Unbiased shifts of Brownian motion

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Unbiased shifts

Setting

Let $B = (B_t)_{t\in\mathbb{R}}$ be a two-sided standard Brownian motion defined as the identity on $(\Omega, \mathcal{A}, \mathbb{P}_0)$ where Ω is the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$, $\mathcal{A}$ is the Kolmogorov product σ-algebra and $\mathbb{P}_0$ is the distribution of B. In particular $B_0 = 0$ a.s.

Note that no external randomization is allowed.

Definition

An unbiased shift of B is a random time T in $\mathbb{R}$ such that:

- $(B_{T+t} - B_T)_{t\in\mathbb{R}}$ is a standard Brownian motion,
- $(B_{T+t} - B_T)_{t\in\mathbb{R}}$ is independent of B_T.

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Remark

i.e. T is an unbiased shift if $(B_{T+t})_{t\in\mathbb{R}}$ is a two-sided standard Brownian motion not necessarily taking the value 0 at time 0.

Example

If $T \geq 0$ is a stopping time, then $(B_{T+t} - B_T)_{t \geq 0}$ is a one-sided Brownian motion independent of B_T. However, the example

$$T := \inf\{t \geq 0 \colon B_t = a\}$$

shows that $(B_{T+t} - B_T)_{t \in \mathbb{R}}$ need not be a two-sided Brownian motion.

Example

Consider a deterministic $T = t_0$.
Then $\tilde{B} := (B_{t_0+t} - B_{t_0})_{t \in \mathbb{R}}$ is a two-sided Brownian motion.
However, it is not independent of B_{t_0} since $B_{t_0} = -\tilde{B}_{-t_0}$.

Remark

We will see later that an unbiased shift need not be a stopping time, even when it is nonnegative.

The case when $B_T = 0$ is of special interest.

Example

Let ℓ^0 be the local time random measure of B at 0.
Let $(T_r)_{r\in\mathbb{R}}$ be the (generalized) inverse of the cumulative mass,

$$T_r := \begin{cases} \sup\{t \geq 0 \colon \ell^0[0,t] = r\}, & r \geq 0, \\ \sup\{t < 0 \colon \ell^0[t,0] = -r\}, & r < 0. \end{cases}$$

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We shall now sketch the theory that leads to this result,
and to many more results on unbiased shifts.

The σ-finite stationary $\mathbb{P}$

Set $\mathbb{P}_x := \mathbb{P}_0(B + x \in \cdot)$, $x \in \mathbb{R}$, and define a σ-finite measure by

$$\mathbb{P} := \int \mathbb{P}_x dx.$$

The stationary increments yield that B is stationary under $\mathbb{P}$:

$$\mathbb{P}(\theta_t B \in \cdot) = \mathbb{P}, \quad t \in \mathbb{R},$$

where $\theta_t : \Omega \mapsto \Omega$ is defined by $(\theta_t \omega)_s := \omega_{t+s}$ for $s \in \mathbb{R}$.

Local time ℓ^x

The local time random measures ℓ^x at $x \in \mathbb{R}$ can be defined such that $\mathbb{P}$-a.e. ℓ^x is diffuse for all x and

$$\ell^x(\theta_t B, C - t) = \ell^x(C), \quad C \in \mathcal{B}, t \in \mathbb{R}, \quad (\ell^x \text{ is invariant})$$
$$\ell^x(B, \cdot) = \ell^0(B - x, \cdot).$$

Palm measures and local time

Recall that the Palm measure $\mathbb{Q}_\xi$ of an invariant random measure ξ with respect to a stationary $\mathbb{P}$ is defined by

$$\mathbb{Q}_\xi(A) := \mathbb{E}\int \mathbf{1}_{[0,1]}(s)\mathbf{1}_A(\theta_s B)\,\xi(ds), \quad A \in \mathcal{A}.$$

Theorem (Geman and and Horowitz '73)

$\mathbb{P}_x$ is the Palm probability measure of the local time ℓ^x

Let ν be a probability measure and put

$$\mathbb{P}_\nu = \int \mathbb{P}_x \nu(dx) \qquad \text{and} \qquad \ell^\nu = \int \ell^x \nu(dx).$$

Corollary

$\mathbb{P}_\nu$ is the Palm probability measure of ℓ^ν

General Palm measures and allocation rules

On this slide we can allow a general setting with a stationary $\mathbb{P}$. In fact B can be a spatial random field, or acted on by a group.

