Infinitely iterated Brownian motion

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Symposium in honour of Olav Kallenberg
http://www.math.uni-frankfurt.de/∼ismi/Kallenberg_symposium/
The talk was given on the blackboard. These slides were created a posteriori and represent a summary of what was presented at the symposium.

The speaker would like to thank the organizers for the invitation.

Definitions

$B_1, B_2, \ldots, B_n$ (standard) Brownian motions with 2-sided time

n-fold iterated Brownian motion: $B_n(B_{n-1}(\cdots B_1(t)\cdots))$

Extreme cases:
I. BMs independent of one another
II. BMs identical: $B_1 = \cdots = B_n$, a.s. (self-iterated BM)

We are interested in case I.

Outline

Physical motivation

Subordination
"Mixing" of time and space dimensions (c.f. relativistic processes)
Branching processes
Higher-order Laplacian PDEs
Modern physics problems

Standard heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u, \quad \text{on } D \times [0, \infty)$$
$$u(t = 0, x) = f(x)$$

is solved probabilistically by

$$u(t, x) = \mathbb{E} f(x + B(t)),$$

where B (possibly stopped) standard BM.

Higher-order Laplacian

Problems of the form

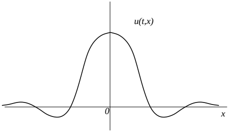
$$\frac{\partial u}{\partial t} = c\Delta^2 u, \quad \text{on } D \times [0, \infty)$$
$$u(t = 0, x) = f(x)$$

arise in vibrations of membranes. Earliest attempt to solve them "probabilistically" is by Yu. V. Krylov (1960).

Caveat: Letting $D = \mathbb{R}$, f a delta function at $x = 0$, and taking Fourier transform with respect to x, gives

$$\widehat{u}(t, \lambda) = \exp(-\lambda^4 t)$$

whose inverse Fourier transform is not positive and so signed measures are needed. Program carried out by K. Hochberg (1978). Caveat: only finite additivity on path space is achieved.



Funaki's approach

$$\frac{\partial u}{\partial t} = \frac{1}{8}\frac{\partial^4 u}{\partial x^4}$$

with initial condition f satisfying some UTC[1], is solved by

$$(t, x) \mapsto \mathbb{E}\widetilde{f}(x + \widetilde{B}_2(B_1(t)))$$

where

$$\widetilde{B}_2(t) = B_2(t)\mathbf{1}_{t\geq 0} + \sqrt{-1}\,B_2(t)\mathbf{1}_{t<0}$$

and $\widetilde{f}$ analytic extension of f from $\mathbb{R}$ to $\mathbb{C}$.

Remark: $\mathbb{E}f(x + B_2(B_1(t)))$ does not solve the original PDE [Allouba & Zheng 2001].

[1] Unspecified Technical Condition–terminolgy due to Aldous

Fractional PDEs

Density $u(t,x)$ of $x + B_2(|B_1(t)|)$ satisfies a fractional PDE of the form

$$\frac{\partial^{1/2^n} u}{\partial t^{1/2^n}} = c_n \frac{\partial^2 u}{\partial x^2}.$$

[Orsingher & Beghin 2004]

Discrete index analogy

We do know several instances of compositions of discrete-index "Brownian motions" (=random walks). For example, let

$$S(t) := X_1 + \cdots + X_t$$

be sum of i.i.d. nonnegative integer-valued RVs. Take $S_1, S_2, \ldots$ be i.i.d. copies of S. Then

$$S_n(S_{n-1}(\cdots S_1(x) \cdots))$$

is the size of the n-th generation Galton-Watson process with offspring distribution the distribution of X_1, starting from x individuals at the beginning.

Other examples...

