MARTINGALE METHODS FOR OBTAINING

GERBER-SHIU TYPE FORMULAS

IN RISK THEORY

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Risk process

$$X_t = x + ct - \sum_{n=1}^{N_t} Y_n$$

with $x \geq 0$, $c > 0$. The *time to ruin*

$$\tau = \inf\{t \geq 0 : X_t < 0\}$$

with $\inf \varnothing = \infty$. *Overshoot* and *surplus prior to ruin*.)

$$X_\tau \quad \text{and} \quad X_{\tau-}$$

defined on $(\tau < \infty)$ only. Gerber-Shiu problem: finding the joint distribution of $(\tau, X_{\tau-}, X_\tau)$.

Basic example: $\mathbf{X}$ is compound Poisson with linear drift, intensity $\lambda > 0$, (Y_n) iid independent of $\mathbf{N}$.

Basic subexample: Y_n exponential rate $\mu > 0$. Ruin certain ($\mathbb{P}_x(\tau < \infty) = 1$ for all $x \geq 0$) iff $\mu \leq \lambda c$ — what is then the $\mathbb{P}_x$-distribution of $X_{\tau-}$?

F the distribution of Y_n,

$$F = \pi F^{+} + (1 - \pi) F^{-}$$

with $0 < \pi \leq 1$, F^{+} a probability on $]0, \infty[$, F^{-} a probability on $]-\infty, 0]$:

$$F^{+}(\mathrm{d}y) = \sum_{r=1}^{m} \alpha_r \mu_r e^{-\mu_r y}\, \mathrm{d}y \quad (y > 0)$$

with $m \in \mathbb{N}$, $0 < \mu_1 < \cdots < \mu_m$, all $\alpha_r \neq 0$, $\sum_r \alpha_r = 1$, density ≥ 0. Laplace transforms

$$\begin{aligned}
L^{+}(\nu) &= \int_0^{\infty} e^{-\nu y}\, F^{+}(\mathrm{d}s) = \sum_{r=1}^{m} \alpha_r \frac{\mu_r}{\mu_r + \nu} \quad (\mathrm{Re}\,\nu \geq 0)\,, \\
L^{-}(\nu) &= \int_0^{\infty} e^{-\nu y}\, F^{-}(\mathrm{d}s) \quad (\mathrm{Re}\,\nu \leq 0)\,, \\
\overline{L}(\nu) &= \pi \overline{L}^{+}(\nu) + (1 - \pi) L^{-}(\nu) \quad (\mathrm{Re}\,\nu \leq 0,\ \nu \neq \text{all } -\mu_r)\,,
\end{aligned}$$

$\overline{L}^{+}$ analytic extension of L^{+}.

MARTINGALE STRUCTURE: $Q = \sum_{n=1}^{\infty} \varepsilon_{(T_n, Y_n)}$ random counting measure on $\mathbb{R}_+ \times \mathbb{R}$, compensating measure (natural filtration) $\Lambda = \lambda\ell \otimes F$, martingales

$$\begin{aligned} M_t &= \int_{]0,t] \times \mathbb{R}} S(s,y)\,(Q(\mathrm{d}s, \mathrm{dy}) - \Lambda(\mathrm{d}s, \mathrm{dy})) \\ &= \sum_{n=1}^{N_t} S(T_n, Y_n) - \lambda \int_0^t \mathrm{d}s \int_{\mathbb{R}} F(\mathrm{d}y)\; S(s,y) \end{aligned}$$

with each process $\mathbf{S}(\cdot, y)$ predictable (here just adapted, *left*-continuous). (True MG if $\mathbf{S}$ uniformly bounded on $]0,t] \times \mathbb{R} \times \Omega$ for all $t > 0$).

