$

(recall that $\inf\emptyset = \infty$) and

$$\widetilde{\sigma}(\omega, \mathbf{e}) := \tau(\omega, \mathbf{e}) \vee \varphi^{-1}(\omega, \mathbf{e}). \tag{1.7}$$

Let us show that

$$\sigma \in \mathcal{T}_{0,\infty} \quad \text{and} \quad \tau(\omega, \mathbf{e}) = \begin{cases} \sigma(\omega) & \text{for } \varphi_{\sigma(\omega)}(\omega) < \mathbf{e} \\ \widetilde{\sigma}(\omega, \mathbf{e}) & \text{for } \varphi_{\sigma(\omega)}(\omega) \geq \mathbf{e}. \end{cases} \tag{1.8}$$

First, note that for every $t > 0$ we have

$$\{\omega \in \Omega \mid \sigma(\omega) < t\} = \bigcup_{s\in\mathbb{Q}\cap(0,t)}\ \bigcup_{u\in\mathbb{Q}_{>0}} \underbrace{\{\omega \in \Omega \mid \tau(\omega, u) \leq s \text{ and } \varphi_s(\omega) < u\}}_{\in\mathcal{F}_s\subset\mathcal{F}_t} \in \mathcal{F}_t.$$

Thus, by the usual conditions of $\mathbb{F}$, we have indeed $\sigma \in \mathcal{T}_{0,\infty}$. That for any $\mathbf{e} \in \mathbb{R}_{>0}$, $\widetilde{\sigma}(\cdot, \mathbf{e}) \in \mathcal{T}_{0,\infty}$ with $\widetilde{\sigma}(\cdot, \mathbf{e}) \geq \varphi^{-1}(\cdot, \mathbf{e})$ is obvious.

Let $(\omega, \mathbf{e}) \in \Omega \times \mathbb{R}_{>0}$ with $\mathbf{e} > \varphi_{\sigma(\omega)}(\omega)$. Let us show that

$$\tau(\omega, \mathbf{e}) = \sigma(\omega). \tag{1.9}$$

First suppose that $\sigma(\omega) = \infty$, so that $\mathbf{e} > \varphi_\infty(\omega)$ and $\varphi^{-1}(\omega, \mathbf{e}) = \infty$. From (1.6) we see that this means $\tau(\omega, u) = \infty$, $\forall u \in \mathbb{Q}_{>0} \cap (\varphi_\infty(\omega), \infty)$. If it were the case that $\tau(\omega, \mathbf{e}) < \infty$, then by (1.2) we would have $\tau(\omega, u) = \tau(\omega, \mathbf{e})$, $\forall u \in (\varphi_{\tau(\omega,\mathbf{e})}(\omega), \infty)$, but combining this with the previous sentence we would arrive at $\tau(\omega, \mathbf{e}) = \infty$. Thus (1.9) holds for $\sigma(\omega) = \infty$.

Now suppose that $\sigma(\omega) < \infty$. By definition of the infimum and the continuity of the paths of φ there is a sequence $(t_n, u_n)_{n\in\mathbb{N}} \subset \mathbb{Q}^2_{>0}$ with $t_n \downarrow \sigma(\omega)$, $\sigma(\omega) \le t_n < \varphi^{-1}(\omega, \mathbf{e})$, $\varphi_{t_n}(\omega) < u_n$ and $\tau(\omega, u_n) \le t_n$ for all $n \in \mathbb{N}$. For any $n \in \mathbb{N}$ it follows from $\varphi_{t_n}(\omega) < u_n$ and $\tau(\omega, u_n) \le t_n$ that $\tau(\omega, u_n) < \varphi^{-1}(\omega, u_n)$ and from $\tau(\omega, u_n) \le t_n$ and $t_n < \varphi^{-1}(\omega, \mathbf{e})$ that $\mathbf{e} > \varphi_{\tau(\omega,u_n)}(\omega)$. Combining these with (1.2) gives

$$\tau(\omega, u_n) = \tau(\omega, \mathbf{e}), \quad \forall n \in \mathbb{N}, \tag{1.10}$$

and since $\tau(\omega, u_n) \le t_n \downarrow \sigma(\omega)$ it follows that

$$\tau(\omega, \mathbf{e}) \le \sigma(\omega).$$

To establish the reversed inequality and thus (1.9) it is on account of (1.10) enough to show $\sigma(\omega) \le \tau(\omega, u_n)$, $\forall n \in \mathbb{N}$. If this were not true we would have an $s \in (\tau(\omega, u_n), \sigma(\omega)) \cap \mathbb{Q}$ for some $n \in \mathbb{N}$. Using this with $\sigma(\omega) \le t_n$ and $\varphi_{t_n}(\omega) < u_n$ it would follow that $\tau(\omega, u_n) \le s$ and $\varphi_s(\omega) \le \varphi_{\sigma(\omega)}(\omega) \le \varphi_{t_n}(\omega) < u_n$, which would by (1.6) result in $\sigma(\omega) \le s$ and thus a contradiction.

Finally, let $(\omega, \mathbf{e}) \in \Omega \times \mathbb{R}_{>0}$ with $\mathbf{e} \le \varphi_{\sigma(\omega)}(\omega)$. We need to show that $\tau(\omega, \mathbf{e}) \ge \varphi^{-1}(\omega, \mathbf{e})$. Assume that $\tau(\omega, \mathbf{e}) < \varphi^{-1}(\omega, \mathbf{e})$, so that we could find an $s \in \mathbb{Q}$ with

$$\varphi_{\tau(\omega,\mathbf{e})}(\omega) < \varphi_s(\omega) < \mathbf{e} \le \varphi_{\sigma(\omega)}(\omega).$$

By the first and second inequality, together with (1.2), we would have that s is in the set on the rhs of (1.6) and thus $\sigma(\omega) \le s$. But this contradicts with the last two inequalities. Thus we have established (1.8).

From (1.8) we see that if either $\varphi_{\tau(\omega,\mathbf{e})}(\omega) < \mathbf{e}$ or $\varphi_{\sigma(\omega)}(\omega) < \mathbf{e}$, then $\tau(\omega, \mathbf{e}) = \sigma(\omega)$. By this property it follows directly from the definition of X that

$$X_{\tau(\omega,\mathbf{e})}(\omega, \mathbf{e}) = X_{\sigma(\omega)}(\omega, \mathbf{e}).$$

The same calculation as in Step 1 shows that $\mathbb{E}_s\left[X_{\tau(\omega,\mathbf{e})}(\omega, \mathbf{e})\right] = \mathbb{E}_s\left[L_{\sigma(\omega)}(\omega)\right]$ and the statement of the proposition follows. $\square$

We conclude with some notation.

Definition 1.5. *(i) By $v : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ we denote the value given by the rhs of (1.3) as a function of the starting price of the stock S.*

(ii) The infinitesimal generator of S we denote by $\mathcal{L}$, that is

$$\mathcal{L} := \frac{\sigma^2}{2} s^2 \frac{\partial^2}{\partial s^2} + (r-\delta) s \frac{\partial}{\partial s}.$$

(iii) For any interval $I \subset \mathbb{R}_{>0}$ we denote by $\tau(I)$ the first exit *time of I, that is $\tau(I) := \inf\{t \geq 0 \,|\, S_t \notin I\}$.*

1.2 Constant default intensity

If the intensity function χ in (1.1) is constant, the problem (1.3) can be reduced to the case without default and a higher discount factor. This shows the following proposition. Its proof follows directly from Proposition 1.3 and [9], Theorem 4.1(i) and is therefore omitted.

Proposition 1.6. *Let $\chi(s) = q$ for some $q \in \mathbb{R}_{\geq 0}$. We denote the associated value function by $\hat{v}_q$, that is*

$$\hat{v}_q(s) := \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}_s \left[e^{-(r+q)\tau} \gamma S_\tau + \int_0^\tau c e^{-(r+q)u}\, du \right]. \tag{1.11}$$

Let $\beta_1^q < 0 < 1 < \beta_2^q$ be the solutions of $\sigma^2 \beta(\beta-1)/2 + (r-\delta)\beta - (r+q) = 0$, so that

$$\beta_1^q \beta_2^q = \frac{-2(r+q)}{\sigma^2} \qquad \text{and} \qquad (\beta_2^q - 1)(1-\beta_1^q) = \frac{2(\delta+q)}{\sigma^2}. \tag{1.12}$$

We have that the optimal stopping time in (1.11) is given by $\tau(0, \hat{s}_q)$, where

$$\hat{s}_q = \frac{\beta_2^q c}{\gamma(r+q)(\beta_2^q - 1)}$$

and furthermore

$$\hat{v}_q(s) = \begin{cases} \gamma \hat{s}_q^{1-\beta_2^q} s^{\beta_2^q} / \beta_2^q + c/(r+q) & \text{on } (0, \hat{s}_q) \\ \gamma s & \text{on } [\hat{s}_q, \infty). \end{cases}$$

Note that $q \mapsto \hat{s}_q$ is continuous and strictly decreasing with limits $\hat{s}_0$ and 0 for $q \downarrow 0$ and $q \to \infty$ respectively, and that

$$\hat{s}_q > \frac{c}{\gamma(\delta+q)}. \tag{1.13}$$

Finally, we have that the pair $(v_q|_{(0,\hat{s}_q)}, \hat{s}_q)$ is the unique solution to the free boundary problem in unknowns $(f, b) \in C^2(0,b) \times \mathbb{R}_{>0}$

$$\begin{cases} (\mathcal{L} - (r+q)) f(s) + c = 0 & \text{on } (0,b) \\ f(b-) = \gamma b, f'(b-) = \gamma \\ f(0+) \in \mathbb{R}_{>0}. \end{cases} \tag{1.14}$$

Remark 1.7. *A common approach to find analytical expressions for the value function and the optimal strategy of optimal stopping problems is to guess candidate expressions by constructing & solving an appropriate free boundary problem, which has a function and boundary point(s) as solution, and to verify the correctness of the guess by showing that the corresponding candidate value process*

(i) dominates the payoff process

(ii) is a supermartingale

(iii) is a martingale when stopped at the first time it hits the payoff process

(cf. Lemma A.1). Uniqueness of solutions of the free boundary problem follows implicitly from this.

In the upcoming sections we will work with free boundary problems that allow only for a semi-explicit characterization of its solution set. The resulting expressions are explicit enough to be useful, but showing by direct means that a solution indeed exists does not always seem easy (like for the free boundary problems involving two boundary points used in Theorem 2.2 (ii) and Theorem 3.3 (ii)). This issue we resolve by proving in the upcoming Subsection 1.3 that v satisfies a set of properties rich enough to allow to conclude that v and the associated optimal exercise level(s) indeed form a solution to the free boundary problem under consideration, thus implicitly yielding existence of solutions.

1.3 Some results for general intensity functions

The following theorem states some properties of v, mainly for use in the examples we consider in the upcoming sections. Note that the sign of the function λ defined below corresponds to the sign of the drift rate in the Itô-decomposition of L and will be used throughout for determining the shape of stopping and continuation regions, using (ii) and (iv) of Theorem 1.9.

Remark 1.8. *As $\lim_{t\to\infty} L_t$ exists a.s. and $\tau \in \mathcal{T}_{0,\infty}$ may take the value $+\infty$, the standard theory of optimal stopping on a compact time interval can directly be translated to our setting. Especially, as L has continuous paths and is of class (D), we already know that the $[0,\infty]$-valued stopping time $\inf\{t \geq 0 \,|\, U_t = L_t\}$ is optimal, where U denotes the Snell envelope of L, cf. the proof of Theorem 1.9 (i).*

Theorem 1.9. *Let the function $\lambda : \mathbb{R}_{>0} \to \mathbb{R}$ be given by $\lambda(s) = c - \gamma(\delta + \chi(s))s$. We have the following.*

(i) v is a continuous function with $\gamma s \leq v(s) \leq \hat{v}_0(s)$ on $\mathbb{R}_{>0}$. The optimal stopping time is attained and given by $\tau^* := \tau(\mathcal{C})$, where $\mathcal{C} = \{s \in \mathbb{R}_{>0} \,|\, v(s) > \gamma s\}$ is the continuation region. *Let $\mathcal{S} = \mathbb{R}_{>0} \setminus \mathcal{C}$ be the* stopping region. *We have $\mathcal{C} \subset (0, \hat{s}_0)$. Furthermore, suppose that $(\chi_n)_{n\in\mathbb{N}}$ is a sequence of intensity functions, with*

associated value functions denoted by v_n, converging to χ in the max-norm. Then v_n converges to v in the max-norm.

