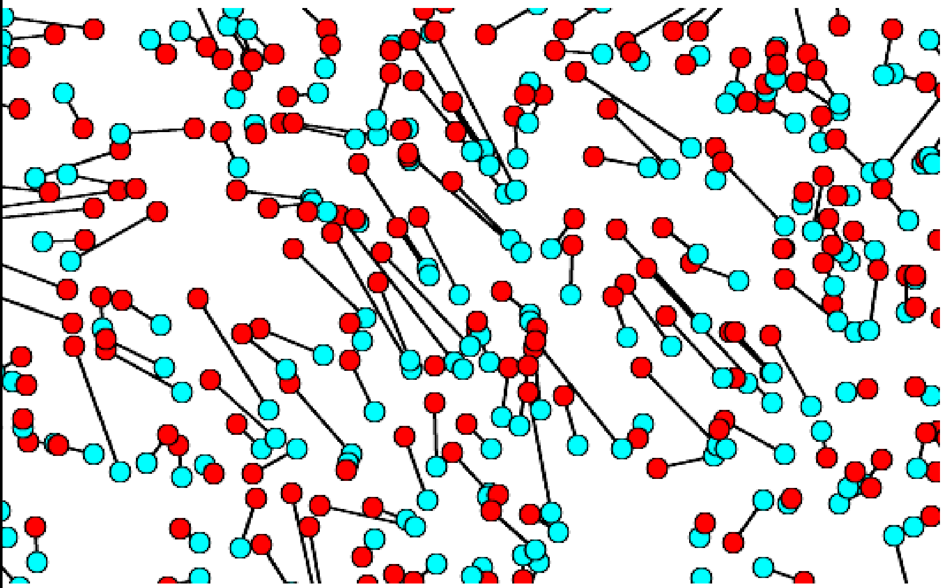
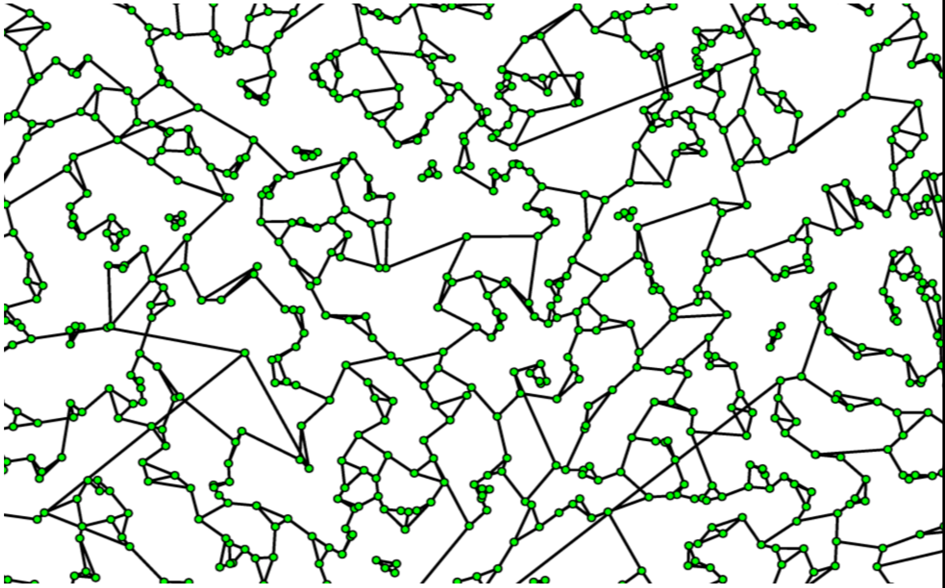
Invariant Matching and Allocation