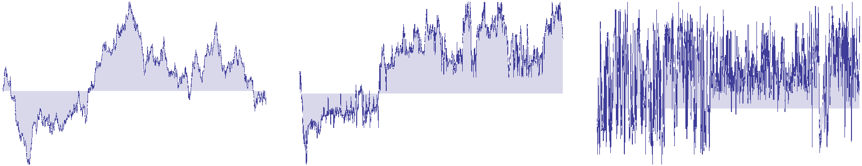
Known results on n-fold iterated BM

- Given a sample path of $B_2 \circ B_1$, we can a.s. determine the paths of B_2 and B_1 (up to a sign) [Burdzy 1992]
- $B_n \circ \cdots \circ B_1$ is not a semimartingale; paths have finite 2^n-variation [Burdzy]
- Modulus of continuity of $B_n \circ \cdots \circ B_1$ becomes bigger as n increases [Eisenbaum & Shi 1999 for $n = 2$]
- As n increases, the paths of $B_n \circ \cdots \circ B_1$ have smaller upper functions [Bertoin 1996 for LIL and other growth results]

Path behavior

$$W_n := B_n \circ \cdots \circ B_1.$$

$n = 1$ (BM) $\qquad n = 2 \qquad n = 3$



Note self-similarity

$$\left\{W_n(\alpha t),\, t \in \mathbb{R}\right\} \overset{\text{(d)}}{=} \left\{\alpha^{2^{-n}} W_n(t),\, t \in \mathbb{R}\right\}$$

Define occupation measure (on time interval $0 \le t \le 1$, w.l.o.g.)

$$\mu_n(A) = \int_0^1 \mathbf{1}\{W_n(t) \in A\}\, dt, \quad A \in \mathscr{B}(\mathbb{R})$$

which has density (local time): $\mu_n(A) = \int_A L_n(x)dx$. We expect that the "smoothness" of L_n increases with n [Geman & Horowitz 1980].

Problems

Does the limit (in distribution) of W_n exist, as $n \to \infty$?

If yes, what is W_∞?

Does the limit of μ_n exist?

What are the properties of W_∞?

Convergence of random measures

Let $\mathcal{M}$ be the space of Radon measures on $\mathbb{R}$ equipped with the topology of vague convergence. Let $\Omega := C(\mathbb{R})^{\mathbb{N}}$, and $\mathbb{P}$ the $\mathbb{N}$-fold product of standard Wiener measures on $C(\mathbb{R})$, be the "canonical" probability space. A measurable $\lambda : \Omega \to \mathcal{M}$ is a random measure. A sequence $\{\lambda_n\}_{n=1}^\infty$ of random measures converges to the random measure λ weakly in the usual sense: For any $F : \mathcal{M} \to \mathbb{R}$, continuous and bounded, we have $\mathbb{E}F(\lambda_n) \to \mathbb{E}F(\lambda)$.

Equivalently [Kallenberg, Conv. of Random Measures], $\int_{\mathbb{R}} f d\lambda_n \to \int_{\mathbb{R}} f d\lambda$, weakly as random variables in $\mathbb{R}$, for all continous $f : \mathbb{R} \to \mathbb{R}$ with compact support ("infinite-dimensional Wold device".)

One-dimensional marginals

Let $\mathcal{E}(\lambda)$ denote an exponential random variable with rate λ and let $\pm\mathcal{E}(\lambda)$ be the product of $\mathcal{E}(\lambda)$ and an independent random sign.

Theorem

For all $t \in \mathbb{R} \setminus \{0\}$,

$$W_n(t) \xrightarrow[n\to\infty]{(\mathrm{d})} \pm\mathcal{E}(2),$$

Corollary

Let $N_1, N_2, \dots$ *be i.i.d. standard normal random variables in* $\mathbb{R}$. *Then*

$$\prod_{n=1}^{\infty} |N_n|^{2^{-n}} \overset{(\mathrm{d})}{=} \mathcal{E}(2).$$

This is a probabilistic manifestation of the duplication formula for the gamma function:

$$\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\,\sqrt{\pi}\,\Gamma(2z).$$

Higher-order marginals

Recall: W_n is 2^{-n}–self-similar.

Also: $W_n(0) = 0$ and W_n has stationary increments.

Let $-\infty < s < t < \infty$. Then

$$\begin{aligned} W_2(t) - W_2(s) &= B_2(B_1(t)) - B_2(B_1(s)) \\ &\stackrel{(d)}{=} B_1(B_1(t) - B_1(s)) && \text{(by conditioning on } B_1) \\ &\stackrel{(d)}{=} B_2(B_1(t-s)) = W_2(t-s) && \text{(by conditioning on } B_2) \end{aligned}$$

By induction, true for all n.