*(ii) Let $I \subset \mathbb{R}_{>0}$ be some interval**. *If $\lambda \leq 0$ on I and $\partial I \subset \mathcal{S}$, then $\bar{I} \subset \mathcal{S}$. If $\lambda > 0$ on I, then $I \subset \mathcal{C}$.*

Now suppose that χ is càdlàg or càglàd and that its set of discontinuities, denoted by D_χ, is finite. Suppose furthermore that $\partial\mathcal{C}$ is finite, i.e. that $\mathcal{C}$ is a finite union of open intervals (from (ii) we see that a sufficient condition for this is that λ changes its sign at most finitely often). Under these assumptions the following holds.

(iii) Set $N_v := (\mathcal{C} \cap D_\chi) \cup \partial\mathcal{C}$. We have that $v \in C^2(\mathbb{R}_{>0} \setminus N_v) \cap C^1(\mathbb{R}_{>0})$ and v satisfies

$$(\mathcal{L} - (r + \chi(s)))v(s) + c \begin{cases} = 0 & \text{on } \mathcal{C} \setminus D_\chi \\ \leq 0 & \text{on } \mathbb{R}_{>0} \setminus N_v. \end{cases}$$

(iv) Let $s_0 \in \mathbb{R}_{>0}$. Suppose that there exists $\epsilon > 0$ such that $\lambda \in C^1(s_0, s_0 + \epsilon)$ and that either $\lambda(s_0+) > 0$ or both $\lambda(s_0+) = 0$ and $\lambda'(s_0+) > 0$. Then $s_0 \in \mathcal{C}$. The same holds if $\lambda \in C^1(s_0 - \epsilon, s_0)$ and either $\lambda(s_0-) > 0$ or both $\lambda(s_0-) = 0$ and $\lambda'(s_0-) < 0$.

Proof. Ad (i). The lower and upper bound for v are obvious. Since $(\exp(-(r-\delta)t)S_t)_{t\geq 0}$ is a martingale and $\delta > 0$, it follows that L is of class (D), i.e. that the family $\{L_\tau \mid \tau \in \mathcal{T}_{0,\infty}\}$ is uniformly integrable. It follows that the Snell envelope U of L is well defined and of class (D), cf. [13], Theorem 3.2 e.g. For any $t \geq 0$ we have

$$\begin{aligned} U_t &= \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,\infty}} \mathbb{E}_s \left[L_\tau \mid \mathcal{F}_t \right] \\ &= \int_0^t c e^{-ru-\varphi_u} \, \mathrm{d}u + e^{-rt-\varphi_t} \\ &\qquad \times \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,\infty}} \mathbb{E}_s \left[e^{-r(\tau-t)-(\varphi_\tau-\varphi_t)} \gamma S_\tau + \int_t^\tau c e^{-r(u-t)-(\varphi_u-\varphi_t)} \, \mathrm{d}u \,\middle|\, \mathcal{F}_t \right] \\ &= \int_0^t c e^{-ru-\varphi_u} \, \mathrm{d}u + e^{-rt-\varphi_t} v(S_t). \end{aligned} \tag{1.15}$$

The above calculation is at least intuitively clear by the Markov property, for a rigorous justification we refer to Theorem 3.4 in [7]. Although the authors work with a payoff of the form $g(X_t)$ for a suitable function g and a Markov process X it also covers this case if we regard L as a function of the Markov process $(t, S_t, \varphi_t, \int_0^t \exp(-ru - \varphi_u) \, \mathrm{d}u)_{t\geq 0}$. Namely, the resulting four-dimensional value function has the form of the rhs of equation (1.15).

*For sets $A \subset \mathbb{R}_{>0}$, ∂A denotes the boundary of A in $\mathbb{R}_{>0}$, i.e. if $A = (a, b)$ with $a \in \mathbb{R}_{\geq 0}$ and $b \in \mathbb{R}_{>0} \cup \{+\infty\}$ then $\partial A = \{a, b\} \cap \mathbb{R}_{>0}$. Furthermore the closure of A in $\mathbb{R}_{>0}$ is denoted by $\bar{A}$, i.e. $\bar{A} = A \cup \partial A$.

Continuity of v follows from Proposition 4.7 in [7]. From general theory on optimal stopping, see Theorem 5.5 in [13] e.g., together with (1.15) it follows that the optimal stopping time in v is attained and given by $\inf\{t \geq 0 \,|\, U_t = L_t\} = \tau(\mathcal{C})$.

Let χ_n tend to χ in the max-norm as $n \to \infty$, denote by ϵ_n the max-norm of $\chi - \chi_n$. Since $\gamma s \leq v(s) \leq \hat{v}_0(s)$ we have $v_n(s) = v(s) = \gamma s$ on $[\hat{s}_0, \infty)$ and we may restrict the set of stopping times over which is maximized in v and v_n to those that are bounded above by $\tau(0, \hat{s}_0)$ on account of $\tau(\mathcal{C}) \leq \tau(0, \hat{s}_0)$. Using this we find by some easy calculations that $|v(s) - v_n(s)| \leq \gamma \hat{s}_0 C(\epsilon_n) + \int_0^\infty c e^{-ru}(1 - e^{-\epsilon_n u})\, du$ for any $s \in (0, \hat{s}_0)$, where $C(\epsilon_n)$ is the maximum value the function $x \mapsto e^{-rx}(1 - e^{-\epsilon_n x})$ attains on $(0, \hat{s}_0]$, yielding the result.

Ad (ii). An application of Itô's formula yields

$$L_t = \gamma s + \int_0^t e^{-ru-\varphi_u} \gamma \sigma S_u \, dW_u + \int_0^t e^{-ru-\varphi_u} \lambda(S_u)\, du. \tag{1.16}$$

Let $s_0 \in I$. First let $\lambda \leq 0$ on I and $\partial I \subset \mathcal{S}$. By (1.15), using that $v(s) = \gamma s$ on ∂I, we find that we may write

$$v(s_0) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}_{s_0}\left[L_\tau^{\tau(I)}\right]. \tag{1.17}$$

Since $\lambda \leq 0$ on I, (1.16) shows that $L^{\tau(I)}$ is a local supermartingale. Since L is of class (D), it follows by Doob's optional sampling that the supremum in (1.17) is attained by $\tau = 0$ and thus indeed $v(s_0) = \gamma s_0$.

Next let $\lambda > 0$ on I. Note that this implies that I is bounded from above since $\lambda \leq 0$ on $[c/(\delta\gamma), \infty)$. It follows that the local martingale part of $L^{\tau(I)}$ in (1.16) is a true martingale. This allows to take any $t > 0$ and use again Doob's optional sampling together with $\lambda > 0$ on I and $\mathbb{P}_{s_0}(\tau(I) > 0) = 1$ to deduce that $v(s_0) \geq \mathbb{E}_{s_0}[L_{t \wedge \tau(I)}] > \gamma s_0$.

Ad (iii). *Step 1.* Note that $\mathcal{C} \setminus D_\chi$ is open in $\mathbb{R}_{>0}$ by continuity of v and since D_χ is finite. Let us show that on this set, v is a C^2-function satisfying $(\mathcal{L} - (r + \chi(s)))v(s) + c = 0$.

For this, take some environment $I = (a, b) \subset \mathcal{C} \setminus D_\chi$ with $a > 0$, $b < \infty$. By the assumptions on χ and since $I \cap D_\chi = \emptyset$ we have $\chi \in C^0(I)$. First consider the homogenous boundary value problem

$$\begin{cases} (\mathcal{L} - (r + \chi(s)))f(s) = 0 & \text{on } I \\ f = 0 & \text{on } \partial I \end{cases} \tag{1.18}$$

and let us show that it only has the trivial solution. Let $f \in C^2(I)$ be any solution and consider the continuous process Z given by $Z_t = \exp(-r(t \wedge \tau(I)) - \varphi_{t \wedge \tau(I)}) f(S_{t \wedge \tau(I)})$ for all $t \geq 0$. Itô's formula shows that Z is a local martingale. Clearly, Z is also a bounded process so that Doob's optional sampling shows that indeed $f(s) = \mathbb{E}_s[Z_0] = \mathbb{E}_s[Z_{\tau(I)}]$ on I, the rhs vanishing on account of $f = 0$ on ∂I.

By the Fredholm Alternative, the fact that (1.18) is only solved by the trivial solution implies that the boundary value problem

$$\begin{cases} (\mathcal{L}-(r+\chi(s)))f(s)+c=0 & \text{on } I \\ f=v & \text{on } \partial I \end{cases}$$

has a solution $f\in C^2(I)$. An application of Lemma A.1 (i) yields for all $s\in I$, using that $\mathbb{P}_s(\tau(I)<\infty)=1$,

$$f(s)=\mathbb{E}_s\left[e^{-r\tau(I)-\varphi_{\tau(I)}}v(S_{\tau(I)})+\int_0^{\tau(I)}ce^{-ru-\varphi_u}\,\mathrm{d}u\right].$$

Since $I\subset\mathcal{C}$, (1.15) shows that for any $s\in I$, $v(s)$ can be written as the rhs of the above formula. Thus $v=f$ on I, yielding the assertion.

Step 2. Let us show that $v\in C^2(\mathbb{R}_{>0}\setminus N_v)\cap C^1(\mathbb{R}_{>0})$. Recall that $N_v=(\mathcal{C}\cap D_\chi)\cup\partial\mathcal{C}$ is by assumption finite, let $a\in N_v$ and $\epsilon>0$ so that with $I:=(a-\epsilon,a+\epsilon)$ we have $I\cap N_v=\{a\}$. Since $v\in C^2(\mathcal{C}\setminus D_\chi)$ (by Step 1) and $v(s)=\gamma s$ on $\mathcal{S}=\mathbb{R}_{>0}\setminus\mathcal{C}$, the assertion follows if we show that $v\in C^1(I)$. In particular, since we already have

$$v\in C^2(a-\epsilon,a]\cup C^2[a,a+\epsilon), \tag{1.19}$$

we know that $v'(a-)$ and $v'(a+)$ both exist and it remains to show that they must coincide. To see (1.19), by construction of I both $(a-\epsilon,a)$ and $(a,a+\epsilon)$ are subsets of either $\mathcal{S}$ or $\mathcal{C}\setminus D_\chi$. Since $v(s)=\gamma s$ on $\mathcal{S}$ there is nothing to show for that case, while on subsets of $\mathcal{C}\setminus D_\chi$ we have from Step 1 that v satisfies $(\mathcal{L}-(r+\chi(s)))v(s)+c=0$, so that by a standard result from the theory of ODEs (cf. e.g. [10], Ch. II, Theorem 1.1 and Theorem 3.1) it follows from the fact that $\chi(a\pm)$ exists and is finite (by our assumption on χ) that also the corresponding $v''(a\pm)$ exists and is finite.