Alexander E. Holroyd, Microsoft Research





Red points

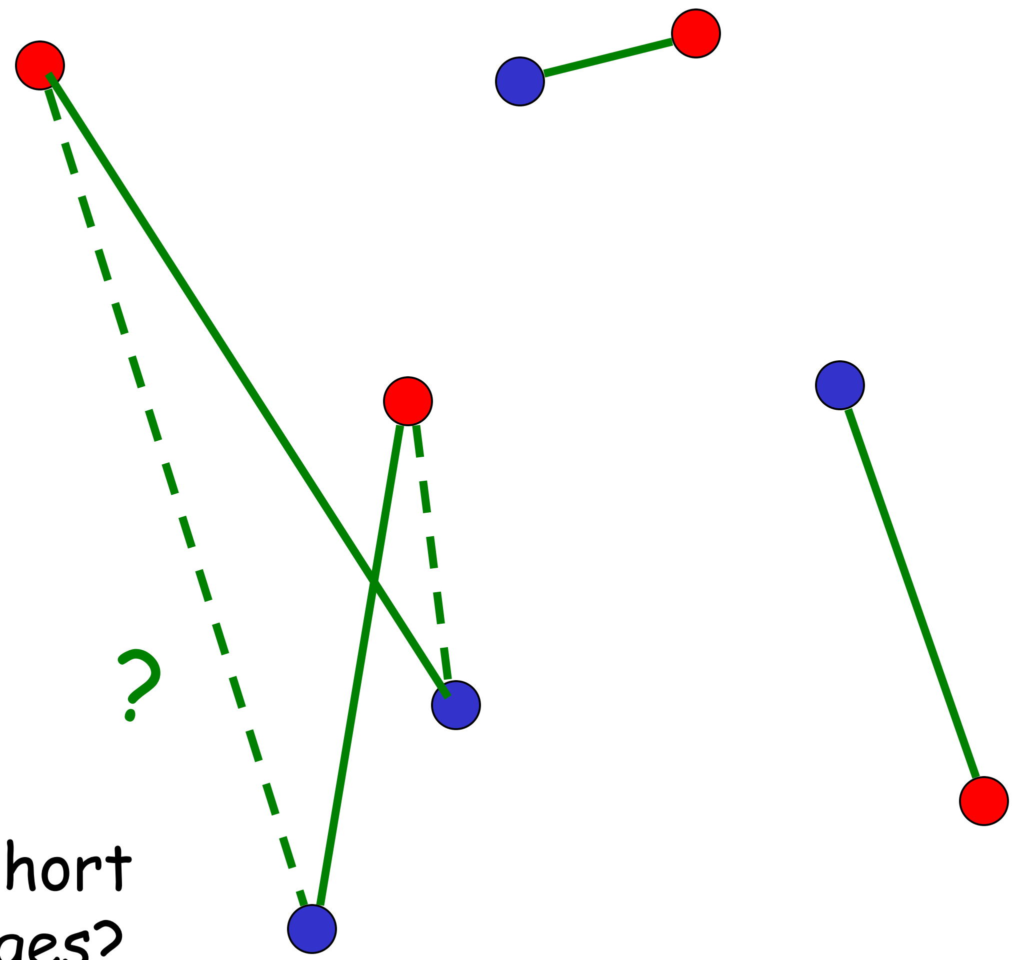
Blue points

Perfect matching

Questions:

Quantitative- how short can we make the edges?

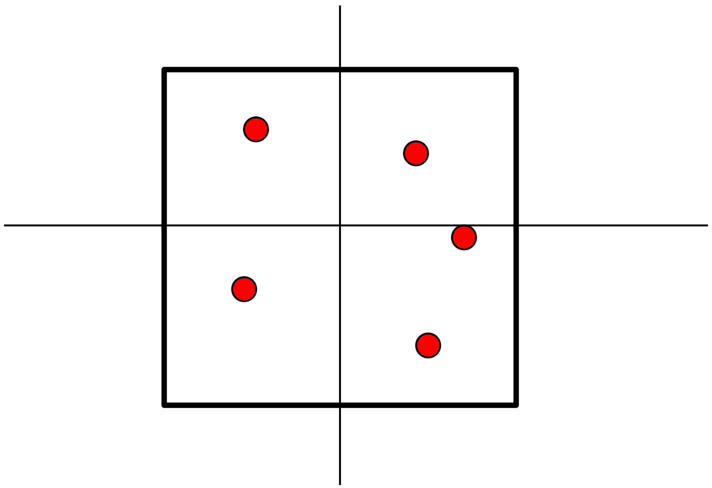
Geometric...



Local/greedy/non-random matching rules?

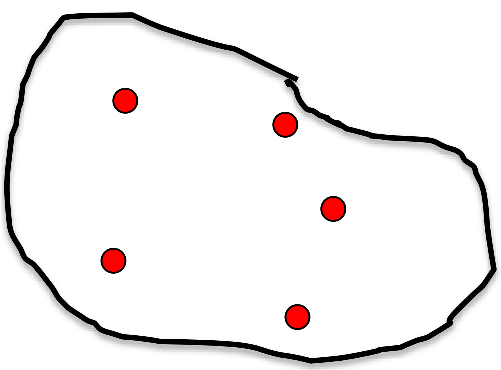
Most basic model of infinitely many random points in $\mathbb{R}^d$:

Intensity-1 homogeneous Poisson point process:



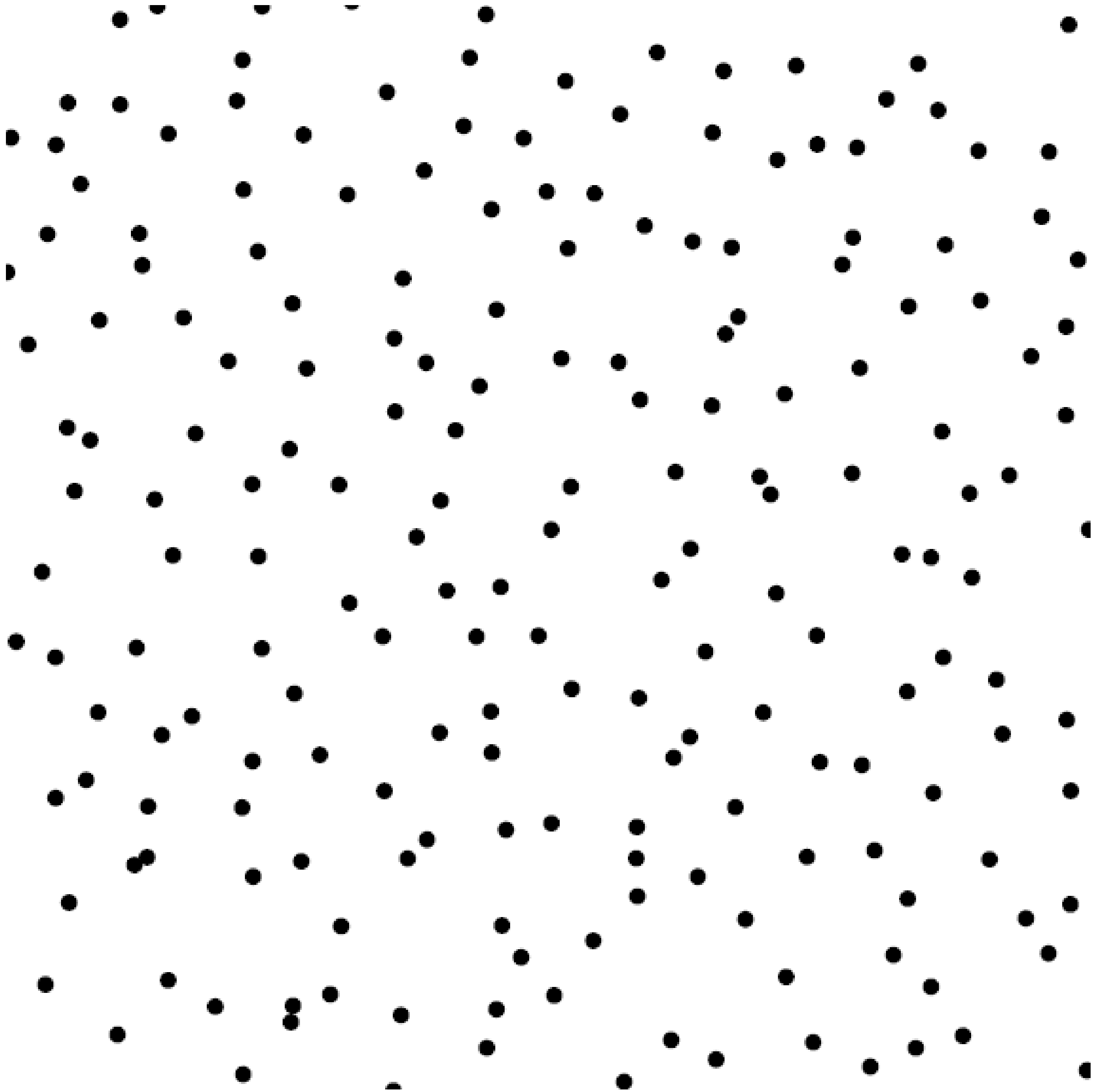
$$\lim_{n \to \infty} \left(\begin{array}{l} \text{n uniformly random points} \\ \text{in cube of volume n} \end{array} \right)$$