Hence, for $s, t \in \mathbb{R} \setminus \{0\}$, $s \neq t$, if weak limit (X_1, X_2) of $(W_n(s), W_n(t))$ exists then it should have the properties that

$$\pm X_1 \stackrel{(d)}{=} \pm X_2 \stackrel{(d)}{=} \pm(X_2 - X_1)$$

The Markovian picture

$\{W_n\}_{n=1}^{\infty}$ is a Markov chain with values in $C(\mathbb{R})$:

$$W_{n+1} = B_{n+1} \circ W_n.$$

However, "stationary distribution" cannot live on $C(\mathbb{R})$.

Look at functionals of W_n, e.g., fix $(x_1, \dots, x_p) \in \mathcal{R}^p$ and consider

$$\mathcal{W}_n := (W_n(x_1), \dots, W_n(x_p)).$$

Here,

$$\mathcal{R}^p := \big\{(x_1, \dots, x_p) \in (\mathbb{R} \setminus \{0\})^p : x_i \neq x_j \text{ for } i \neq j\big\}.$$

Then $\{\mathcal{W}_n\}_{n=1}^{\infty}$ is a Markov chain in $\mathcal{R}^p$ with transition kernel

$$\mathsf{P}(x, A) = \mathbb{P}((B(x_1), \dots, B(x_p)) \in A), \quad x \in R^p, \quad A \subset \mathcal{R}^p (\text{Borel}).$$

Theorem

$\{\mathcal{W}_n\}_{n=1}^{\infty}$ *is a positive recurrent Harris chain.*

There is Lyapunov function $V : \mathcal{R}^p \to \mathbb{R}_+$,

$$V(x_1, \ldots, x_p) := \max_{1 \le i \le p} |x_i| + \sum_{0 \le i < j \le p} \frac{1}{\sqrt{|x_i - x_j|}}$$

($x_0 := 0$, by convention), such that, for C_1, C_2 universal positive constants,

$$(\mathsf{P} - I)V \le -C_1 \sqrt{V}, \quad \text{on } \{V > C_2\}.$$

Corollary

$\{\mathcal{W}_n\}_{n=1}^{\infty}$ *has a unique stationary distribution* ν_p *on* $\mathcal{R}^p$*.*

The family $\nu_1, \nu_2, \ldots$ is consistent:

$$\int_y \nu_{p+1}(dx_1 \cdots dx_{k-1}\, dy\, dx_k \cdots dx_p) = \nu_p(dx_1 \cdots dx_p).$$

Kolmogorov's extension theorem $\Rightarrow$ there exists unique probability measure ν on $\mathbb{R}^{\mathbb{N}}$ (product σ-algebra) consistent with all the ν_p. Also, $\nu_1 \stackrel{(d)}{=} \pm\mathcal{E}(2)$.

Define $\{W_\infty(x),\, x \in \mathbb{R}\}$, a family of random variables (a random element of $\mathbb{R}^{\mathbb{N}}$ with the product σ-algebra), such that

$$\big(W_\infty(x_1), \ldots, W_\infty(x_p)\big) \stackrel{(d)}{=} \nu_p, \quad \text{whenever } x = (x_1, \ldots, x_p) \in \mathcal{R}^p,$$

letting $W_\infty(0) = 0$. Then

$$W_n \xrightarrow[n\to\infty]{\text{fidis}} W_\infty$$

Properties

- If $x, y, 0$ are distinct, $W_\infty(x) \overset{(d)}{=} W_\infty(y) \overset{(d)}{=} W_\infty(x) - W_\infty(y) \overset{(d)}{=} \pm\mathcal{E}(2)$,
- If $(x_1, \ldots, x_p) \in \mathcal{R}^p$ and $1 \leq \ell \leq p$, then

$$\big(W_\infty(x_i) - W_\infty(x_\ell)\big)_{\substack{1\leq i\leq p \\ i\neq\ell}} \overset{(d)}{=} \nu_{p-1}$$

- The collection $\big(W_\infty(x),\, x \in \mathbb{R} \setminus \{0\}\big)$ is an exchangeable family of random variables: its law is invariant under permutations of finitely many coordinates

By the de Finetti/Ryll-Nardzewski/Hewitt-Savage theorem [Kallenberg, Foundations of Modern Probability, Theorem 11.10], these random variables are i.i.d., conditional on the invariant σ-algebra.

Exchangeability and directing random measure

Recall that

$$\mu_n(A) = \int_0^1 \mathbf{1}\{W_n(t) \in A\}\, dt, \quad A \in \mathscr{B}(\mathbb{R})$$

occupation measure of the n-th iterated process.