So let us show that $v'(a-)$ and $v'(a+)$ are equal. Recall that the Snell envelope U can be expressed as (1.15). On account of (1.19) we may apply the change-of-variables formula (A.1) from the proof of Lemma A.1 (cf. the remarks preceding that formula) and we obtain

$$\begin{aligned} U_t^{\tau(I)} &= U_0+M_t+\int_0^{t\wedge\tau(I)}\mathbf{1}_{\{S_u\neq a\}}e^{-ru-\varphi_u}\left[(\mathcal{L}-(r+\chi(S_u)))\,v(S_u)+c\right]\,\mathrm{d}u \\ &\quad+\int_0^{t\wedge\tau(I)}\mathbf{1}_{\{S_u=a\}}e^{-ru-\varphi_u}\left(v'(a+)-v'(a-)\right)\,\mathrm{d}L_u^a,\quad t\geq 0, \end{aligned}$$

where M is given by

$$M_t=\frac{\sigma}{2}\int_0^{t\wedge\tau(I)}e^{-ru-\varphi_u}\left(v'(S_u+)+v'(S_u-)\right)S_u\,dW_u,\quad t\geq 0.$$

Note that M is a true martingale on account of the boundedness of I and of v' on I.

Let us start S at a. First consider $a\in\mathcal{C}\cap D_\chi$. By construction we have $I\subset\mathcal{C}$, so that $(\mathcal{L}-(r+\chi(s)))v(s)+c=0$ on $I\setminus\{a\}$ by Step 1. This means that in the above decomposition of $U^{\tau(I)}$ the drift part consists solely of the integral with respect to local time. Thus if we had $v'(a-) \neq v'(a+)$, then $U^{\tau(I)}$ would be a strict super- or submartingale. But this is impossible since $U^{\tau(I)}$ is a martingale, which follows directly from the well known fact that U^{τ^*} is a martingale (see e.g. [13], Corollary 5.3) and $\tau(I) \leq \tau^*$ on account of $I \subset \mathcal{C}$.

Next consider $a \in \partial\mathcal{C} \subset \mathcal{S}$. Since $v(a) = \gamma a$ while $v(s) \geq \gamma s$ on I we have $v'(a-) \leq v'(a+)$. So if $v'(a-) = v'(a+)$ did not hold, then $v'(a+) - v'(a-) > 0$. But the above decomposition of U shows that this would mean that the process Z given by $Z_t = \int_0^t \mathbf{1}_{\{S_u=a\}} dU_t^{\tau(I)}$ for all $t \geq 0$ is a strict submartingale. However, this is impossible since it is well known that U is a supermartingale, see e.g. [13], Lemma 3.3.

Step 3. Let us show that $(\mathcal{L} - (r + \chi(s)))v(s) + c \leq 0$ on $\mathbb{R}_{>0} \setminus N_v$ (which is an open set since N_v is finite). Taking the result from Step 1 into account, it is enough to show that $(\mathcal{L} - (r + \chi(s)))v(s) + c \leq 0$ on the inner of $\mathcal{S}$ (denoted by $\mathrm{inn}(\mathcal{S})$). This however is clear. Namely, since $v(s) = \gamma s$ on $\mathcal{S}$ we have for any $s \in \mathrm{inn}(\mathcal{S})$ that $(\mathcal{L} - (r + \chi(s)))v(s) + c = \lambda(s)$. Suppose that we had $\lambda(s) > 0$. By the assumptions on χ we would have $\lambda > 0$ on either $(s - \epsilon, s)$ or $(s, s + \epsilon)$ for some $\epsilon > 0$, but this means by (ii) that $s \in \bar{\mathcal{C}}$, which contradicts with $s \in \mathrm{inn}(\mathcal{S})$.

Ad (iv). Set $I := (s_0, s_0 + \epsilon)$. Let us assume that $s_0 \in \mathcal{S}$ and derive a contradiction. We have either Case 1: $\lambda(s_0+) > 0$ or Case 2: $\lambda(s_0+) = 0$ and $\lambda'(s_0+) > 0$. Since $\lambda \in C^1(I)$ (and thus also $\chi \in C^1(I)$) we may assume w.l.o.g. that $\lambda > 0$ on I, which means that $I \subset \mathcal{C}$ (cf. (ii)) and that $(\mathcal{L} - (r + \chi(s)))v(s) + c = 0$ holds on I (cf. (iii)), with, since we assumed $s_0 \in \mathcal{S}$, $v(s_0+) = \gamma s_0$ and $v'(s_0+) = \gamma$.

For Case 1, taking the limit for $s \downarrow s_0$ in $(\mathcal{L} - (r + \chi(s)))v(s) + c = 0$ we find, using $v(s_0+) = \gamma s_0$, $v'(s_0+) = \gamma$ and $\lambda(s_0+) > 0$, that $v''(s_0+) < 0$. But again using $v(s_0+) = \gamma s_0$ and $v'(s_0+) = \gamma$ this would imply $v(s) < \gamma s$ on $(s_0, s_0 + \epsilon')$ for some $\epsilon' > 0$, yielding the required contradiction.

For Case 2, taking the same limit as in Case 1 this time yields $v''(s_0+) = 0$ on account of $\lambda(s_0+) = 0$. On I, differentiating the equation $(\mathcal{L} - (r + \chi(s)))v(s) + c = 0$ once (which is possible since $\chi \in C^1(I)$) we find

$$\frac{\sigma^2}{2} s^2 v'''(s) = (\delta - r - \sigma^2) s v''(s) + (\chi(s) + \delta) v'(s) + \chi'(s) v(s).$$

Furthermore $\lambda'(s) = -(\delta + \chi(s))\gamma - \chi'(s)\gamma s$, so we may take the limit for $s \downarrow s_0$ in the above equation and use $v(s_0+) = \gamma s_0$, $v'(s_0+) = \gamma$, $v''(s_0+) = 0$ and $\lambda'(s_0+) > 0$ to derive that $v'''(s_0+) < 0$. But this would again imply $v(s) < \gamma s$ on $(s_0, s_0 + \epsilon')$ for some $\epsilon' > 0$ and yield a contradiction.

□

Remark 1.10. *Theorem 1.9 (iii) shows that v is C^1 across the points in $\partial\mathcal{C}$ and $\mathcal{C} \cap D_\chi$. For $\partial\mathcal{C}$ this is just the usual smooth pasting condition at the boundary between continuation and stopping region. For $s_0 \in \mathcal{C} \cap D_\chi$ we may use the differential equation that governs v around s_0 to compute $v''(s_0+) - v''(s_0-) = 2v(s_0)(\chi(s_0+) - \chi(s_0-))/\sigma^2$. That is to say* *that a jump of χ at s_0 causes a jump of v'' in the same direction, but it does not affect the continuity of v'.*

2 Piecewise constant intensity function

In this section we will address in more detail the case that χ is given by $\chi(s) = \mathbf{1}_{\{s \leq \bar{s}\}} p$ for parameters $p, \bar{s} > 0$. The process φ from (1.1) is now given by

$$\varphi_t = \int_0^t p \mathbf{1}_{\{S_u \leq \bar{s}\}} \, \mathrm{d}u$$

for all $t \geq 0$ and we denote the associated value function by v_p, that is

$$v_p(s) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}_s[L_\tau] = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}_s \left[e^{-r\tau - \varphi_\tau} \gamma S_\tau + \int_0^\tau c e^{-ru - \varphi_u} \, \mathrm{d}u \right],$$

with continuation and stopping regions denoted by $\mathcal{C}_p$ and $\mathcal{S}_p$, resp.

Throughout we will make repeated use of the functions $\hat{v}_q$ and associated optimal stopping levels $\hat{s}_q$ which were discussed in Proposition 1.6. Note that for any $p > 0$, from Theorem 1.9 (i) & (iii) and $\chi(s) \leq p$ we know that v_p is a non-decreasing $C^1(\mathbb{R}_{>0})$-function with $\hat{v}_p \leq v_p \leq \hat{v}_0$.

The drift rate λ of L takes the form $\lambda(s) = c - \gamma s(\delta + p\mathbf{1}_{\{s \leq \bar{s}\}})$. If $\lambda(\bar{s}+) \leq 0$, i.e. $\bar{s} \geq c/(\gamma\delta)$, $\mathcal{S}_p$ has the same structure as the optimal stopping region of $\hat{v}_p$, see Theorem 2.1 below.

On the other hand, if $\lambda(\bar{s}+) > 0$, i.e. $\bar{s} \in (0, c/(\gamma\delta))$, λ is strictly positive on $(0, c/(\gamma(\delta + p))) \cup (\bar{s}, c/(\gamma\delta))$, which for p large enough causes $\mathcal{S}_p$ to be the union of two disjoint intervals, one contained in $(c/(\gamma(\delta + p)), \bar{s})$ and one contained in $(c/(\gamma\delta), \hat{s}_0)$. See Theorem 2.2 below, resp. Figures 1 & 2 in Section 4.

Theorem 2.1. *Suppose that $\bar{s} \geq c/(\gamma\delta)$. For $q \in \mathbb{R}_{\geq 0}, b \in \mathbb{R}_{>0}$ let*

$$c_1^q(b) := \frac{(\beta_2^q - 1)\gamma b - \beta_2^q c/(r+q)}{(\beta_2^q - \beta_1^q) b^{\beta_1^q}} \quad \text{and} \quad c_2^q(b) = \frac{(\beta_1^q - 1)\gamma b - \beta_1^q c/(r+q)}{(\beta_1^q - \beta_2^q) b^{\beta_2^q}},$$

where β_1^q, β_2^q are defined like $\hat{s}_p$ and $\hat{v}_p$ in Proposition 1.6. We have

(i) If $\hat{s}_p \leq \bar{s}$, then $v_p = \hat{v}_p$.

(ii) If $\hat{s}_p > \bar{s}$, then $\mathcal{S}_p = [b_p, \infty)$, where $b_p \in (\hat{s}_p, \hat{s}_0)$ is the unique solution on $(\bar{s}, \infty)$ of the following equation in b

$$(\beta_1^0 - \beta_2^p) c_1^0(b) \bar{s}^{\beta_1^0} + (\beta_2^0 - \beta_2^p) c_2^0(b) \bar{s}^{\beta_2^0} - \frac{\beta_2^p c p}{r(r+p)} = 0 \tag{2.1}$$

and

$$v_p(s) = \begin{cases} \left(c_1^0(b_p)\bar{s}^{\beta_1^0} + c_2^0(b_p)\bar{s}^{\beta_2^0} + cp/(r(r+p))\right)(s/\bar{s})^{\beta_2^p} + c/(r+p) & \textit{on } (0,\bar{s}) \\ c_1^0(b_p)s^{\beta_1^0} + c_2^0(b_p)s^{\beta_2^0} + c/r & \textit{on } [\bar{s}, b_p) \\ \gamma s & \textit{on } [b_p,\infty). \end{cases} \tag{2.2}$$

Proof. Ad (i). Recall from Proposition 1.6 that the free boundary system (1.14) has a unique solution (f_*, b_*), with $b_* = \hat{s}_p$ and $f_* = \hat{v}_p|_{(0,b_*)}$, and that by extending f_* by setting $f_*(s) = \gamma s$ on $[b_*,\infty)$ we get $f_* \in C^2(\mathbb{R}_{>0} \setminus \{b_*\}) \cap C^1(\mathbb{R}_{>0})$ and $f_* = \hat{v}_p$. Let us show that $v_p = f_*$.

By assumption we have $b_* = \hat{s}_p \leq \bar{s}$ and thus $\chi(s) = p$ on $(0, b_*)$, which yields by (1.14)

$$(\mathcal{L} - (r + \chi(s)))f_*(s) + c = 0 \quad \text{on } (0, b_*), \tag{2.3}$$

while on (b_*,∞) a direct calculation with $f_*(s) = \gamma s$ and $\hat{s}_p > c/(\gamma(\delta + p))$ (cf. (1.13)) gives

$$(\mathcal{L} - (r + \chi(s)))f_*(s) + c = \lambda(s) \leq 0 \quad \text{on } (b_*,\infty). \tag{2.4}$$

Applying Lemma A.1 (i), thereby using (2.3) and $f_*(s) = \gamma s$ on $[b_*,\infty)$, and Lemma A.1 (ii), thereby using (2.3), (2.4) and $f_*(s) = \hat{v}_p(s) \geq \gamma s$ on $\mathbb{R}_{>0}$, we find that $f_*(s) = \sup_{\tau\in\mathcal{T}_{0,\infty}} \mathbb{E}_s[L_\tau]$ on $\mathbb{R}_{>0}$. Thus indeed $v_p = f_*$.