Equivalently,

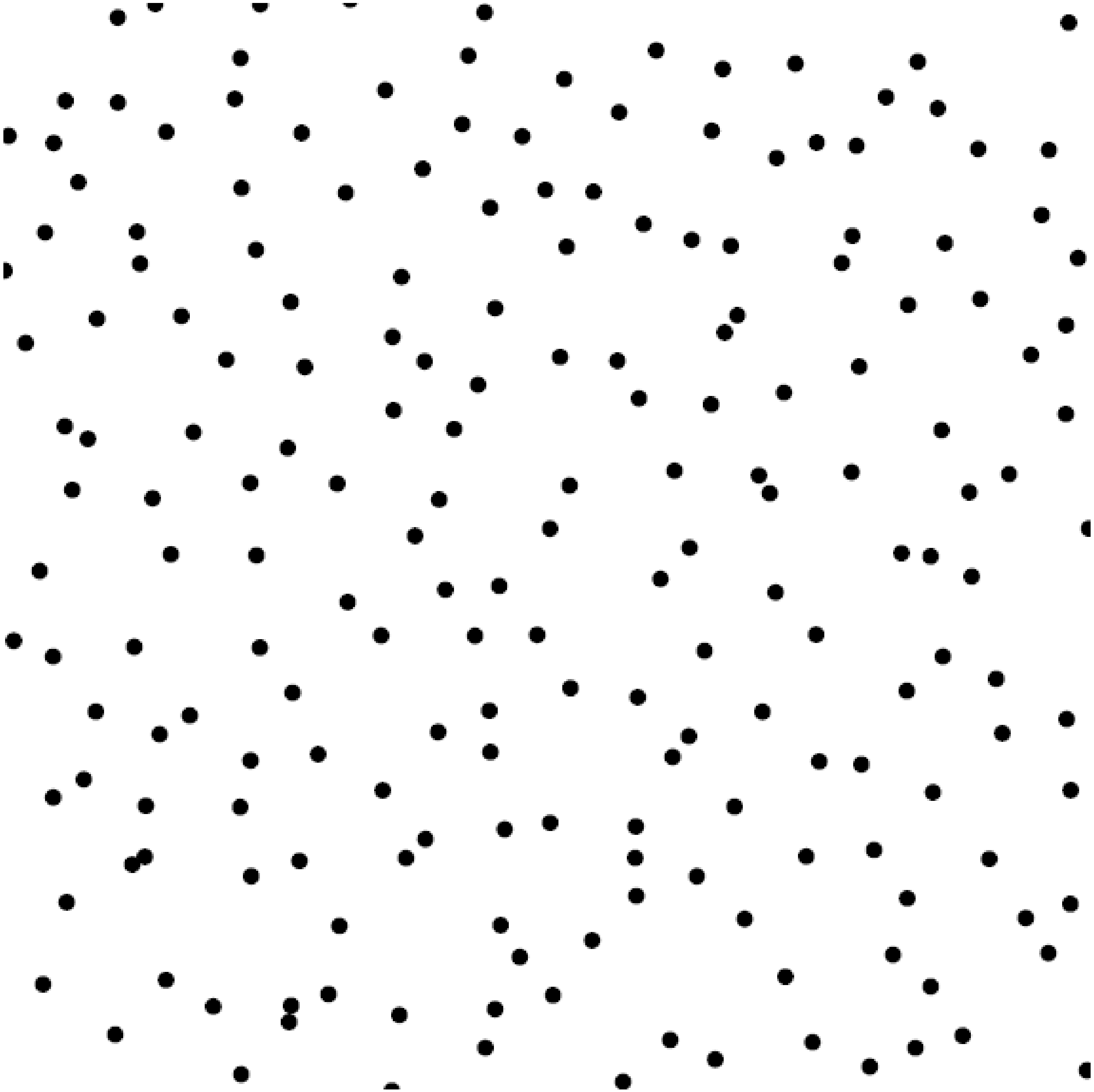


# of points K in any set A
~ Poisson distribution of mean μ=vol(A)

$$P(K = k) = e^{-\mu} \frac{\mu^k}{k!}$$



Poisson Process on R^2



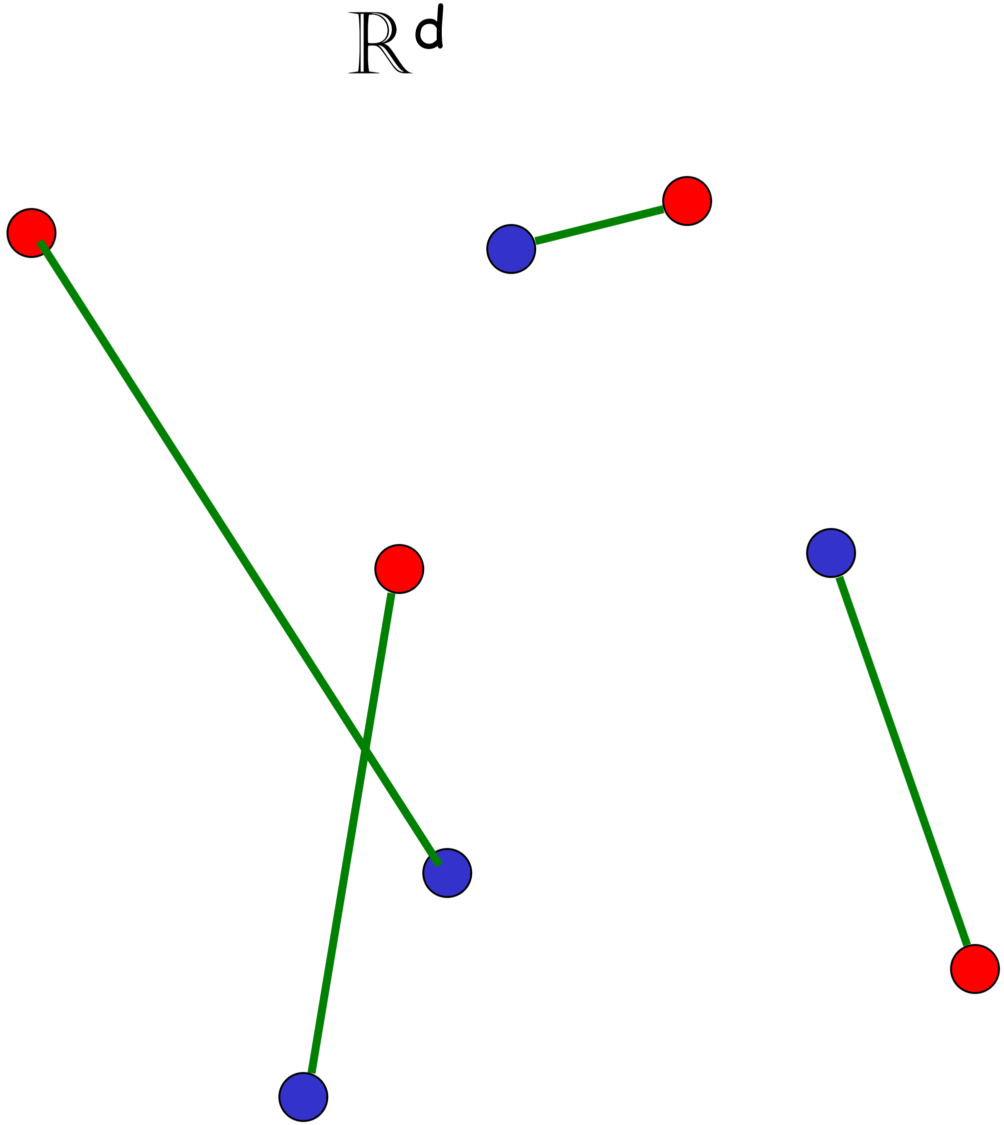
Zeros of random analytic function (Sodin, Tsirelson 2004)