Define the allocation rule associated with a random time T by

$$\tau_T : s \to T \circ \theta_s + s.$$

An allocation rule τ balances two random measures ξ and η if

$$\xi\{s : \tau(s) \in \cdot\} = \eta \quad \mathbb{P}\text{-a.e.}$$

Theorem (Last and Thorisson '09 – general stationary setting)

Let ξ and η be invariant random measures with positive and finite intensities. Then ξ and η have the same intensity and

$$\mathbb{Q}_\xi(\theta_T B \in \cdot) = \mathbb{Q}_\eta \qquad \text{(shift-coupling of Palm versions)}$$

if and only if τ_T balances ξ and η.

Unbiased shifts and balancing allocation rules

Consider again the Brownian motion B under $\mathbb{P}_0$.

Definition (Skorokhod embedding problem)

A random time T embeds ν if B_T has distribution ν.

Until now this has only been considered in one-sided time and usually T is assumed to be a stopping time.

Here is a key characterication result in two-sided time:

Theorem

T is an unbiased shift embedding $\nu \iff \tau_T$ balances ℓ^0 and ℓ^ν

Proof: T is unbiased shift embedding $\nu \iff \mathbb{P}_0(\theta_T B \in \cdot) = \mathbb{P}_\nu$. Since $\mathbb{P}_0$ and $\mathbb{P}_\nu$ are Palm probability measures of ℓ^0 and ℓ^ν, the shift-coupling theorem (previous slide) yields the theorem.

Existence of unbiased shifts

Theorem

For each $r \in \mathbb{R}$, the time T_r imbeds δ_0 and is an unbiased shift.

Proof: For $r \geq 0$, the allocation rule associated with T_r,

$$\tau_{T_r}(s) = \sup\{t > s \colon \ell^0([s,t]) = r\}, \quad s \in \mathbb{R},$$

balances ℓ^0 and ℓ^0 for $r > 0$. Similarly for $r < 0$.

Theorem

Let ν be a probability measure on $\mathbb{R}$ with $\nu\{0\} = 0$. Then

$$T^\nu := \inf\{t > 0 \colon \ell^0([0,t]) = \ell^\nu([0,t])\}$$

embeds ν and is an unbiased shift.

Remark

This stopping time T^ν was introduced in Bertoin and Le Jan '92 as a solution of the Skorokhod embedding problem.
The unbiasedness seems to be a new observation.

The crucial ingredient in the proof

This theorem:

Theorem (from previous slide)

Let ν be a probability measure on $\mathbb{R}$ with $\nu\{0\} = 0$. Then

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embeds ν and is an unbiased shift.

follows from the next one by applying joint ergodicity of the $\mathbb{P}_x$ and the balancing characterization of unbiased shifts.

Theorem (holds in a general stationary setting on $\mathbb{R}$)

Let ξ and η be invariant diffuse and orthogonal random measures on $\mathbb{R}$ with the same conditional intensity given the invariant σ-algebra. Then the allocation rule τ defined by

$$\tau(s) := \inf\{t > s \colon \xi([s,t]) = \eta([s,t])\}, \quad s \in \mathbb{R},$$

balances ξ and η.

Existence of unbiased shifts – one more theorem

Theorem

For each $r \in \mathbb{R}$, the time T_r imbeds δ_0 and is an unbiased shift.