Ad (ii). From $\hat{v}_p \leq v_p \leq \hat{v}_0$, $\hat{s}_p > \bar{s}$ and λ being negative on $(\bar{s},\infty)$ it follows with Theorem 1.9 (ii) that $\mathcal{S}_p = [b_p,\infty)$ for some $b_p \in [\hat{s}_p, \hat{s}_0]$.

Step 1. From Theorem 1.9 (iii) it follows that the pair $(v_p|_{(0,b_p)}, b_p)$ solves the following free boundary problem in unknowns $(f, b) \in C^2((0, b) \setminus \{\bar{s}\}) \cap C^1(0, b) \times (\bar{s},\infty)$.

$$\begin{cases} f(0+) \in \mathbb{R}_{\geq 0} & \\ (\mathcal{L} - (r + p))f(s) + c = 0 & \text{on } (0,\bar{s}) \\ f(\bar{s}-) = f(\bar{s}+), f'(\bar{s}-) = f'(\bar{s}+) & \\ (\mathcal{L} - r)f(s) + c = 0 & \text{on } (\bar{s}, b) \\ f(b-) = \gamma b, f'(b-) = \gamma. & \end{cases}$$

Let us show that this system has in fact a unique solution (f_*, b_*), with b_* equal to the unique solution to (2.1) and f_* given by the first two lines in the rhs of (2.2). Clearly, for any $b > \bar{s}$, f_b solves both differential equations in the above system iff

$$f_b(s) = \begin{cases} C_1 s^{\beta_1^p} + C_2 s^{\beta_2^p} + c/(r + p) & \text{on } (0,\bar{s}) \\ C_3 s^{\beta_1^0} + C_4 s^{\beta_2^0} + c/r & \text{on } (\bar{s}, b) \end{cases}$$

for constants $C_1,\ldots,C_4$. Since $\beta_1^p < 0 < \beta_2^p$, we have $f_b(0+) \in \mathbb{R}_{\geq 0}$ iff $C_1 = 0$. Furthermore some straightforward calculations show that the four boundary conditions at $s = \bar{s}$ and

$s = b$ translate into explicit expressions for $C_2 = C_2(b), \dots, C_4 = C_4(b)$ in terms of b and the requirement that b solves (2.1). Using the identities (1.12), differentiating the lhs of (2.1) with respect to b yields the expression

$$\frac{2(\delta\gamma - cb^{-1})}{\sigma^2(\beta_2^0 - \beta_1^0)} \left[(\beta_1^0 - \beta_2^p) \left(\frac{\bar{s}}{b}\right)^{\beta_1^0} - (\beta_2^0 - \beta_2^p) \left(\frac{\bar{s}}{b}\right)^{\beta_2^0} \right],$$

and on account of $\beta_2^p > 0$, $\beta_1^0 < 0 < 1 < \beta_2^0$ and $\bar{s} \geq c/(\delta\gamma)$ this quantity is strictly negative for $b \in (\bar{s}, \infty)$, thus it follows that (2.1) can have at most one solution on $(\bar{s}, \infty)$. So (f_*, b_*) is indeed uniquely determined with $b_* = b_p$. Plugging $b_* = b_p$ in the above expressions for $C_2(b), \dots, C_4(b)$ shows that $f_* = f_{b_p}$ is indeed given by the formulae in the rhs of (2.2).

Step 2. We noted above already that $b_p \in [\hat{s}_p, \hat{s}_0]$. It remains to show that $b_p < \hat{s}_0$ and $b_p > \hat{s}_p$. To see the former, note that from Proposition 1.6 and Step 1 we have that the pairs $(\hat{v}_0|_{(\bar{s},\hat{s}_0)}, \hat{s}_0)$ and $(v_p|_{(\bar{s},b_p)}, b_p)$ both solve the following system in unknowns $(f, b) \in C^2(\bar{s}, b) \times (\bar{s}, \infty)$

$$\begin{cases} (\mathcal{L} - r)f(s) + c = 0 & \text{on } (\bar{s}, b) \\ f(b-) = \gamma b, f'(b-) = \gamma. \end{cases}$$

If we fix some $b \in (\bar{s}, \infty)$, the corresponding f in the above system is obviously uniquely determined. This means that if we had $b_p = \hat{s}_0 =: b_*$, then also $v_p = \hat{v}_0$ on $(\bar{s}, b_*)$, but this is clearly impossible since for any $s \in (\bar{s}, b_*)$, there is a positive $\mathbb{P}_s$-probability that S spends a Lebesgue positive amount of time in $(0, \bar{s})$ before reaching the optimal stopping level b_*, implying $v_p(s) < \hat{v}_0(s)$. Thus $b_p < \hat{s}_0$.

To see $b_p > \hat{s}_p$, suppose that we had $b_p = \hat{s}_p =: b_*$. From Step 1 we know that v_p satisfies $(\mathcal{L} - r)v_p(s) + c = 0$ on $(\bar{s}, b_*)$ with $v_p(b_*-) = \gamma b_*$ and $v_p'(b_*-) = \gamma$ while from Proposition 1.6 we know that $\hat{v}_p$ satisfies $(\mathcal{L} - (r+p))\hat{v}_p(s) + c = 0$ on $(0, b_*)$ with $\hat{v}_p(b_*-) = \gamma b_*$ and $\hat{v}_p'(b_*-) = \gamma$. Taking the limit for $s \uparrow b_*$ in both differential equations and making use of the mentioned boundary conditions at $s = b_*-$, it readily follows on account of the different potentials that $v_p''(b_*-) < \hat{v}_p''(b_*)$. This however means, taking into account that by the boundary conditions v_p and $\hat{v}_p$ and their derivatives coincide at $s = b_*-$, that $\hat{v}_p > v_p$ on $(b_* - \epsilon, b_*)$ for some $\epsilon > 0$, thus yielding a contradiction to $\hat{v}_p \leq v_p$. $\square$

Theorem 2.2. *Suppose that $\bar{s} \in (0, c/(\delta\gamma))$. There exists a unique $\bar{p} \in (0, \infty)$ with $\hat{s}_{\bar{p}} \in (0, \bar{s})$ such that the following holds.*

(i) For $p \in (0, \bar{p})$ we have $\mathcal{S}_p = [b_p, \infty)$, where $b_p \in (c/(\delta\gamma) \vee \hat{s}_p, \hat{s}_0)$ is the unique solution of equation (2.1) on $(c/(\delta\gamma), \infty)$ and v_p is given by the rhs of (2.2).

(ii) For $p \in [\bar{p}, \infty)$ we have $\mathcal{S}_p = [\hat{s}_p, a_p] \cup [b_p, \infty)$, with $a_{\bar{p}} = \hat{s}_{\bar{p}}$, where the pair (a_p, b_p) is the unique solution of the following system of equations in unknowns $(a, b) \in$

$[\hat{s}_p, \bar{s}) \times (c/(\delta\gamma), \hat{s}_0)$

$$c_1^p(a)\bar{s}^{\beta_1^p} + c_2^p(a)\bar{s}^{\beta_2^p} + c/(r+p) = c_1^0(b)\bar{s}^{\beta_1^0} + c_2^0(b)\bar{s}^{\beta_2^0} + c/r \tag{2.5}$$

$$\beta_1^p c_1^p(a)\bar{s}^{\beta_1^p} + \beta_2^p c_2^p(a)\bar{s}^{\beta_2^p} = \beta_1^0 c_1^0(b)\bar{s}^{\beta_1^0} + \beta_2^0 c_2^0(b)\bar{s}^{\beta_2^0} \tag{2.6}$$

and

$$v_p(s) = \begin{cases} \hat{v}_p(s) & \text{on } (0, \hat{s}_p) \\ \gamma s & \text{on } [\hat{s}_p, a_p]. \\ c_1^p(a_p)s^{\beta_1^p} + c_2^p(a_p)s^{\beta_2^p} + c/(r+p) & \text{on } (a_p, \bar{s}] \\ c_1^0(b_p)s^{\beta_1^0} + c_2^0(b_p)s^{\beta_2^0} + c/r & \text{on } (\bar{s}, b_p) \\ \gamma s & \text{on } [b_p, \infty). \end{cases} \tag{2.7}$$

Proof. Let us prove the assertion for

$$\bar{p} := \inf\{p > 0 \,|\, \mathcal{S}_p \cap (0, \bar{s}) \neq \emptyset\}. \tag{2.8}$$

Obviously, there can be at most one $\bar{p}$ for which (i) and (ii) hold both. Since $v_p \geq \hat{v}_p$ and $\hat{s}_0 > c/(\delta\gamma) > \bar{s}$ (by assumption and (1.13)) we have $\bar{p} > 0$ and $\hat{s}_{\bar{p}} \in (0, \bar{s})$. Obviously $\bar{p} < \infty$.

Ad (i). Since $\mathcal{S}_p \cap (0, \bar{s}) = \emptyset$ and $\lambda > 0$ on $(\bar{s}, c/(\delta\gamma))$ we have by Theorem 1.9 (ii) & (iv) that $\mathcal{S}_p \cap (0, c/(\delta\gamma)] = \emptyset$. Furthermore, since $\lambda < 0$ on $(c/(\delta\gamma), \infty)$ we get by again Theorem 1.9 (ii) and $\hat{v}_p \leq v_p \leq \hat{v}_0$ that $\mathcal{S}_p = [b_p, \infty)$ for some $b_p \in [\hat{s}_p, \hat{s}_0]$ with $b_p > c/(\delta\gamma)$. For the remaining statements of (i) the same proof as for Theorem 2.1 (ii) applies.

Ad (ii). *Step 1.* Take some $s_0 \in \mathcal{S}_p \cap (0, \bar{s})$, which is non-empty by the assumption (note that, due to the continuity of $p \mapsto v_p$ in the max-norm (cf. Theorem 1.9 (i)), the infimum in (2.8) is attained). From $v_p(s_0) = \gamma s_0$ and $s_0 \leq \bar{s}$, it follows that

$$v_p(s) = \hat{v}_p(s), \quad \forall s \leq s_0. \tag{2.9}$$

Namely, starting S at $s \in (0, s_0]$ the process never enter the default-free region $(\bar{s}, \infty)$ when optimally stopped at $\mathcal{S}_p$. Thus the optimal payoff of $v_p(s_0)$ coincides with the corresponding payoff of the potentially smaller claim when default occurs everywhere with rate p. Now, (2.9) means that $\mathcal{S}_p \cap (0, s_0] = [\hat{s}_p, s_0]$. Since s_0 is an arbitrary element of $\mathcal{S}_p \cap (0, \bar{s})$, this implies in particular that $\mathcal{S}_p \cap (0, \bar{s})$ is an interval. Since $\lambda(\bar{s}+) > 0$ we have by Theorem 1.9 (iv) that $\bar{s} \notin \mathcal{S}_p$, which also means that we have $\mathcal{S}_p \cap (0, \bar{s}) = \mathcal{S}_p \cap (0, \bar{s}]$, the latter being closed in $\mathbb{R}_{>0}$. In conclusion we arrive at $\mathcal{S}_p \cap (0, \bar{s}) = [\hat{s}_p, a_p]$ for some $a_p \in [\hat{s}_p, \bar{s})$.

