Some Failures, Problems, and Conjectures from a Lifetime in Probability

Mittag-Leffler Workshop

24–28 June 2013

Olav Kallenberg

PROBLEM 1:

Extension theorem for symmetric arrays

— — — ◇ ◇ ◇ — — —

A sequence of random elements $\xi_1, \xi_2, \dots$ is said to be

- **exchangeable** if

$$(\xi_{p_1}, \xi_{p_2}, \dots) \stackrel{d}{=} (\xi_1, \xi_2, \dots)$$

  for all permutations (p_n) of N,

- **contractable** if the same relation holds for all subsequences (p_n) of N.

The two conditions are in fact equivalent (Ryll-Nardzewski).

A random array $X = (X_n) = (X_{n_1,\dots,n_d})$ is said to be

- **jointly exchangeable** if $X \circ p \overset{d}{=} X$ for all permutations $p = (p_n)$ of N,
- **jointly contractable** if $X \circ p \overset{d}{=} X$ for all subsequences $p = (p_n)$ of N,

where $X \circ p = (X_{p \circ n})$ with $p \circ n = (p_{n_1}, \dots, p_{n_d})$.

For $d > 1$ the conditions are no longer equivalent. We may take the n_j to be distinct or strictly increasing, respectively.

1. Extension problem for symmetric arrays

- A random array X is **exchangeble** (Aldous, Hoover) iff

$$X_n = f(\hat{\xi}_n), \quad n_1, \ldots, n_d \text{ distinct},$$

- A random array X is **contractable** iff

$$X_n = f(\hat{\xi}_n), \quad n_1 < \cdots < n_d,$$

for some measurable function f on $[0,1]^{2^d}$ and some i.i.d. $U(0,1)$ random variables ξ_k, $k \in \mathrm{N}^{d'}$, $d' \leq d$. Examples:

$$\begin{aligned} X_{ij} &= f(\xi_{ij}, \xi_i, \xi_j, \xi_\emptyset), \\ X_{ijk} &= f(\xi_{ijk}, \xi_{ij}, \xi_{ik}, \xi_{jk}, \xi_i, \xi_j, \xi_k, \xi_\emptyset) \end{aligned}$$

1. Extension problem for symmetric arrays

Extension theorem: A random array $X = (X_n)$, indexed by $n = (n_1, \ldots, n_d)$ with $n_1 < \cdots < n_d$, is contractable iff it can be extended to an exchangeable array, indexed by $n = (n_1, \ldots, n_d)$ with distinct $n_1, \ldots, n_d$.

- The extension is not unique, not even in distribution,
- The only known proof is by comparing the coding representations, which take ~ 50 pages to prove.

Problem: Find a direct proof.

PROBLEM 2:

Representation of exchangeable random sheets

— — — ◇ ◇ ◇ — — —

A **continuous linear random functional** (CLRF) on a Hilbert space H is a process X with

- $X(af+bg) = aXf + bXg$ a.s., $a, b \in \mathrm{R}$, $f, g \in H$,
- $\|f_n\| \to 0$ in H implies $Xf_n \xrightarrow{P} 0$.

It is said to be **rotatable** if $X \circ U \stackrel{d}{=} X$, i.e.

$$X(Uf) \stackrel{d}{=} Xf, \quad f \in H,$$

for every unitary operator U on H. A CLRF X on $H^{\otimes d}$ is **jointly rotatable** if $X \circ U^{\otimes d} \stackrel{d}{=} X$ for any unitary operator U on H:

$$X(U^{\otimes d} f) \stackrel{d}{=} Xf, \quad f \in H^{\otimes d}.$$

A CLRF X on $H^{\otimes d}$ is **jointly rotatable** (Aldous, OK) iff a.s.

$$Xf = \sum_{\pi \in \mathcal{O}_d} \Big(\bigotimes\nolimits_{k \in \pi} \zeta_{|k|} \Big) (\alpha_\pi \otimes f), \quad f \in H^{\otimes d}.$$

- $\mathcal{O}_d$ — class of partitions π of $\{1, \ldots, d\}$ into *ordered* subsets k of size $|k|$,
- $\zeta_1, \ldots, \zeta_d$ — independent, isonormal Gaussian processes on $H^{\otimes 2}, \ldots, H^{\otimes (d+1)}$,
- $\bigotimes_k \zeta_{|k|}$ — multiple Wiener-Itô integrals,
- α_π — random elements in $H^{\otimes \pi}$, $\pi \in \mathcal{O}_d$.