Theorem

μ_n converges weakly (in the space $\mathcal{M}$) to a random measure μ_∞. Moreover, μ_∞ takes values in the set $\mathcal{M}_1 \subset \mathcal{M}$ of probability measures.

Theorem

Let μ_∞ be a random element of $\mathcal{M}_1$ with distribution as specified by the weak limit above. Conditionally on μ_∞, let $\{V_\infty(x), x \in \mathbb{R} \setminus \{0\}\}$ be a collection of i.i.d. random variables each with distribution μ_∞. Then

$$\{W_\infty(x)\,, x \in \mathbb{R} \setminus \{0\}\} \overset{\text{(fidis)}}{=} \{V_\infty(x)\,, x \in \mathbb{R} \setminus \{0\}\}.$$

Intuition

Here are some non-rigorous statements:

- The limiting process ("infinitely iterated Brownian motion") is merely a collection of independent and identically distributed random variables with a random common distribution. (Exclude the origin!)
- Each W_n is short-range dependent. But the limit is long-range dependent. However, the long-range dependence is due to unknown a priori "parameter" (μ_∞).
- Wheras $W_n(t)$ grows, roughly, like $O(t^{1/2^n})$, for large t, the limit $W_\infty(t)$ is "bounded" (explanation coming up).

Properties of μ_∞

- μ_∞ has bounded support, almost surely.
- $$\widehat{\mu}_\infty(\omega, \xi) := \int_{\mathbb{R}} \exp(\sqrt{-1}\,\xi x)\,\mu_\infty(\omega, dx).$$
$$\mathbb{E}\widehat{\mu}_\infty(\cdot, \xi) = \int_\Omega \widehat{\mu}_\infty(\omega, \xi)\,\mathbb{P}(d\omega) = \frac{4}{4+\xi^2}.$$
- Density $L_\infty(\omega, x)$ of $\mu_\infty(\omega, dx)$ exists, for $\mathbb{P}$-a.e. ω.

We may think of $L_\infty(x)$ as the local time at level x on the time interval $0 \le t \le 1$ of the limiting "process." (This is not a rigorous statement.)

Properties of L_∞

- L_∞ is a.s. continuous.
- $\int_{\mathbb{R}} L_\infty(x)^q \, dx < \infty$, a.s., for all $1 \le q < \infty$.
- For all small $\varepsilon > 0$, the density L_∞ is locally $(1/2 - \varepsilon)$–Hölder continuous.

Oscillation

Let

$$\Delta_n(t) := \sup_{0 \le s,t \le t} \left| W_n(s) - W_n(t) \right|$$

be the oscillation of the n-th iterated process W_n on the time interval $[0, t]$.

Theorem

The limit in distribution of the random variable $\Delta_n(t)$, as $n \to \infty$, exists and is a random variable which does not depend on t:

$$\Delta_n(t) \xrightarrow[n\to\infty]{\text{(d)}} \prod_{i=0}^{\infty} D_i^{2^{-i}},$$

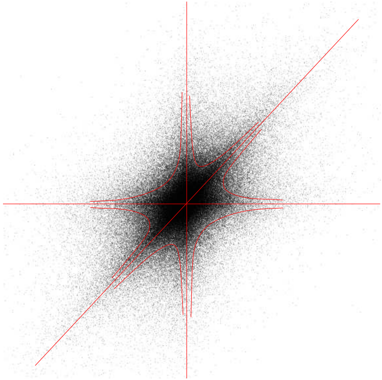
where $D_0, D_1, \ldots$ are i.i.d. copies of $\Delta_1(1)$ (the oscillation of a standard BM on the time interval $[0, 1]$.)

Joint distributions

Recall that, for $s, t \in \mathbb{R} \setminus \{0\}$, $s \neq t$, the joint law ν_2 of $(W_\infty(s), W_\infty(t))$ satisfies the remarkable property

$$\pm W_\infty(s) \overset{\text{(d)}}{=} \pm W_\infty(t) \overset{\text{(d)}}{=} \pm(W_n(s) - W_n(t)).$$

We have no further information on what this 2-dimensional law is. The following scatterplot[2] gives an idea of the level sets of the joint density:



[2] Thanks to A. Holroyd for the simulation!