Furthermore, since $\lambda > 0$ on $(\bar{s}, c/(\delta\gamma))$, $\lambda < 0$ on $(c/(\delta\gamma), \infty)$ and $v_p \leq \hat{v}_0$ we get from Theorem 1.9 (ii) & (iv) that $\mathcal{S}_p \cap [\bar{s}, \infty) = [b_p, \infty)$ for some $b_p \in (c/(\delta\gamma), \hat{s}_0]$. The same argument as in Step 2 in the proof of Theorem 2.1 (ii) shows that $b_p < \hat{s}_0$. We have thus established the shape of $\mathcal{S}_p$ as in the statement.

Step 2. Let us show that $a_{\bar p} = \hat s_{\bar p}$, i.e. that $\mathcal{S}_{\bar p} \cap (0, \bar s)$ consists of a single point. From (i) we know that for $p \in (0, \bar p)$ we have $v_p(s) = C_p s^{\beta_2^p} + c/(r+p)$ on $(0, \bar s)$. By the continuity of $p \mapsto \beta_2^p$ and of $p \mapsto v_p$ in max-norm (cf. Theorem 1.9 (i)), C_p has a limit value $C_{\bar p}$ as $p \to \bar p$ and we can write $v_{\bar p}(s) = C_{\bar p} s^{\beta_2^{\bar p}} + c/(r+\bar p)$ on $(0, \bar s)$. Clearly, $C_{\bar p}$ has to be strictly positive. Since $\beta_2^{\bar p} > 1$, this formula shows that $v_{\bar p}$ can touch $s \mapsto \gamma s$ in only a single point.

Step 3. Finally, let us establish the stated formulae for (a_p, b_p) and $v_p|_{(a_p,b_p)}$. Consider the following free boundary problem in unknowns $(f, a, b) \in C^2((a, \bar s) \cup (\bar s, b)) \cap C^1(a, b) \times [\hat s_p, \bar s) \times (c/(\delta\gamma), \hat s_0)$

$$\begin{cases} f(a+) = \gamma a, f'(a+) = \gamma & \\ (\mathcal{L} - (r+p))f(s) + c = 0 & \text{on } (a, \bar s) \\ f(\bar s-) = f(\bar s+), f'(\bar s-) = f'(\bar s+) & \\ (\mathcal{L} - r)f(s) + c = 0 & \text{on } (\bar s, b) \\ f(b-) = \gamma b, f'(b-) = \gamma. & \end{cases} \tag{2.10}$$

From Theorem 1.9 (iii) (and Step 1 for the intervals which contain a_p and b_p) we know that the triplet $(v_p|_{(a_p,b_p)}, a_p, b_p)$ solves this system. Let (f, a, b) be any solution to this system. By considering the initial value problems consisting of the first and last two lines of the system resp., it is straightforward to check that we have

$$f|_{(a,\bar s)}(s) = c_1^p(a) s^{\beta_1^p} + c_2^p(a) s^{\beta_2^p} + c/(r+p) \text{ and } f|_{(\bar s,b)}(s) = c_1^0(b) s^{\beta_1^0} + c_2^0(b) s^{\beta_2^0} + c/r \tag{2.11}$$

(recall that $c^{\cdot}_{1,2}$ are defined in Theorem 2.1). As is readily checked, the remaining pasting conditions at $s = \bar s$, i.e. the third line of (2.10), are satisfied iff (a, b) satisfies the system of equations (2.5)-(2.6). Thus (a_p, b_p) indeed satisfies (2.5)-(2.6) and, taking into account that $v_p = \hat v_p$ on $(0, \hat s_p]$ (by Step 1), v_p is indeed given by (2.7).

It remains to show that (2.5)-(2.6) has at most one solution. For this, let $(a, b) \in [\hat s_p, \bar s) \times (c/(\delta\gamma), \hat s_0)$ be any solution. Defining the function f on (a, b) by (2.11), with the understanding that $f(\bar s) := f|_{(a,\bar s)}(\bar s-) = f|_{(\bar s,b)}(\bar s+)$, we have that (f, a, b) is a solution to system (2.10). Let us first show that

$$f(s) > \gamma s \quad \text{on } (a, b). \tag{2.12}$$

From $a \geq \hat s_p$, $\delta < r$ and $c/(\gamma(\delta + p)) < \hat s_p$ (cf. (1.13)) it follows that $c_1^p(a) \geq 0$ and $c_2^p(a) > 0$, using this with $\beta_1^p < 0 < 1 < \beta_2^p$ a straightforward calculation shows that $f|_{(a,\bar s)}'' > 0$, so that with the boundary conditions in $s = a+$ from system (2.10) we see that $f|_{(a,\bar s)}(s) > \gamma s$. For $f|_{(\bar s,b)}$, using system (2.10) we can take the limit for $s \uparrow b$ in $(\mathcal{L} - r)f_b(s) + c = 0$ and use the boundary conditions at $s = b-$ together with $b > c/(\delta\gamma)$ to see that $f|_{(\bar s,b)}''(b-) > 0$. Furthermore, on account of $\beta_1^p < 0 < 1 < \beta_2^p$ a simple computation shows that $f|_{(\bar s,b)}''$ might have at most one zero. From this structure of $f|_{(\bar s,b)}''$, the boundary conditions at $s = b-$ and $f|_{(\bar s,b)}(\bar s+) = f|_{(a,\bar s)}(\bar s-) > \gamma \bar s$ it readily follows that $f|_{(\bar s,b)}(s) > \gamma s$ and thus (2.12) indeed holds.

Now, extend f to a $C^2((\hat s_p, \infty) \setminus \{a, \bar s, b\}) \cap C^1(\hat s_p, \infty)$-function by setting $f(s) = \gamma s$ on $[\hat s_p, a] \cup [b, \infty)$. Using that f satisfies $(\mathcal{L} - (r + \chi(s)))f(s) + c = 0$ on $(a, \bar s) \cup (\bar s, b)$(cf.

system (2.10)), that $(\mathcal{L} - (r + \chi(s)))f(s) + c = \lambda(s) \leq 0$ on $(\hat{s}_p, a) \cup (b, \infty) \subset (c/(\gamma(\delta + p)), \bar{s}) \cup (c/(\delta\gamma), \infty)$ and that $f(s) \geq \gamma s$ on $[\hat{s}_p, \infty)$ (cf. (2.12)), we get from Lemma A.1 (i) & (ii) that

$$f(s) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}_s \left[L_\tau^{\tau(\hat{s}_p, \infty)} \right] \quad \text{on } [\hat{s}_p, \infty).$$

A second solution (a_2, b_2) of (2.5)-(2.6) would by the same means as above allow to construct an associated solution function f_2 on $[\hat{s}_p, \infty)$ that also equals the rhs of the above formula. Thus $f_2 = f$, and since $f(s) > \gamma s$ iff $s \in (a, b)$ and $f_2(s) > \gamma s$ iff $s \in (a_2, b_2)$ (cf. (2.12)) this indeed implies $(a_2, b_2) = (a, b)$, as required. □

3 Power based intensity function

In this section we look at an intensity function of the form $\chi(s) = s^{-\alpha}$ for $\alpha > 0$ and we denote the associated value function by v_α. This means that we get $\varphi_t = \int_0^t S_u^{-\alpha} \, \mathrm{d}u$, $t \geq 0$, and

$$v_\alpha(s) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}_s[L_\tau] = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}_s \left[e^{-r\tau - \varphi_\tau} \gamma S_\tau + \int_0^\tau c e^{-ru - \varphi_u} \, \mathrm{d}u \right], \tag{3.1}$$

we denote by $\mathcal{C}_\alpha$ and $\mathcal{S}_\alpha$ the associated continuation and stopping regions resp. The drift rate is now given by $\lambda(s) = c - \gamma s(\delta + \chi(s)) = c - \delta\gamma s - \gamma s^{1-\alpha}$. It turns out that depending on whether $\alpha < 1$, $\alpha = 1$ or $\alpha > 1$, v_α behaves quite differently, see Theorem 3.3 below and Figures 3 & 4 in Section 4.

Proposition 3.1. *For any $\alpha > 0$, v_α is a non-decreasing $C^1(\mathbb{R}_{>0})$-function with $\gamma s \leq v_\alpha(s) \leq \hat{v}_0(s)$ on $\mathbb{R}_{>0}$ and $v_\alpha(0+) = 0$.*

Proof. $\gamma s \leq v_\alpha(s) \leq \hat{v}_0(s)$ on $\mathbb{R}_{>0}$ and $v_\alpha \in C^1(\mathbb{R}_{>0})$ are immediate from Theorem 1.9 (i) & (iii). That v_α is non-decreasing is obvious by writing (3.1) as

$$v_\alpha(s) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}_1 \left[e^{-r\tau - s^{-\alpha}\varphi_\tau} \gamma s S_\tau + \int_0^\tau c e^{-ru - s^{-\alpha}\varphi_u} \, \mathrm{d}u \right].$$

Using this expression and that $(\exp(-rt)S_t)_{t \geq 0}$ is a class (D) supermartingale we find by Doob's optional sampling theorem

$$v_\alpha(s) \leq \gamma s + \mathbb{E}_1 \left[\int_0^\infty c \exp\left(-ru - s^{-\alpha}\varphi_u\right) \, \mathrm{d}u \right]$$

and thus by dominated convergence it follows that $v_\alpha(0+) = 0$. □

Investigating where λ is positive (if anywhere) requires a few calculations and is done next.

Proposition 3.2. *Assume (for convenience) that $\delta < 1$. We have the following cases.*

(i) Let $\alpha \in (0,1)$. Then λ is strictly decreasing with $\lambda(0+) = c$. We denote its zero by s_r.

(ii) Let $\alpha = 1$. Then λ is strictly decreasing with $\lambda(0+) = c - \gamma$. For $c > \gamma$ we denote its zero by s_r.

(iii) Let $\alpha > 1$. Then λ attains a strict maximum in $s_0 := (\delta/(\alpha-1))^{-1/\alpha}$. The set $J := \{\alpha > 1 \,|\, \lambda(s_0) > 0\}$ satisfies

$$J = \begin{cases} \emptyset & \text{if } c/\gamma \leq \delta \\ (\alpha_r, \infty) & \text{if } c/\gamma \in (\delta, 1] \\ (1, \alpha_l) \cup (\alpha_r, \infty) & \text{if } c/\gamma \in (1, \delta+1) \\ (1, \infty) & \text{if } c/\gamma \geq \delta + 1, \end{cases}$$

where (if applicable) $\alpha_l \in (1, \delta+1)$ and $\alpha_r \in (\delta+1, \infty)$ are the zeros of

$$\Psi(\alpha) = \alpha \left(\frac{\delta}{\alpha - 1} \right)^{(\alpha-1)/\alpha} - \frac{c}{\gamma}.$$

If $\alpha \in J$, λ has two zeros $s_l < s_0 < s_r$.