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Let ν be a probability measure on $\mathbb{R}$ with $\nu\{0\} = 0$. Then

$$T^\nu := \inf\{t > 0 \colon \ell^0([0, t]) = \ell^\nu([0, t])\}$$

embeds ν and is an unbiased shift.

Theorem

$\forall \nu \ \exists$ *stopping time $T \geq 0$ which is unbiased shift embedding ν.*

$\nu\{0\} < 1 \Rightarrow$ *all unbiased T embedding ν satisfy $\mathbb{P}_0(T = 0) = 0$.*

$\nu = \delta_0 \Rightarrow \forall p \in [0, 1] \ \exists$ *an unbiased shift $T \geq 0$ embedding ν*
and satisfying $\mathbb{P}_0(T = 0) = p$

Moment properties

Theorem

Suppose $\nu\{0\} = 0$ *and the stopping time* $T \geq 0$ *is an unbiased shift embedding* ν. *Then*

$$\mathbb{E}_0 T^{1/4} = \infty.$$

If additionally $\int |x|\nu(dx) < \infty$ *and*

$$T = T^\nu = \inf\{t > 0 \colon \ell^0([0,t]) = \ell^x([0,t])\}.$$

Then

$$\forall \beta \in [0, 1/4) : \quad \mathbb{E}_0 T^\beta < \infty.$$

Theorem

If T *is an unbiased shift embedding* $\nu \neq \delta_0$, *then* $\mathbb{E}_0 \sqrt{|T|} = \infty$.

Consider the one-sided stable matching τ between independent Poisson processes ξ and η. Then $\mathbb{E}_{\mathbb{Q}_\xi} \sqrt{\tau(0)} = \infty$. The stable marriage of Lebesgue and Poisson (Holroyd and Peres ’05) has also this property.

Moment properties – one more theorem

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Theorem

If T *is an unbiased shift embedding* $\nu \neq \delta_0$, *then* $\mathbb{E}_0 \sqrt{|T|} = \infty$.

Theorem

Let T *be a nontrivial unbiased shift embedding* δ_0.
Then it is possible that $\mathbb{E} e^{\lambda |T|} < \infty$ *for some* $\lambda > 0$.
But if $T \geq 0$ *and* $\mathbb{P}_0(T > 0) > 0$ *then* $\mathbb{E}_0 T = \infty$.

Minimality

Definition

An unbiased shift $T \geq 0$ embedding ν is called minimal if for each unbiased shift S embedding ν and satisfying $\mathbb{P}_0(0 \leq S \leq T) = 1$ we have that $\mathbb{P}_0(S = T) = 1$.

Theorem

If $\nu\{0\} = 0$ then the Bertoin-Le Jan stopping time T^ν is a minimal unbiased shift.

Levy processes

The central results extend to recurrent Levy processes under mild regularity conditions:

Theorem

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Theorem

For each $r \in \mathbb{R}$, the time T_r imbeds δ_0 and is an unbiased shift.

Theorem

Let ν be a probability measure on $\mathbb{R}$ with $\nu\{0\} = 0$. Then

$$T^\nu := \inf\{t > 0 \colon \ell^0([0,t]) = \ell^\nu([0,t])\}$$

embeds ν and is an unbiased shift.

References

- Mandelbrot (1982). *The Fractal Geometry of Nature*, Freeman and Co., San Francisco.
- Bertoin and Le Jan (1992). Representation of measures by balayage from a regular recurrent point. *Annals of Probability* **20**, 538–548.
- Holroyd and Peres (2005). Extra heads and invariant allocations. *Annals of Probability* **33**, 31–52.
- Holroyd, Pemantle, Peres, and Schramm (2009). Poisson Matching. *Annales de l'institut Henri Poincaré (B)* **45**, 266–287.
- Last and Thorisson (2009). Invariant transports of stationary random measures and mass-stationarity. *Annals of Probability* **37**, 790–813.
- Last, Mörters and Thorisson (2013). Unbiased shifts of Brownian Motion. *Annals of Probability* (to appear).