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Invariant transports of stationary random measures

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1. The Monge-Kantorovich problem

Setting

Let ξ and η be measures on $\mathbb{R}^d$ such that

$$0 < \xi(\mathbb{R}^d) = \eta(\mathbb{R}^d) < \infty.$$

Let $c(x, y)$ be the cost of transporting one unit of mass from $x \in \mathbb{R}^d$ to $y \in \mathbb{R}^d$.

Problem (Monge 1781)

Minimize

$$\int c(x, \tau(x))\xi(dx)$$

among all transport maps $\tau : \mathbb{R}^d \to \mathbb{R}^d$ satisfying $\tau^*(\xi) = \eta$, that is

$$\int \mathbf{1}\{\tau(x) \in B\}\xi(dx) = \eta(B), \quad B \subset \mathbb{R}^d.$$

Such a τ is called admissable.

Remark

If ξ and η have the same number of atoms of equal size, the Monge Problem corresponds to optimal matching.

Remark

Admissable transports need not exist, for instance if ξ and η have atoms of different sizes.

Remark

If ξ and η are absolutely continuous and $c(x, y) = \|x - y\|^p$ for some $p > 1$ then (under moment assumptions on ξ and η) there is a unique solution of the Monge problem.

Definition (Coupling)

Let $\Pi(\xi, \eta)$ denote the set of all (finite) measures π on $\mathbb{R}^d \times \mathbb{R}^d$ such that $\pi(\cdot \times \mathbb{R}^d) = \xi$ and $\pi(\mathbb{R}^d \times \cdot) = \eta$. Any such π is called a coupling of ξ and η.

Problem (Kantorovich 1940)

Minimize

$$\int c(x, y)\pi(d(x, y))$$

among all $\pi \in \Pi(\xi, \eta)$.

Remark

Any $\pi \in \Pi(\xi, \eta)$ can be identified with a stochastic kernel $T(x, dy)$ from $\mathbb{R}^d$ to $\mathbb{R}^d$ such that

$$\int T(x, B)\xi(dx) = \eta(B), \quad B \subset \mathbb{R}^d.$$

Such a T is called transport kernel.

Remark

If the costs are finite for some transport kernel, then there exists a solution to the Monge-Kantorovich problem.

2. Invariant random measures

Setting

G denotes a (multiplicative) LCSC group with Borel σ-field $\mathcal{G}$, neutral element e, Haar measure λ, and modular function Δ.

Definition

(i) Let **M** denote the space of all locally finite measures on G.

(ii) The σ-field $\mathcal{M}$ is the smallest σ-field of subsets of **M** making the mappings $\mu \mapsto \mu(B)$ for all Borel sets $B \in \mathcal{G}$ measurable.

(iii) A random measure ξ on G is a measurable mapping $\xi : \Omega \to \mathbf{M}$, where $(\Omega, \mathcal{A}, \mathbb{P})$ is a given σ-finite measure space.

Setting

Consider measurable mappings $\theta_g : \Omega \to \Omega$, $g \in G$, satisfying $\theta_e = \mathrm{id}_\Omega$ and the flow property

$$\theta_g \circ \theta_h = \theta_{gh}, \quad g, h \in G$$

The mapping $(\omega, g) \mapsto \theta_g \omega$ is assumed measurable. The measure $\mathbb{P}$ is assumed stationary under the flow, that is

$$\mathbb{P} \circ \theta_g = \mathbb{P}, \quad g \in G.$$

Definition

A random measure ξ is invariant if

$$\xi(\theta_g \omega, gB) = \xi(\omega, B), \quad \omega \in \Omega,\ g \in G,\ B \in \mathcal{G}.$$

Definition

Let $w : G \to \mathbb{R}_+$ be a measurable with $\int w(g)\,\lambda(dg) = 1$. Let ξ be an invariant random measure on G. The measure

$$\mathbb{P}_\xi(A) := \mathbb{E}_\mathbb{P} \int \mathbf{1}\{\theta_g^{-1} \in A\} w(g)\,\xi(dg), \quad A \in \mathcal{A},$$

is called the Palm measure of ξ.

3. Transport properties of Palm measures

Definition

A measurable mapping $\tau : \Omega \times G \to G$ is called allocation if

$$\tau(\theta_g \omega, gh) = g\tau(\omega, h), \quad \omega \in \Omega,\ g, h \in G.$$

Definition

An allocation balances two random measures ξ and η if $\mathbb{P}$-a.e.

$$\int \mathbf{1}\{\tau(g) \in \cdot\}\, \xi(dg) = \eta(\cdot)$$

Theorem (Mecke ’75, Holroyd and Peres ’05, L. and Thorisson ’09, L. 10)

Consider two invariant random measures ξ and η and let τ be an allocation. Then τ balances ξ and η iff

$$\mathbb{E}_{\mathbb{P}_\xi} f(\theta^{-1}_{\tau(e)})\Delta(\tau(e)^{-1}) = \mathbb{E}_{\mathbb{P}_\eta} f,$$

for all measurable $f : \Omega \to \mathbb{R}_+$. In particular, if G is unimodular, this is equivalent with

$$\mathbb{P}_\xi(\theta^{-1}_{\tau(e)} \in A) = \mathbb{P}_\eta(A), \quad A \in \mathcal{A}.$$