Proof. Cases (i) and (ii) are obvious. Also case (iii) is easily checked. Note that $\lambda(s_0) > 0$ iff $\Psi(\alpha) > 0$, so $J = \{\alpha > 1 \,|\, \Psi(\alpha) > 0\}$. Taking into account the easily verified facts $\Psi(1+) = 1$, Ψ is increasing on $(1, \delta+1)$, Ψ is decreasing on $(\delta+1, \infty)$ and $\Psi(\infty) = \delta$, together with $\delta < 1$, the characterization of J follows. □

Finally we turn to obtaining semi-explicit formulae for v_α and the optimal exercise level(s):

Theorem 3.3. *Assume $\delta < 1$. Let I_ν and K_ν denote the modified Bessel functions of the first and second kind resp., of order ν. Set*

$$\nu = \frac{2}{\alpha\sigma} \sqrt{2r + \left(\frac{\sigma}{2} - \frac{r-\delta}{\sigma} \right)^2},$$

let the functions $\phi_1, \phi_2 : \mathbb{R}_{>0} \to \mathbb{R}$ be defined by

$$\phi_1(s) = s^{1/2-(r-\delta)/\sigma^2} I_\nu \left(\frac{2\sqrt{2}}{\alpha\sigma} s^{-\alpha/2} \right) \quad \text{and} \quad \phi_2(s) = s^{1/2-(r-\delta)/\sigma^2} K_\nu \left(\frac{2\sqrt{2}}{\alpha\sigma} s^{-\alpha/2} \right)$$

and the functions $c_1, c_2 : \mathbb{R}_{>0} \to \mathbb{R}$ by

$$c_1(b) = \frac{2\gamma}{\alpha} b^{2(r-\delta)/\sigma^2} \left(b\phi_2'(b) - \phi_2(b) \right) \quad \text{and} \quad c_2(b) = \frac{2\gamma}{\alpha} b^{2(r-\delta)/\sigma^2} \left(\phi_1(b) - b\phi_1'(b) \right).$$

Furthermore, define $F : \mathbb{R}_{>0} \to \mathbb{R}$ by

$$F(b) = -\frac{2}{\sigma^2}\int_0^b \xi^{(r-\delta)/\sigma^2-3/2} K_\nu\left(\frac{2\sqrt{2}}{\alpha\sigma}\xi^{-\alpha/2}\right)\lambda(\xi)\, d\xi. \tag{3.2}$$

We have the following.

(i) Suppose that $\alpha \in (0,1)$ or both $\alpha = 1$ and $c > \gamma$. Then $\mathcal{S}_\alpha = [b_\alpha, \infty)$, where b_α is the unique solution of $F(b) = 0$ on (s_r, ∞) and

$$v_\alpha(s) = \begin{cases} \frac{4c}{\alpha\sigma^2}\phi_1(s)\int_0^s \phi_2(\xi)\xi^{2(r-\delta)/\sigma^2-2}\, d\xi & \\ \quad +\phi_2(s)\left(c_2(b_\alpha) + \frac{4c}{\alpha\sigma^2}\int_s^{b_\alpha}\phi_1(\xi)\xi^{2(r-\delta)/\sigma^2-2}\, d\xi\right) & \text{on } (0, b_\alpha) \\ \gamma s & \text{on } [b_\alpha, \infty). \end{cases} \tag{3.3}$$

In particular, $v_\alpha(0+) = 0$ and $v'_\alpha(0+) = \infty$ for $\alpha < 1$ while $v'_\alpha(0+) = c$ for $\alpha = 1$.

(ii) Suppose that $\alpha = 1$ and $c \le \gamma$. Then $\mathcal{S}_\alpha = \mathbb{R}_{>0}$ and $v_\alpha(s) = \gamma s$ on $\mathbb{R}_{>0}$.

(iii) Suppose that $\alpha > 1$. If $\alpha \notin J$ then $\mathcal{S}_\alpha = \mathbb{R}_{>0}$ and $v_\alpha(s) = \gamma s$ on $\mathbb{R}_{>0}$. Let $\alpha \in J$. Then $\mathcal{S}_\alpha = (0, a_\alpha] \cup [b_\alpha, \infty)$, where the pair (a_α, b_α) is the unique solution of the following systems of equations in unknowns $(a, b) \in (0, s_l) \times (s_r, \infty)$

$$c_1(b)\phi_1(a) + c_2(b)\phi_2(a) + \frac{4c}{\alpha\sigma^2}\int_a^b \frac{\phi_1(\xi)\phi_2(a) - \phi_1(a)\phi_2(\xi)}{\xi^{2-2(r-\delta)/\sigma^2}}\, d\xi = \gamma a \tag{3.4}$$

$$c_1(b)\phi'_1(a) + c_2(b)\phi'_2(a) + \frac{4c}{\alpha\sigma^2}\int_a^b \frac{\phi_1(\xi)\phi'_2(a) - \phi'_1(a)\phi_2(\xi)}{\xi^{2-2(r-\delta)/\sigma^2}}\, d\xi = \gamma \tag{3.5}$$

and

$$v_\alpha(s) = \begin{cases} \gamma s & \text{on } (0, a_\alpha] \\ \phi_1(s)\left(c_1(b_\alpha) - \frac{4c}{\alpha\sigma^2}\int_s^{b_\alpha}\phi_2(\xi)\xi^{2(r-\delta)/\sigma^2-2}\, d\xi\right) & \\ \quad +\phi_2(s)\left(c_2(b_\alpha) + \frac{4c}{\alpha\sigma^2}\int_s^{b_\alpha}\phi_1(\xi)\xi^{2(r-\delta)/\sigma^2-2}\, d\xi\right) & \text{on } (a_\alpha, b_\alpha) \\ \gamma s & \text{on } [b_\alpha, \infty). \end{cases} \tag{3.6}$$

Proof. First suppose that either both $\alpha = 1$ and $c \le \gamma$ or $\alpha \in (1,\infty) \cap J^c$. These are exactly the cases for which λ is everywhere non-positive, cf. Proposition 3.2. It follows from Theorem 1.9 (ii), thereby using $[\hat{s}_0, \infty) \subset \mathcal{S}_\alpha$ that indeed $\mathcal{S}_\alpha = \mathbb{R}_{>0}$ and thus $v_\alpha(s) = \gamma s$ on $\mathbb{R}_{>0}$. It remains to consider α for which λ is not everywhere non-positive, in particular we can assume for the sequel that s_r is well defined and so is s_l if $\alpha > 1$, cf. Proposition 3.2.

The remainder consists of three steps, in the first two we study two free boundary problems by analytical means and in the last one we use these to deduce the statements.

Step 1. Consider the free boundary system in unknowns $(f, b) \in C^2(0,b) \times (s_r, \infty)$:

$$\begin{cases} (\mathcal{L} - (r + \chi(s)))f(s) + c = 0 \quad \text{on } (0,b) \\ f(b-) = \gamma b, f'(b-) = \gamma \\ f(0+) = 0. \end{cases} \tag{3.7}$$

Let us show that it has a (unique) solution pair (f_*, b_*) iff $F(s_r) < 0$ and that if $F(s_r) < 0$ holds, b_* is the unique solution of $F(b) = 0$ on (s_r, ∞), f_* is given by the first two lines of (3.3) with b_α replaced by b_* and $f_*'(0+)$ equals ∞, c, 0 for $\alpha < 1$, $\alpha = 1$, $\alpha > 1$, resp.

First, for any $b \in (s_r, \infty)$ we have from Lemma A.2 (i) and the general theory on ODEs that the initial value problem consisting of the first two lines of system (3.7), so without the condition $f(0+) = 0$, admits a unique solution $f = f_b$, with

$$f_b(s) = \phi_1(s)\left(c_1(b) - \frac{4c}{\alpha\sigma^2}\int_s^b \frac{\phi_2(\xi)}{\xi^{2-2(r-\delta)/\sigma^2}}\,\mathrm{d}\xi\right) + \phi_2(s)\left(c_2(b) + \frac{4c}{\alpha\sigma^2}\int_s^b \frac{\phi_1(\xi)}{\xi^{2-2(r-\delta)/\sigma^2}}\,\mathrm{d}\xi\right) \tag{3.8}$$

for $s \in (0, b)$. So in order to find the solutions to system (3.7) we need to find those $b \in (s_r, \infty)$ for which $f_b(0+) = 0$. Using $\phi_1(0+) = \infty$, $\phi_2(0+) = 0$ and (A.4) from Lemma A.2 (ii) & (iii) we see that this holds iff the first of the two bracketed terms in (3.8) vanishes as $s \downarrow 0$, which using the formula for c_1 boils down to

$$\gamma b^{2(r-\delta)/\sigma^2}\left(b\phi_2'(b) - \phi_2(b)\right) - \frac{2c}{\sigma^2}\int_0^b \xi^{2(r-\delta)/\sigma^2-2}\phi_2(\xi)\,\mathrm{d}\xi = 0. \tag{3.9}$$

On account of $\phi_2(0+) = \phi_2'(0+) = 0$ by Lemma A.2 (ii) the above lhs vanishes as $b \downarrow 0$. Furthermore, differentiating this lhs, thereby using that by definition $(\mathcal{L} - (r + \chi(s)))\phi_2(s) = 0$ (Lemma A.2 (i)), gives the quantity $-2b^{2(r-\delta)/\sigma^2-2}\phi_2(b)\lambda(b)/\sigma^2$. So (3.9) may be written as $F(b) = 0$ with F given by (3.2) and furthermore this derivation shows, together with $\phi_2 > 0$ (Lemma A.2 (i)) and $\lambda < 0$ on (s_r, ∞) (cf. Proposition 3.2) that F is strictly increasing on (s_r, ∞), thus it has a (unique) root on this interval iff $F(s_r) < 0$.

If $F(s_r) < 0$ holds and b_* denotes this unique root, the pair (f_*, b_*) is thus the unique solution to system (3.7), where $f_* = f_{b_*}$ takes the required form by adjusting (3.8) for $b = b_*$. A straightforward computation with $\phi_2'(0+) = 0$ and (A.5) from Lemma A.2 (ii) & (iii) yields that $f_*'(0+)$ equals ∞, c, 0 for $\alpha < 1$, $\alpha = 1$, $\alpha > 1$, resp.

Step 2. Suppose in this step that $\alpha > 1$, so that s_l is well defined. Consider the free boundary problem in unknowns $(f, a, b) \in C^2(a, b) \times (0, s_l) \times (s_r, \infty)$:

$$\begin{cases} (\mathcal{L} - (r + \chi(s)))f(s) + c = 0 & \text{on } (a, b) \\ f(b-) = \gamma b, f'(b-) = \gamma \\ f(a+) = \gamma a, f'(a+) = \gamma. \end{cases} \tag{3.10}$$

Let us show that the set of solutions consist of all pairs (a, b) that satisfy (3.4)-(3.5), with associated solution function f given by the first two lines of the rhs of (3.6) with a_α and b_α replaced by a and b resp., and that for any solution we have $f(s) > \gamma s$ on (a, b).

By the same arguments as in Step 1, for any $(a, b) \in (0, s_l) \times (s_r, \infty)$ there exists a unique $C^2(a, b)$-function that satisfies the initial value problem consisting of the first two lines of system (3.10), namely $f_b(s)$ as given by (3.8), considered for $s \in (a, b)$. Using this formula for f_b shows readily that $f_b(a+) = \gamma a$ and $f_b'(a+) = \gamma$ hold iff (a, b) satisfies (3.4)-(3.5).

Let now (f, a, b) be any solution to the system. For proving that $f(s) > \gamma s$ on (a, b) we only need the system itself and the sign of λ from Proposition 3.2. First consider f on $[s_r, b)$. Taking the limit $s \uparrow b$ in $(\mathcal{L} - (r + \chi(s)))f(s) + c = 0$ and using the boundary conditions in $s = b-$ we find that $\sigma^2 b^2 f''(b-)/2 = -\lambda(b) > 0$. Thus, by again the boundary conditions at $s = b-$

$$f'(s) \uparrow \gamma \text{ and } f(s) - \gamma s \downarrow 0 \text{ as } s \uparrow b. \tag{3.11}$$

Now suppose that $s_0 \in [s_r, b)$ exists with $f(s_0) = \gamma s_0$ and let it w.l.o.g. be the largest such point, so that $f(s) > \gamma s$ on (s_0, b). It follows that there has to a point in (s_0, b) where f' is larger than γ, otherwise namely $f(s_0) = \gamma s_0$ is impossible on account of (3.11). Let $s_1 \in (s_0, b)$ be the largest point where f' attains the value γ, so that $f''(s_1) \leq 0$, $f'(s_1) = \gamma$ and $f(s_1) > \gamma s_1$. But if we plug the latter two into $(\mathcal{L} - (r + \chi(s_1)))f(s_1) + c = 0$ it follows that $\sigma^2 s_1^2 f''(s_1)/2 \geq -\lambda(s_1) > 0$, and thus a contradiction is obtained.