The joint distribution of (τ, X_τ): for $\theta \geq 0$, $h : \mathbb{R} \to \mathbb{C}$ suitably nice, want MG of the form

$$M_t = e^{-\theta(\tau \wedge t)} h\left(X_{\tau \wedge t}\right).$$

Can obtain ($X_0 \equiv x \geq 0$)

$$e^{-\theta_{\tau \wedge t}} h\left(X_{\tau \wedge t}\right) = h(x) + \int_0^t V_s \, \mathrm{d}s + M_t.$$

For $t \leq \tau$, Identify $\mathbf{S}$ by matching jumps, then identify $\mathbf{V}$ by matching derivatives d/dt:

$$e^{-\theta_{\tau \wedge t}} h\left(X_{\tau \wedge t}\right) = h(x) + \int_0^t \mathbf{1}_{\tau \geq s} e^{-\theta s} \left(\mathcal{A} h\left(X_s\right) - \theta h(X_s)\right) \mathrm{d}s + M_t$$

with $\mathcal{A}$ infinitesimal generator for $\mathbf{X}$,

$$\mathcal{A} h(x) = c h'(x) + \lambda \int_{\mathbb{R}} \left(h(x-y) - h(x)\right) F\left(\mathrm{d}y\right) \quad \left(x \geq 0\right),$$

h bounded, h on $]0, \infty[$ 'nice', h on $]-\infty, 0]$ measurable.

For $\theta > 0$, $\left(e^{-\theta(\tau\wedge t)}h\left(X_{\tau\wedge t}\right)\right)_{t\geq 0}$ is a martingale and

$$\mathbb{E}_x\left[e^{-\theta\tau}h\left(X_\tau\right);\tau<\infty\right]=h(x)\quad(x\geq 0)$$

provided h is a *partial eigenfunction* for $\mathcal{A}$ corresponding to the eigenvalue θ,

$$\mathcal{A}h(x)=\theta h(x)\quad(x\geq 0)\,.$$

With F^+ as above, the desired partial eigenfunction essentially has the form

$$h(x)=\sum_{k=1}^{m}d_k e^{\gamma_k x}\quad(x\geq 0)$$

where for $\theta > 0$ the $\gamma_k = \gamma_k(\theta)$ are the precisely m solutions γ to the *Cramér-Lundberg equation*

$$c\gamma-(\lambda+\theta)+\lambda\overline{L}\left(\gamma\right)=0 \tag{1}$$

with $\operatorname{Re}\gamma < 0$ and the d_k are found as the solutions to the linear system (2) below.

Theorem 1 *Let $\phi :]0, \infty[\to \mathbb{R}$ be bounded and measurable.*

(i) For $\theta > 0$

$$\mathbb{E}_x \left[e^{-\theta\tau} \phi\left(-X_\tau\right); \tau < \infty \right] = \sum_{k=1}^{m} d_k e^{\gamma_k x} \quad (x \geq 0)$$

provided the solutions γ_k to (1) are distinct and with the d_k the unique solutions to the system

$$\sum_{k=1}^{m} \frac{\mu_r}{\mu_r + \gamma_k} d_k = I_r\left(\phi\right) \quad (1 \leq r \leq m) \tag{2}$$

where

$$I_r\left(\phi\right) = \int_0^\infty \mu_r e^{-\mu_r z} \phi\left(z\right) \, \mathrm{d}z.$$

(ii) For $\theta = 0$

$$\mathbb{E}_x \left[\phi\left(-X_\tau\right); \tau < \infty \right] = \sum_{k=1}^{m} d_k e^{\gamma_k} \quad (x \geq 0)$$

where if $c/\lambda \leq \mathbb{E} Y_n$ (ruin is certain) $\gamma_1, \ldots, \gamma_{m-1}$ are the precisely $m-1$ (assumed distinct) solutions to (1) with $\theta = 0$ having $\operatorname{Re} \gamma_k < 0$ and $\gamma_m = 0$ while if $c/\lambda > \mathbb{E} Y_n$ (ruin is uncertain) $\gamma_1, \ldots, \gamma_m$ are the precisely m (assumed distinct) solutions to (1) with $\theta = 0$ having $\operatorname{Re} \gamma_k < 0$. In both cases find the d_k by solving (2).