Poisson process $\mathcal{R}$
of red points

Independent
Poisson process $\mathcal{B}$
of blue points

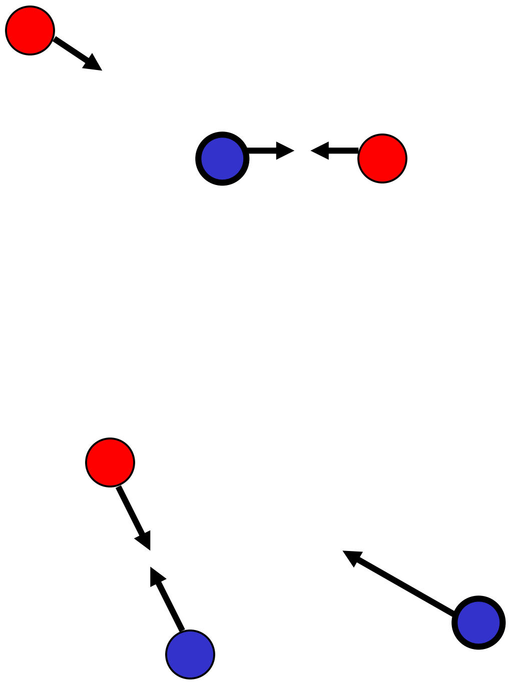
(Random) perfect
matching scheme $\mathcal{M}$

Assume ($\mathcal{R}$, $\mathcal{B}$, $\mathcal{M}$)
translation-invariant
in distribution



Example: Gale-Shapley stable matching.

- Match all *mutually closest* red/blue pairs.



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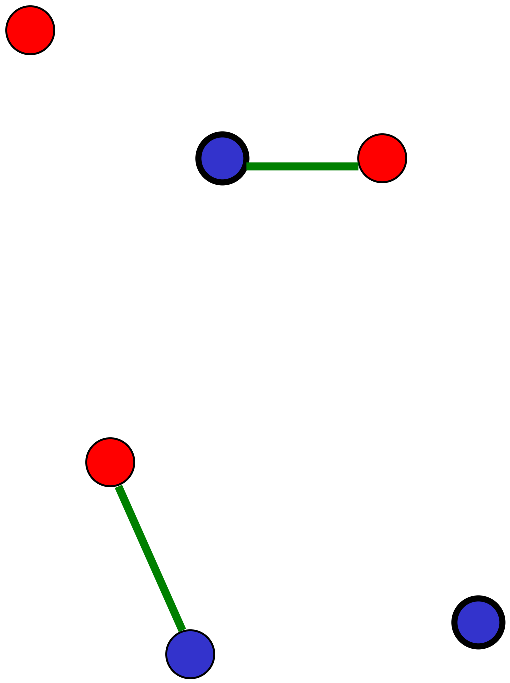
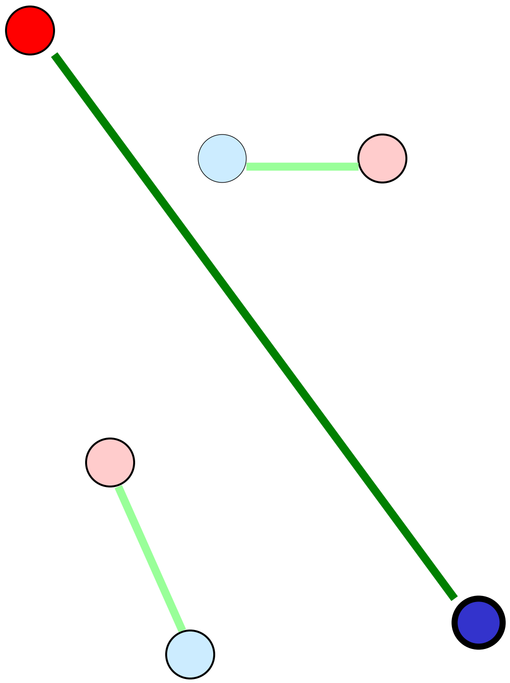