The same idea, but "reflected", can be used to show that $f(s) > \gamma s$ on $(a, s_l]$. So it remains to show that $f(s) > \gamma s$ also holds on (s_l, s_r). Suppose that this assertion does not hold and let, using $f(s_r) > \gamma s_r$, s_2 be the largest solution of $f(s) = \gamma s$ on (s_l, s_r), so that $f(s_2) = \gamma s_2$ and $f'(s_2) \geq \gamma$. Plugging this in $(\mathcal{L} - (r + \chi(s_2)))f(s_2) + c = 0$ and using that $\lambda(s_2) > 0$ it follows that $f''(s_2) < 0$, so that

$$f'(s) \downarrow f'(s_2) \geq \gamma \text{ and } f(s) - \gamma s \uparrow 0 \text{ as } s \uparrow s_2. \tag{3.12}$$

It follows that there has to be a point in (s_l, s_2) where f' is smaller than γ, otherwise namely (3.12) would imply that $f(s_l) < \gamma s_l$ and we know already that $f(s_l) > \gamma s_l$. In particular, taking again (3.12) into account, f' has to have a largest point $s_3 \in (s_l, s_2)$ where it equals γ, i.e. $f''(s_3) \geq 0$, $f'(s_3) = \gamma$ and $f(s_3) \leq \gamma s_3$. But as before we can plug the latter two into $(\mathcal{L} - (r + \chi(s_3)))f(s_3) + c = 0$ and use $\lambda(s_3) > 0$ to derive that $f''(s_3) < 0$ and obtain a contradiction.

Step 3. Ad (i). Let $\alpha \in (0, 1)$ or both $\alpha = 1$ and $c < \gamma$. Using the behaviour of λ from Proposition 3.2, it follows from Theorem 1.9 (i), (ii) & (iv) that $\mathcal{S}_\alpha = [b_\alpha, \infty)$ for some $b_\alpha \in (s_r, \hat{s}_0]$. Next, applying Theorem 1.9 (iii) and using $v_\alpha(0+) = 0$ (by Proposition 3.1), it follows that the pair $(v_\alpha|_{(0,b_\alpha)}, b_\alpha)$ solves the free boundary system (3.7). As seen in Step 1 this system has a unique solution pair (f_*, b_*). Thus we may identify $b_\alpha = b_*$ and $v_\alpha|_{(0,b_\alpha)} = f_*$ and using the properties of (f_*, b_*) derived in Step 1, the results follow.

Ad (iii). Let $\alpha > 1$ and $\alpha \in J$. Using the behaviour of λ from Proposition 3.2, it follows from Theorem 1.9 (i), (ii) & (iv) that there are two possibilities, either $\mathcal{S}_\alpha = [b_\alpha, \infty)$ or $\mathcal{S}_\alpha = (0, a_\alpha] \cup [b_\alpha, \infty)$ for some $a_\alpha \in (0, s_l)$ and $b_\alpha \in (s_r, \hat{s}_0]$. Let us show that the former can not hold. Namely, if this were the case, it would by the same means as in the previous paragraph follow that $(v_\alpha|_{(0,b_\alpha)}, b_\alpha)$ is the unique solution to system (3.7). But the result from Step 1 would now imply that $v_\alpha'(0+) = 0$ and since $v_\alpha(0+) = 0$ (cf. Proposition 3.1) this would contradict with $v_\alpha(s) \geq \gamma s$ on $\mathbb{R}_{>0}$.

So indeed $\mathcal{S}_\alpha = (0, a_\alpha] \cup [b_\alpha, \infty)$. From Theorem 1.9 (iii) we have that the triplet

$(v_\alpha|_{(a_\alpha,b_\alpha)}, a_\alpha, b_\alpha)$ solves the system (3.10) from Step 2. It follows by Step 2 that (a_α, b_α) solves equations (3.4)-(3.5) and that v_α is given by (3.6). So it remains to show that (3.4)-(3.5) has at most one solution. For this, let (a, b) be any solution to those equations and let, by Step 2, $f \in C^2(a, b)$ be the function for which the triplet (f, a, b) solves system (3.10). We extend f to a $C^2(\mathbb{R}_{>0} \setminus \{a, b\}) \cap C^1(\mathbb{R}_{>0})$-function by setting $f(s) = \gamma s$ on $(0, a] \cup [b, \infty)$. Applying Lemma A.1 (i), thereby using that $f(s) = \gamma s$ on $(0, a] \cup [b, \infty)$ and $(\mathcal{L} - (r + \chi(s)))f(s) + c = 0$ on (a, b), and Lemma A.1 (ii), thereby using that $f(s) \geq \gamma s$ on $\mathbb{R}_{>0}$ (cf. Step 2) and $(\mathcal{L} - (r + \chi(s)))f(s) + c = \lambda(s) \leq 0$ on $(0, a) \cup (b, \infty) \subset (0, s_l) \cup (s_r, \infty)$ (cf. Proposition 3.2), it follows that $f(s) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}_s[L_\tau] = v_\alpha(s)$ on $\mathbb{R}_{>0}$. Since, using Step 2, $f(s) > \gamma s$ iff $s \in (a, b)$ and $v_\alpha(s) > \gamma s$ iff $s \in (a_\alpha, b_\alpha)$, this indeed implies $(a, b) = (a_\alpha, b_\alpha)$. □

4 Some plots

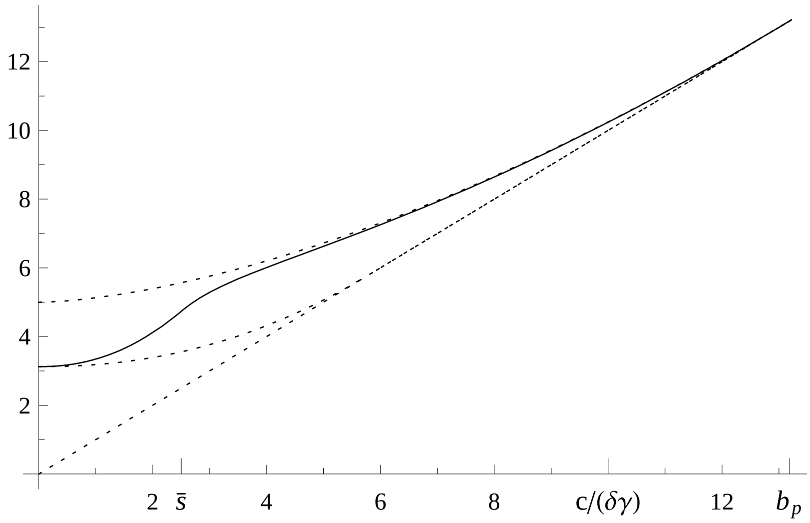

Figure 1: Situation as in Theorem 2.2 (i), with $\sigma = 0.2$, $r = 0.1$, $\delta = 0.05$, $\gamma = 1$, $c = 0.5$, $\bar{s} = 2.5$ and $p = 0.06$. The solid line is v_p, the three dotted lines are (from the bottom up) $s \mapsto \gamma s$, $\hat{v}_p$ and $\hat{v}_0$ resp. Note that $\hat{v}_0$ is the value in the standard case without default.

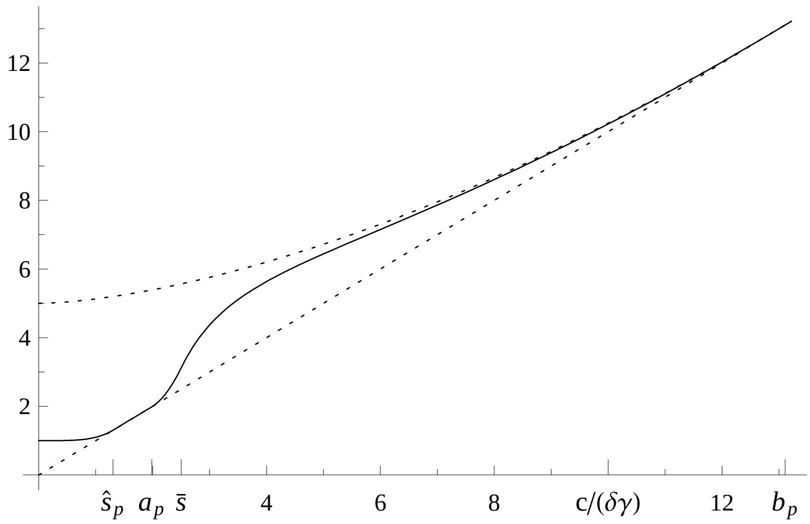

Figure 2: Situation as in Theorem 2.2 (ii), with the same parameters as in Figure 1, except with $p = 0.4$. The solid line is v_p, the two dotted lines are (from the bottom up) $s \mapsto \gamma s$ and $\hat{v}_0$ resp. Note that $\hat{v}_0$ is the value in the standard case without default.

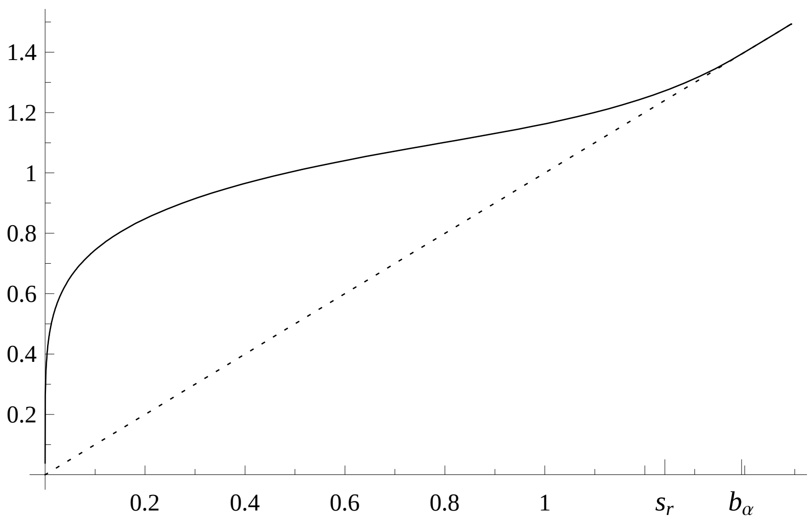

Figure 3: Situation as in Theorem 3.3 (i), with $\sigma = 0.2$, $r = 0.1$, $\delta = 0.05$, $\gamma = 1$, $c = 1.25$ and $\alpha = 0.2$. The solid line is v_α, the dotted one is $s \mapsto \gamma s$.

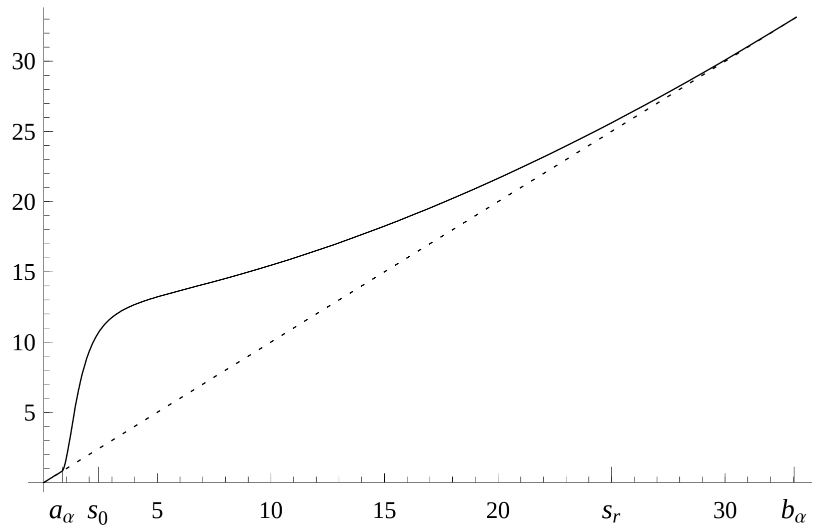

Figure 4: Situation as in Theorem 3.3 (ii), with the same parameters as in Figure 3, except with $\alpha = 5$. The solid line is v_α, the dotted one is $s \mapsto \gamma s$.