MORE SOPHISTICATED MODELS: MJ AAP 2005. $(\mathbf{X},\mathbf{J})$ Markov with $\mathbf{J}$ Markov chain on finite state space; between jumps $\mathbf{X}$ moves as a $\mathbf{J}$-dependent Brownian motion (could be degenerate); intensity for $\mathbf{X}$ to jump state-dependent, $\mathbf{J}$ regenerates at each jump; ruin by 'creeping' possible.
MJ JAP 2012, $(\mathbf{X},\mathbf{Z})$ Markov, $c \geq 0$ (only ruin by jump)

$$X_t = x + \int_0^t c\,(Z_s)\;\mathsf{d}s - \sum_{n=1}^{N_t} U_n,$$

$\mathbf{Z}$-dependent jump intensity. Generator

$$\begin{aligned}\mathcal{A}f\,(x,z) \;&=\; \mathcal{A}_{\mathbf{Z}^\circ} f\,(x,\cdot)\,(z) + c(z)\frac{\partial}{\partial x} f\,(x,z) \\ &\quad +\lambda\,(z)\int_{\mathbb{R}} F\,(\mathsf{d}y)\int_E B\,(\mathsf{d}v)\;(f\,(x-y,v) - f\,(x,z))\,.\end{aligned}$$

Results of form

$$\mathbb{E}_{x,z}\left[e^{-\theta\tau+\zeta X_\tau};\tau<\infty\right] = \sum_{k=1}^{m} d_k\,(z)\,e^{\gamma_k x} \quad (x \geq 0,\ \text{all } z)\,.$$

GERBER-SHIU: attempt a decomposition

$$e^{-\theta\tau\wedge t} g\left(X_{\tau\wedge t-}, X_{\tau\wedge t}\right) = g(x,x) + \int_0^t V_s \,\mathsf{d}s + M_t$$

with $\mathbf{M}$ a martingale? Of course not, non-sensical!

Consider processes $\mathbf{Z}$ of the form

$$Z_t = \begin{cases} e^{-\theta t} h\left(X_t\right) & (t < \tau) \\ e^{-\theta \tau} g\left(X_{\tau-}, X_\tau\right) & (t \geq \tau) \,. \end{cases}$$

Decomposition

$$Z_t = h(x) + \int_0^t \mathbf{1}_{\tau \geq s} e^{-\theta s} \Big(c h'\left(X_s\right) - \left(\lambda + \theta\right) h\left(X_s\right) + Q\left(X_s\right) \Big) \, \mathrm{d}s + M_t$$

where for $x \geq 0$

$$\begin{aligned} Q(x) \;\; = \;\; & \lambda \left(1 - \pi\right) \int_{]-\infty,0]} F^{-}\left(\mathrm{d}y\right) \, h\left(x - y\right) \\ & + \lambda \pi \int_0^x F^{+}\left(\mathrm{d}y\right) \, h\left(x - y\right) + \lambda \pi \int_x^\infty F^{+}\left(\mathrm{d}y\right) \, g\left(x, x - y\right) . \end{aligned}$$

If h, g are bounded with h nice and

$$ch'(x) - (\lambda + \theta) h(x) + Q(x) = 0 \quad (x \geq 0),$$

then $\mathbf{Z}$ is a martingale and

$$\mathbb{E}_x \left[e^{-\theta\tau} g(X_{\tau-}, X_\tau); \tau < \infty \right] = h(x) \quad (x \geq 0).$$

With F^+ as before and simple choices for g, h can be found as a linear combination of $2m$ exponentials:

Theorem 2 *Let γ_k be the solutions to (1) as before (whether $\theta > 0$ or $\theta = 0$), let $\phi : [0, \infty[\to \mathbb{R}$ be bounded and measurable and let $\rho \geq 0$. Then*

$$\mathbb{E}_x \left[e^{-\theta\tau - \rho X_{\tau-}} \phi\left(-X_\tau\right); \tau < \infty \right] = \sum_{k=1}^{m} d_k e^{\gamma_k x} + \sum_{r=1}^{m} A_r e^{-(\mu_r + \rho)x} \quad (x \geq 0)$$

where

$$A_r = \frac{\lambda\pi\alpha_r I_r\left(\phi\right)\rho}{c\left(\mu_r + \rho\right)\rho + \left(\lambda + \theta\right)\rho - \lambda\rho\overline{L}\left(-\left(\mu_r + \rho\right)\right)}$$

and the d_k are the unique solutions to the system

$$\sum_{k=1}^{m} \frac{\mu_r}{\mu_r + \gamma_k} d_k = \sum_{r_0=1}^{m} \frac{\mu_r A_{r_0}}{\left(\mu_{r_0} + \rho\right) - \mu_r} \quad \left(1 \leq r \leq m\right).$$