Example: Gale-Shapley stable matching.

- Match all *mutually closest* red/blue pairs.
- Remove them
- Repeat indefinitely



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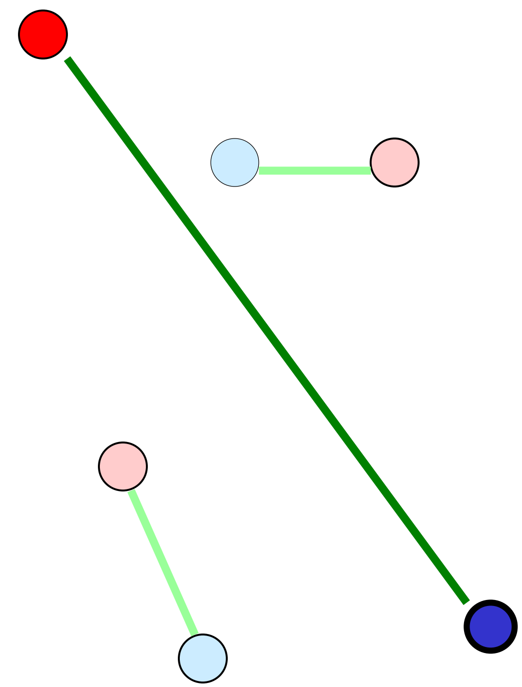
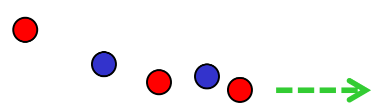


Why does every point get matched?

R:={∃ unmatched red point}
B:={∃ unmatched red point}

Ergodicity ⇒ P(R),P(B) ∈ {0,1}

Algorithm ⇒ P(R∩B)=0
(no ∞ descending chains)