Basis representation for $d = 2$:

$$X_{ij} = \alpha\, \zeta_{ij} + \beta\, (\zeta_i\, \zeta_j - \delta_{ij}), \quad i, j \in \mathbb{N}.$$

A *random sheet* X on R_+^d (continuous process vanishing on all coordinate hyperplanes) has **jointly exchangeable increments** iff a.s.

$$X_t = \sum_{\pi \in \mathcal{P}_d} \sum_{\kappa \in \hat{\mathcal{O}}_\pi} \Big(\lambda^{\kappa^c} \otimes \bigotimes\nolimits_{k \in \pi} \zeta_{|k|} \Big) (\alpha_{\pi,\kappa} \otimes [0, \hat{t}_\pi]), \quad t \in \mathrm{R}_+^d.$$

- $\mathcal{P}_d$ — class of partitions π of $\{1, \ldots, d\}$,
- $\hat{\mathcal{O}}_\pi$ — set of partial partitions of π into ordered subsets κ,
- λ^{κ^c} — Lebesgue measure on $\mathrm{R}_+^{\kappa^c}$,
- $\hat{t}_\pi$ — vector with components $\hat{t}_{\pi,J} = \min_{j \in J} t_j$, $J \in \pi$.

Similar representations hold for **separately exchangeable sheets** on R_+^d. A scaling argument yields the corresponding result for sheets on $[0,1]^d$, with the ζ_π replaced by suitably tied-down versions $\hat{\zeta}_\pi$.

Failure: I don't know how to adapt the scaling argument to the jointly exchangeable case.

Conjecture: The previous representation remains valid for jointly exchangeable random sheets on $[0,1]^d$, with the ζ_π replaced by their tied-down versions.

PROBLEM 3:

Approximations in fractal random measures

— — — ◇ ◇ ◇ — — —

Let ξ be a random measure on R^d supported by a **"fractal"** random set Ξ. Consider the restriction ξ^ε of Lebesgue measure to the ε-neighborhood of the support:

$$\xi^\varepsilon = \Xi^\varepsilon \cdot \lambda^d, \quad \varepsilon > 0.$$

Problem (Lebesgue approximation): When is there a normalizing function m_ε such that as $\varepsilon \to 0$

$$m_\varepsilon \, \xi^\varepsilon \xrightarrow{v} \xi, \quad \text{a.s. or in probability ?}$$

Assuming ξ to be the Hausdorff measure of Ξ, can we obtain m from the corresponding gauge function ϕ ?

Examples: Local time (Kingman), superprocesses (Tribe, OK).

Let B_x^ε be the ε-ball centered at x:

$$B_x^\varepsilon = \{y \in \mathrm{R}^d;\ |x-y| < \varepsilon\}, \quad x \in \mathrm{R}^d,\ \varepsilon > 0.$$

Problem (hitting approximation): When are there normalizing functions m_ε and $m_{n,\varepsilon}$ such that

$$P\{\xi B_x^\varepsilon > 0\} \sim m_\varepsilon \, E\xi B_x^\varepsilon, \qquad \text{a.e. } E\xi,$$

$$P \bigcap_{k \le n} \{\xi B_{x_k}^\varepsilon > 0\} \sim m_{n,\varepsilon} \, E \prod_{k \le n} \xi B_{x_k}^\varepsilon, \qquad \text{a.e. } E\xi^n\,?$$

Are those normalizations related to the previous one?

Examples: Point processes, local time, superprocesses

Let $\mathcal{L}(\eta \,\|\, \xi)_x$ be the **Palm distribution** of η with respect to ξ, evaluated at the point $x \in \mathbb{R}^d$.

Problem (Palm approximation): Give conditions ensuring

$$P[\,\eta \in \cdot \mid \xi B_x^\varepsilon > 0\,] \to \mathcal{L}(\eta \,\|\, \xi)_x, \qquad \text{a.e. } E\xi,$$

$$P[\,\eta \in \cdot \mid \prod_{k \le n} \xi B_{x_k}^\varepsilon > 0\,] \to \mathcal{L}(\eta \,\|\, \xi)_{x_1,\dots,x_n}, \qquad \text{a.e. } E\xi^n.$$

Examples: Point processes, local time, superprocesses

PROBLEM 4:

Tree structures in superprocesses

— — — ◇ ◇ ◇ — — —

A **superprocess** ξ (measure-valued branching diffusion process) may be thought of as a randomly evolving random cloud.

The evolution of ξ is determined by the *historical process*, consisting of **historical trees** rooted at the sites of the *ancestors* at time 0. When ξ_0 is non-random, ξ is infinitely divisible, and the tree structure gives the associated **cluster representation**.

For Dawson–Watanabe processes ξ, the genealogy of the historical tree is encoded by a *Brownian excursion*, and ξ itself may be generated by a *Brownian snake* (Le Gall).

Dynkin found recursive formulas for the **moment measures** of a general superprocess, obtainable from a discrete branching structure, composed of finite **moment trees**. For a DW-process, the trees can be generated by a *forward*, *backward*, or *sideways* recursion (Etheridge).

Problem: Construct moment trees for a general superprocess, examine their recursive properties, and clarify their relations to the historical tree.

For stationary, discrete-time branching processes in R^d, the univariate Palm distributions are given by a certain **backward tree**, useful to derive criteria for persistence or extinction.

The **Palm distributions** are obtained by disintegration of the *Campbell measures*, which are extensions of the moment measures. This suggests the existence of **Palm trees**.