Remark 3 *Note that for $\rho = 0$, $\lambda\rho\overline{L}\left(-\left(\mu_r + \rho\right)\right) = -\lambda\pi\alpha_r\mu_r$ so $A_r = 0$ while $\mu_r A_r / \rho \Big|_{\rho=0} = I_r\left(\phi\right)$.*

Example 4 *Downward jumps only, exponential at rate* μ*:* $m = 1$*,* $\pi = 1$*. Ruin certain iff* $c/\lambda \leq 1/\mu$ *and then* $-X_\tau \perp (\tau, X_{\tau-})$ *and* $-X_\tau$ *is* $\exp(\mu)$*. For* $\theta = 0$ *use* $\gamma = 0$ *so*

$$A = \frac{\lambda I(\phi)\rho}{(\lambda + c\rho)(\mu + \rho)}, \quad d = A\frac{\mu}{\rho},$$

Taking $\phi \equiv 1$*,*

$$\mathbb{E}_x\left[e^{-\rho X_{\tau-}}; \tau < \infty\right] = \frac{\lambda/c}{\lambda/c + \rho}\left(\frac{\mu}{\mu + \rho} + \frac{\rho}{\mu + \rho}e^{-(\mu+\rho)x}\right) \quad (x \geq 0).$$

Result: let $U \perp V$*,* $U \sim \exp(\mu)$*,* $V \sim \exp(\lambda/c)$*. The* $\mathbb{P}_x$*-distribution of* $X_{\tau-}$ *is that of*

$$V + \min(U, x).$$

Second result: if ruin is uncertain, $c/\lambda > 1/\mu$*, the conditional* $\mathbb{P}_x$*-distribution of* $X_{\tau-}$ *given* $\tau < \infty$ *is that of*

$$U + \min(V, x).$$

NON iid JUMPS, THE ALBRECHER-BOXMA MODEL. $(\mathbf{X}, \mathbf{J})$ Markov with $\mathbf{J}$ Markov chain on finite state space. Generator

$$\mathcal{A}f(x,i) = c_i \frac{\partial}{\partial x} f(x,i) + \lambda_i \sum_{j \in E} p_{ij} \int_{\mathbb{R}} F_j(\mathsf{d}y) \left(f(x-y,j) - f(x,y) \right).$$

Assume $F_j = \pi_j F_j^{+} + \left(1 - \pi_j\right) F_j^{-}$,

$$F_j^{+}(\mathsf{d}y) = \sum_{r=1}^{m_j} \alpha_{jr} \mu_{jr} e^{-\mu_{jr}} \, \mathsf{d}y.$$

Real challenge: which Cramér-Lundberg equation? Sometimes the following is true: define

$$q_j(\gamma) = \frac{\lambda_j}{c_j \gamma - \lambda_j - \theta}, \quad h_{ij}(\gamma) = p_{ij} q_j(\gamma) \overline{L}_j(\gamma)$$

Then for $\theta > 0$,

$$\mathbb{E}_{x,i}\left[b_{J_\tau} e^{-\theta\tau}\phi(-X_\tau); \tau < \infty\right] = \sum_{k\in K} q_i(\gamma_k)\, v_i^{(k)} \beta_k e^{\gamma_k x} \quad (x \geq 0, i \in E)$$

with K an index set of $\overline{m} = \sum_j m_j$ elements, the γ_k the precisely $\overline{m}$ solutions (assumed distinct) to the equation

$$\det(I + H(\gamma)) = 0$$

with $\operatorname{Re}\gamma < 0$, $\left(v_i^{(k)}\right)_{i\in E}$ is for each k a non-zero right eigenvector for the matrix $H(\gamma_k)$ corresponding to the eigenvalue -1 and the β_k are the unique solutions to the system

$$\sum_{k\in K} \frac{1}{\mu_{jr} + \gamma_k} q_j(\gamma_k)\, v_j^{(k)} \beta_k = b_j I_{jr}(\phi) \quad \left(j \in E, 1 \leq r \leq m_j\right).$$