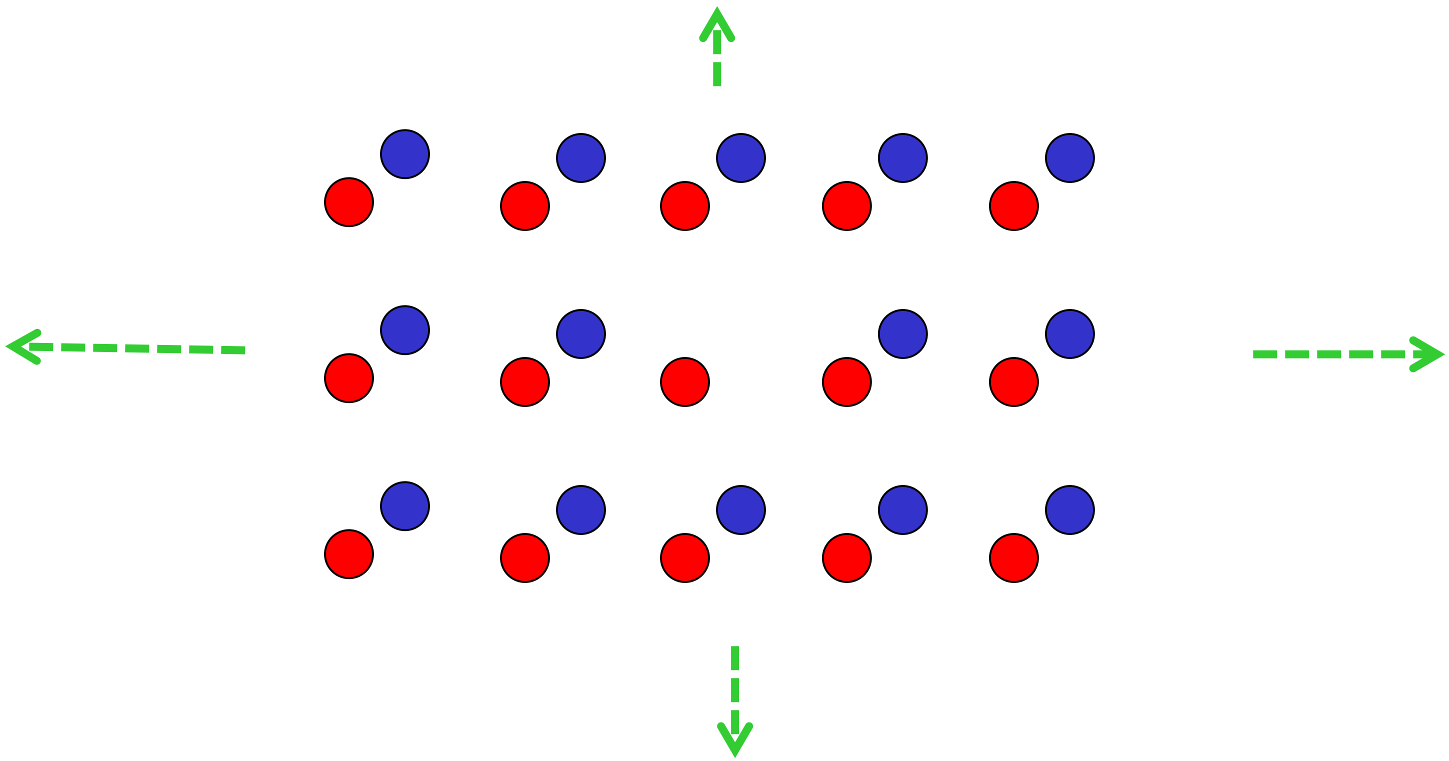


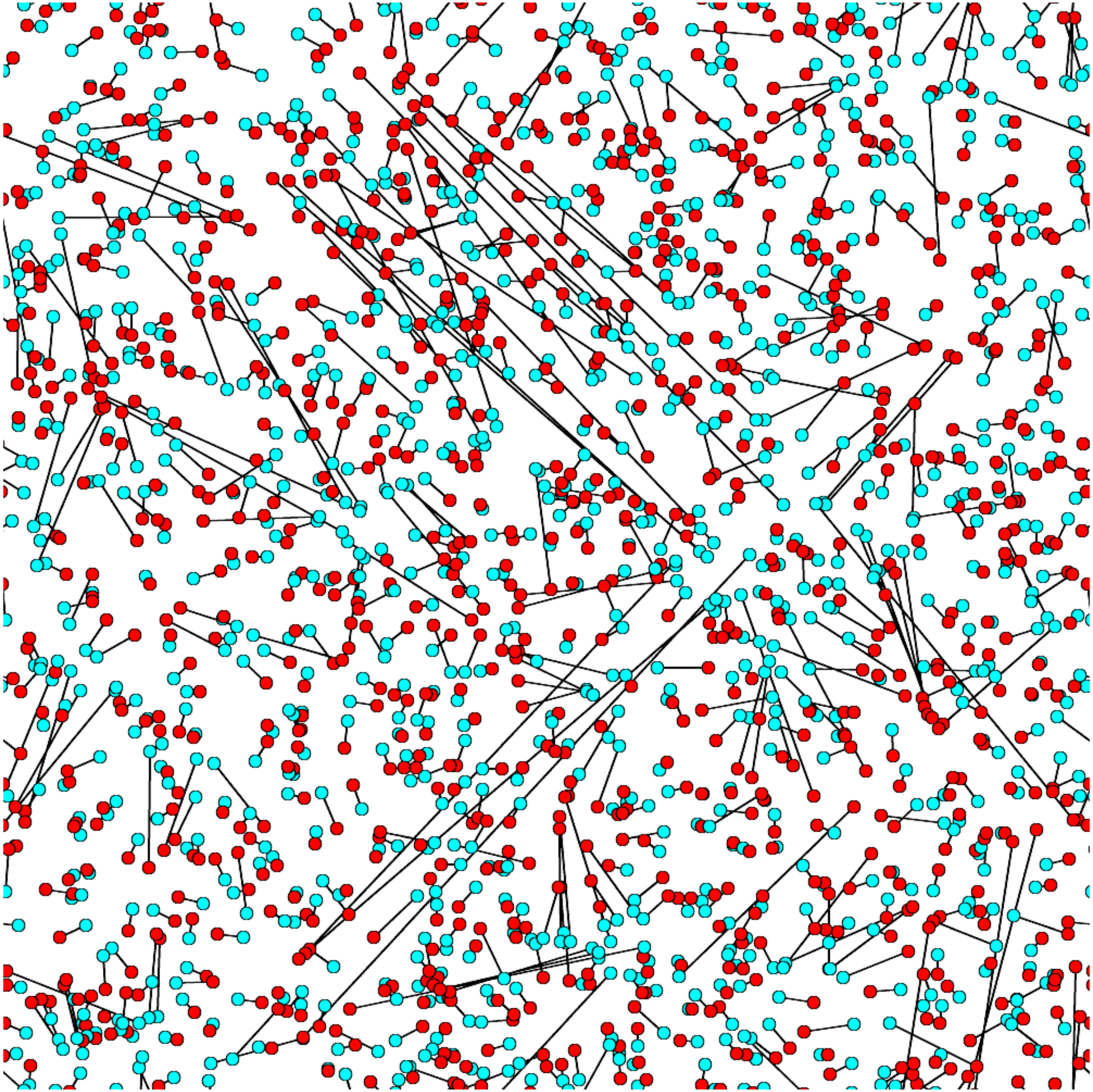
Symmetry ⇒ cannot have P(R)=0 , P(B)=1
(Also true for any jointly ergodic processes of equal intensity – mass transport)