A Appendix

Lemma A.1. *Let $I \subset \mathbb{R}_{>0}$ be an interval and $N_f \subset I$ be a finite set. Let f be a continuous function on $\bar{I}$ such that $f \in C^2(I \setminus N_f) \cap C^1(I)$ and the limits $f''(a\pm)$ exist and are finite for all $a \in N_f$. We have the following.*

(i) Suppose that f is bounded and satisfies $(\mathcal{L} - (r + \chi(s)))f(s) + c = 0$ on $I \setminus N_f$. Then

$$f(s) = \mathbb{E}_s \left[\mathbf{1}_{\{\tau(I)<\infty\}} e^{-r\tau(I)-\varphi_{\tau(I)}} f(S_{\tau(I)}) + \int_0^{\tau(I)} c e^{-ru-\varphi_u}\, du \right], \quad \forall s \in I.$$

(ii) Suppose that f satisfies

$$\begin{cases} (\mathcal{L} - (r + \chi(s)))f(s) + c \leq 0 & \text{on } I \setminus N_f \\ f(s) \geq \gamma s & \text{on } \bar{I}. \end{cases}$$

Then

$$f(s) \geq \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}_s \left[L_\tau^{\tau(I)} \right], \quad \forall s \in I.$$

Proof. First consider a function that satisfies the weaker requirement $h \in C^2(I \setminus D) \cap C(I)$ for some finite set D, with existing and finite limits $h''(a\pm)$ for all $a \in D$. Applying the change-of-variable formula from [14] we may write

$$\begin{aligned} h(S_t) = h(s) &+ \int_0^t \mathbf{1}_{\{S_u \notin D\}} h'(S_u)\, dS_u + \frac{1}{2}\int_0^t \mathbf{1}_{\{S_u \notin D\}} h''(S_u)\, \mathrm{d}\langle S, S\rangle_u \\ &+ \frac{1}{2}\sum_{a\in D}\int_0^t (h'(a+) - h'(a-))\, \mathrm{d}L_u^a, \quad t \geq 0, \end{aligned} \tag{A.1}$$

where L^a denotes the local time of S at a and is defined as the càdlàg version of

$$L_t^a = \mathbb{P} - \lim_{\epsilon\downarrow 0} \frac{1}{\epsilon}\int_0^t \mathbf{1}_{\{a \leq S_u < a+\epsilon\}}\, \mathrm{d}\langle S, S\rangle_u, \quad t \geq 0.$$

Now let $f \in C^2(I \setminus N_f) \cap C^1(I)$ be as in the statement of the lemma, take some $s \in I$ and let S start in s. Define the process Z by

$$Z_t = e^{-r(t\wedge\tau(I)) - \varphi_{t\wedge\tau(I)}} f(S_{t\wedge\tau(I)}) + \int_0^{t\wedge\tau(I)} c e^{-ru-\varphi_u}\, \mathrm{d}u, \quad t \geq 0. \tag{A.2}$$

Using the above formula (A.1) we can write $Z = Z_0 + M + A$, where $Z_0 = f(s)$, M is a local martingale given by

$$M_t = \int_0^{t\wedge\tau(I)} e^{-ru-\varphi_u} \sigma S_u f'(S_u)\, dW_u, \quad t \geq 0,$$

and the drift A is given by

$$A_t = \int_0^{t\wedge\tau(I)} \mathbf{1}_{\{S_u \notin N_f\}} e^{-ru-\varphi_u} \left[(\mathcal{L} - (r + \chi(S_u))) f(S_u) + c\right] \mathrm{d}u, \quad t \geq 0.$$

Ad (i). Since f satisfies $(\mathcal{L} - (r+\chi(s)))f(s) + c = 0$ on $I \setminus N_f$, A vanishes and thus Z is a local martingale. Furthermore, since f is bounded and continuous on $\bar{I}$, (A.2) shows that Z is a bounded process and that as $t \to \infty$

$$Z_t \to Z_\infty := \mathbf{1}_{\{\tau(I)<\infty\}} e^{-r\tau(I) - \varphi_{\tau(I)}} f(S_{\tau(I)}) + \int_0^{\tau(I)} c e^{-ru-\varphi_u}\, \mathrm{d}u, \quad \mathbb{P}_s - \text{a.s.}$$

By dominated convergence it follows that $\mathbb{E}_s[Z_0] = \mathbb{E}_s[Z_\infty]$, yielding the result.

Ad (ii). On account of $f(s) \geq \gamma s$ on $\bar{I}$ we see from (A.2) that

$$Z_t \geq L_t^{\tau(I)}, \quad \forall t \geq 0. \tag{A.3}$$

Since f satisfies $(\mathcal{L} - (r + \chi(s)))f(s) + c \leq 0$ on $I \setminus N_f$, A is non-increasing and thus Z is a local supermartingale. Since Z is non-negative by (A.3), it is a true supermartingale. This implies for any $\tau \in \mathcal{T}_{0,\infty}$ and $t > 0$, using Doob's optional sampling and (A.3), that $f(s) = \mathbb{E}_s[Z_0] \geq \mathbb{E}_s\left[Z_{t\wedge\tau}\right] \geq \mathbb{E}_s\left[L_{t\wedge\tau}^{\tau(I)}\right]$. Since L is of class (D), we can let $t \to \infty$ and obtain $f(s) \geq \mathbb{E}_s\left[L_\tau^{\tau(I)}\right]$, which yields the result.

□

Lemma A.2. *Let $\alpha > 0$ and let the functions ϕ_1 and ϕ_2 be defined as in Theorem 3.3. Furthermore let $b \in \mathbb{R}_{>0}$ and set $A := 2(r-\delta)/\sigma^2$. We have the following.*

(i) *ϕ_1 and ϕ_2 are positive, linear independent $C^\infty(\mathbb{R}_{>0})$-functions such that $(\mathcal{L} - (r + s^{-\alpha}))\phi_{1,2}(s) = 0$. The Wronskian of (ϕ_1, ϕ_2) is given by $s \mapsto \alpha s^{-A}/2$.*

(ii) *We have $\phi_1(0+) = \infty$ and $\phi_2(0+) = 0$, $\phi_2'(0+) = 0$.*

(iii) *Let $s \downarrow 0$. Then*

$$\phi_2(s) \int_s^b \xi^{A-2} \phi_1(\xi)\, d\xi \to 0, \quad \phi_1(s) \int_0^s \xi^{A-2} \phi_2(\xi)\, d\xi \to 0 \tag{A.4}$$

and

$$\phi_1'(s) \int_0^s \xi^{A-2} \phi_2(\xi)\, d\xi + \phi_2'(s) \int_s^b \xi^{A-2} \phi_1(\xi)\, d\xi \to \begin{cases} \infty & \text{if } \alpha \in (0,1) \\ \sigma^2/4 & \text{if } \alpha = 1 \\ 0 & \text{if } \alpha > 1. \end{cases} \tag{A.5}$$

Proof. The facts about Bessel functions used in this proof can be found in [1] p. 358 – 378 e.g. Recall from Theorem 3.3 that

$$\phi_1(s) = s^{1/2-(r-\delta)/\sigma^2} I_\nu \left(\frac{2\sqrt{2}}{\alpha\sigma} s^{-\alpha/2} \right) \text{ and } \phi_2(s) = s^{1/2-(r-\delta)/\sigma^2} K_\nu \left(\frac{2\sqrt{2}}{\alpha\sigma} s^{-\alpha/2} \right),$$

where

$$\nu = \frac{2}{\alpha\sigma} \sqrt{2r + \left(\frac{\sigma}{2} - \frac{r-\delta}{\sigma} \right)^2}.$$

Ad (i). For arbitrary μ the functions I_μ and K_μ are by definition two positive, linear independent solutions of the modified Bessel equation $x^2 y''(x) + x y'(x) - (x^2 + \mu^2) y(x) = 0$ on $\mathbb{R}$. By some standard substitutions and calculations it can be checked that consequently $(\mathcal{L} - (r + s^{-\alpha}))\phi_{1,2}(s) = 0$. The Wronskian of (ϕ_1, ϕ_2) can by standard means be derived from the Wronskian of (I_μ, K_μ), which is known to equal $s \mapsto -1/s$.

Ad (ii). We use the asymptotic expansions for $z \to \infty$, denoted in the form $f(z) \sim g(z) \sum_{k \geq 0} c_k z^{-k}$ and normalized such that $c_0 = 1$, from Table 1.

From the exponential increase and decay of $I_\nu(z)$ and $K_\nu(z)$ as $z \to \infty$ as seen in Table 1 it readily follows that $\phi_1(0+) = \infty$ and $\phi_2(0+) = 0$. Computing $\phi_2'(s)$ and using the expansions of K_ν and K_ν' from the Table 1 we find $\phi_2'(0+) = 0$.

Ad (iii). We only show (A.5) since the same means can be used to show the easier (A.4). First consider the term $\phi_1'(s) \int_0^s \xi^{A-2} \phi_2(\xi)\, \mathrm{d}\xi$. For arbitrary x, the expression

f	g	Coefficient c_1
$I_\nu(z)$	$e^z/\sqrt{2\pi z}$	$-(4\nu^2-1)/8$
$I'_\nu(z)$	$e^z/\sqrt{2\pi z}$	$-(4\nu^2+3)/8$
$K_\nu(z)$	$e^{-z}\sqrt{\pi/(2z)}$	$(4\nu^2-1)/8$
$K'_\nu(z)$	$-e^{-z}\sqrt{\pi/(2z)}$	$(4\nu^2+3)/8$

Table 1: Asymptotic expansions

$$\int_0^s \xi^x \exp\left(-\frac{2\sqrt{2}}{\alpha\sigma}\xi^{-\alpha/2}\right) \mathrm{d}\xi$$

allows for an asymptotic expansion for $s \downarrow 0$ by repetitively applying partial integration. Combining this with the expansions of $\phi_1'(s)$ and $\phi_2(s)$ for $s \downarrow 0$ obtained by using Table 1, we arrive after some algebra at

$$\phi_1'(s)\int_0^s \xi^{A-2}\phi_2(\xi)\,\mathrm{d}\xi = -\frac{\alpha\sigma}{4\sqrt{2}}s^{\alpha/2-1} + \frac{\alpha^2\sigma^2}{8}s^{\alpha-1} + \mathcal{O}(s^{3\alpha/2-1}), \quad s \downarrow 0. \tag{A.6}$$

Next consider the expression $\phi_2'(s)\int_s^b \xi^{A-2}\phi_1(\xi)\,\mathrm{d}\xi$. Since $\phi_2'(0+) = 0$ (cf. (ii)), its limit value for $s \downarrow 0$ is independent of the value of b. Now we can repetitively apply partial integration on the expression

$$\int_s^b \xi^x \exp\left(\frac{2\sqrt{2}}{\alpha\sigma}\xi^{-\alpha/2}\right) \mathrm{d}\xi$$

and use the expansions of $\phi_1(s)$ and $\phi_2'(s)$ for $s \downarrow 0$ obtained by using Table 1 to deduce after some algebra that for all b small enough

$$\phi_2'(s)\int_s^b \xi^{A-2}\phi_1(\xi)\,\mathrm{d}\xi = \frac{\alpha\sigma}{4\sqrt{2}}s^{\alpha/2-1} + \frac{\alpha^2\sigma^2}{8}s^{\alpha-1} + \mathcal{O}(s^{3\alpha/2-1}) + \phi_2'(s)C(b), \quad s \downarrow 0, \tag{A.7}$$

where $C(b)$ is a sum of terms that come from the partial integration. Thus, if we add (A.6) and (A.7) we see that their leading terms cancel against each other and (A.5) indeed follows. □

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