Problem: Find a discrete branching structure determining the multivariate Palm distributions of a superprocess. Look for recursive properties, and clarify the relations to the historical, backward, and moment trees.

PROBLEM 5:

Multiple point process integrals

— — — ◇ ◇ ◇ — — —

Consider existence criteria and continuity properties of the **multiple stochastic integrals**

$$X_1 \cdots X_d f = \int \cdots \int f \, dX_1 \cdots dX_d,$$

where $X = (X_1, \ldots, X_d)$ is a positive or symmetric **Lévy process**. When X is a Brownian motion, those are **multiple Wiener-Itô integrals**.

The purely discontinuous case may be reduced to that of positive or symmetric multiple **Poisson integrals** $\xi^d f$ or $\tilde{\xi}^d f$. Here necessary and sufficient conditions are known (OK & Szulga). In particular, the integrals $\tilde{\xi}^d f$, $\xi^d f^2$, $\tilde{\xi}_1 \ldots \tilde{\xi}_d f$, and $\xi_1 \cdots \xi_d f^2$ exist simultaneously, and precise criteria are given recursively in terms of ordinary *Lebesgue integrals*.

A point process ξ is said to be a **Cox process directed by** η, if it is conditionally Poisson given η with $E[\xi|\eta] = \eta$. All results for multiple Poisson integrals carry over to the Cox case.

Two point processes ξ_1 and ξ_2 on R_+ are said to be **tangential** if they have the same compensator $\hat{\xi}_1 = \hat{\xi}_2$ a.s.

Coxification: For every simple, quasi-left-continuous (qlc) point process ξ on R_+ with compensator $\hat{\xi}$, there exists a tangential Cox process ξ' directed by $\hat{\xi}$.

Decoupling: For *independent* point processes $\xi_1, \ldots, \xi_d$ with compensators $\hat{\xi}_1, \ldots, \hat{\xi}_d$, the existence and continuity properties of the multiple integrals $\xi_1 \cdots \xi_d f$ and $\tilde{\xi}_1 \cdots \tilde{\xi}_d f$ carry over to any tangential processes $\xi'_1, \ldots, \xi'_d$. In particular, they agree when the ξ'_i are Cox and directed by $\hat{\xi}_1, \ldots, \hat{\xi}_d$.

Failure: I don't know how to extend the decoupling argument to the integrals $\xi^d f$ and $\tilde{\xi}^d f$:

Conjecture: The existence and continuity criteria for multiple Poisson integrals $\xi^d f$ and $\tilde{\xi}^d f$ extend to any simple, qlc point processes ξ, with all Lebesgue integrals replaced by integrals with respect to the compensator $\hat{\xi}$.

PROBLEM 6:

Stationary line and flat processes

— — — ◇ ◇ ◇ — — —

Stationarity and invariance (Davidson, Krickeberg): Let η be a stationary (first order) random measure on the space of lines in R^2, such that a.s. η^2 charges no pairs of parallel lines. Then η is a.s. shift invariant. — Many multivariate extensions and asymptotic results are known (Papangelou, OK).

Cox equivalence: For any stationary line process in R^2 with no pairs of parallel lines, there exists a Cox line process with invariant directing measure and the same first and second moments (Davidson).

Rollo Davidson (1944–1970) thought that every stationary line process in the plane with a.s. no pairs of parallel lines might be Cox. Counterexamples have since been found.

For any simple point process ξ, the associated **Papangelou intensity measure** $\hat{\xi}$ is given by

$$\sum\nolimits_j E[\,\xi B_{nj} \,|\, B_{nj}^c \cdot \xi\,] \to \hat{\xi} B \;\text{ a.s.}, \quad n \to \infty,$$

for any Borel set B with dissection (B_{nj}). We may regard $\hat{\xi}$ as *dual* to the set of Palm distributions.

Invariance and Cox structure (Papangelou, OK):

- If $\hat{\xi}$ is a.s. invariant, then ξ is Cox with invariant directing measure.
- If $\hat{\xi}_1, \hat{\xi}_2, \ldots$ are asymptotically invariant, then $\xi_1, \xi_2, \ldots$ are asymptotically Cox with invariant directing measures.

Stationarity and Cox structure: Let ξ be a stationary process of k-flats in $\mathbb{R}^d$. If the associated Papangelou intensity measure $\hat{\xi}$ is suitably regular, it is a.s. shift invariant, and so ξ is a Cox process with invariant directing measure (Papangelou).

Failure: I don't know any general conditions on ξ that imply the required regularity of $\hat{\xi}$.

Problem: Let ξ be a stationary process of k-flats in $\mathbb{R}^d$. Find regularity conditions directly on ξ, ensuring it to be Cox with invariant directing measure. Similarly for the corresponding asymptotic property.

Problem summary:

1. Extension theorem for symmetric arrays
2. Representation of exchangeable random sheets
3. Approximations in fractal random measures
4. Tree structures in superprocesses
5. Multiple point process integrals
6. Stationary line and flat processes