Definition

A transport-kernel is a kernel T from $\Omega \times G$ to G such that $T(\omega, x, \cdot)$ is a locally finite measure for all $(\omega, g) \in \Omega \times G$ which is invariant, that is

$$T(\theta_g \omega, gh, gB) = T(\omega, h, B), \quad g, h \in G, \, \omega \in \Omega, \, B \in \mathcal{B}(G).$$

Definition

Let ξ and η be random measures. A transport kernel balances ξ and η if

$$\int T(\omega, x, \cdot)\, \xi(\omega, dx) = \eta(\omega, \cdot) \quad \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Theorem (Holroyd and Peres '05, L. and Thorisson '09)

Consider two invariant random measures ξ and η and let T be a transport kernel. Then T is balances ξ and η iff

$$\mathbb{E}_{\mathbb{P}_\xi} \int f(\theta_g^{-1})\Delta(g^{-1})T(e, dg) = \mathbb{E}_{\mathbb{P}_\eta} f,$$

for all measurable $f : \Omega \to \mathbb{R}_+$.

4. Transport formulas

Theorem (L. and Thorisson '09)

Consider two invariant random measures ξ and η and let T and T^* be transport-kernels satisfying

$$\iint \mathbf{1}\{(g,h) \in \cdot\} T(g,dh)\xi(dg) = \iint \mathbf{1}\{(g,h) \in \cdot\} T^*(h,dg)\eta(dh)$$

$\mathbb{P}$-a.e. Then we have for any measurable function $f : \Omega \times G \to \mathbb{R}_+$ that

$$\mathbb{E}_{\mathbb{P}_\xi} \int f(\theta_g^{-1}, g^{-1})\Delta(g^{-1})T(e,dg) = \mathbb{E}_{\mathbb{P}_\eta} \int f(\theta_e, g)T^*(e,dg).$$

Corollary (Neveu '77)

Let ξ, η be invariant random measure on G. Then we have for any measurable function $f : \Omega \times G \to \mathbb{R}_+$ that

$$\mathbb{E}_{\mathbb{P}_\xi} \int f(\theta_g^{-1}, g^{-1})\Delta(g^{-1})\eta(dg) = \mathbb{E}_{\mathbb{P}_\eta} \int f(\theta_e, g)\xi(dg).$$

Corollary (mass transport principle)

Let $t : \Omega \times G \times G \to \mathbb{R}_+$ be measurable and invariant. Then

$$\mathbb{E} \iint \mathbf{1}\{g \in B\} t(h, g)\Delta(g^{-1})\Delta(h)\eta(dh)\xi(dg)$$
$$= \mathbb{E} \iint \mathbf{1}\{g \in B\} t(g, h)\eta(dg)\xi(dh),$$

for any $B \in \mathcal{G}$ with positive and finite Haar measure.

5. Existence of balancing transport kernels

Definition

The intensity of an invariant random measure ξ is the number

$$\mathbb{E} \int w(g)\, \xi(dg),$$

where $\int w \, d\lambda = 1$.

Definition

The invariant σ-field $\mathcal{I} \subset \mathcal{A}$ is the class of all sets $A \in \mathcal{A}$ satisfying $\theta_g A = A$ for all $g \in G$.

Theorem

Suppose that ξ and η are invariant random measures with positive and finite intensities. Then there exists a transport-kernel balancing ξ and η and satisfying

$$\int \Delta(g^{-1})T(e, dg) = 1$$

iff

$$\mathbb{E}[\xi(B)|\mathcal{I}] = \mathbb{E}[\eta(B)|\mathcal{I}] \quad \mathbb{P}\text{-a.e.}$$

for some $B \in \mathcal{B}(G)$ satisfying $0 < \lambda(B) < \infty$.

6. The Mecke characterization

Theorem (Mecke ’67, Rother and Zähle ’90)

Let ξ be an invariant random measure and $\mathbb{Q}$ a measure on $(\Omega, \mathcal{A})$. The measure $\mathbb{Q}$ is a Palm measure of ξ with respect to some σ-finite stationary measure iff $\mathbb{Q}$ is σ-finite, $\mathbb{Q}\{\xi(G) = 0\} = 0$, and

$$\mathbb{E}_{\mathbb{Q}} \int f(\theta_g^{-1}, g^{-1})\Delta(g^{-1})\xi(dg) = \mathbb{E}_{\mathbb{Q}} \int f(\theta_e, g)\xi(dg)$$

holds for all measurable $f : \Omega \times G \to \mathbb{R}_+$.

7. Extensions

Remark

Many of the previous results can be extended to the case of invariant random measures on some state spaces S and T, on which acts G in a proper way. See Kallenberg (2007,2011) and Gentner and Last (2011).

8. References

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