So P(R)=P(B)=0

<u>(Philosophical) question</u>: what almost sure property of Poisson process did we use to deduce that every point gets matched??

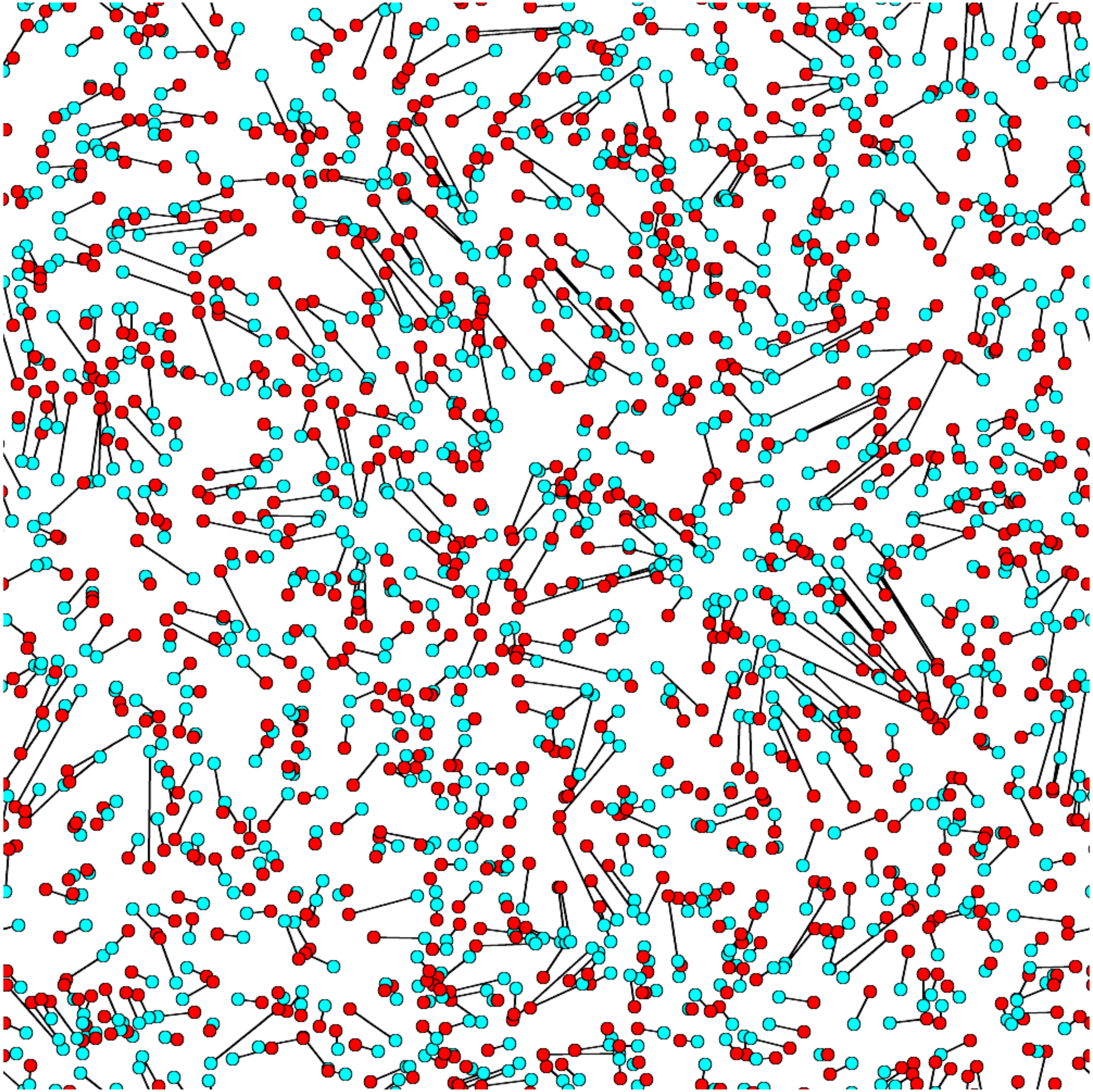
A non-invariant example where this fails(!):





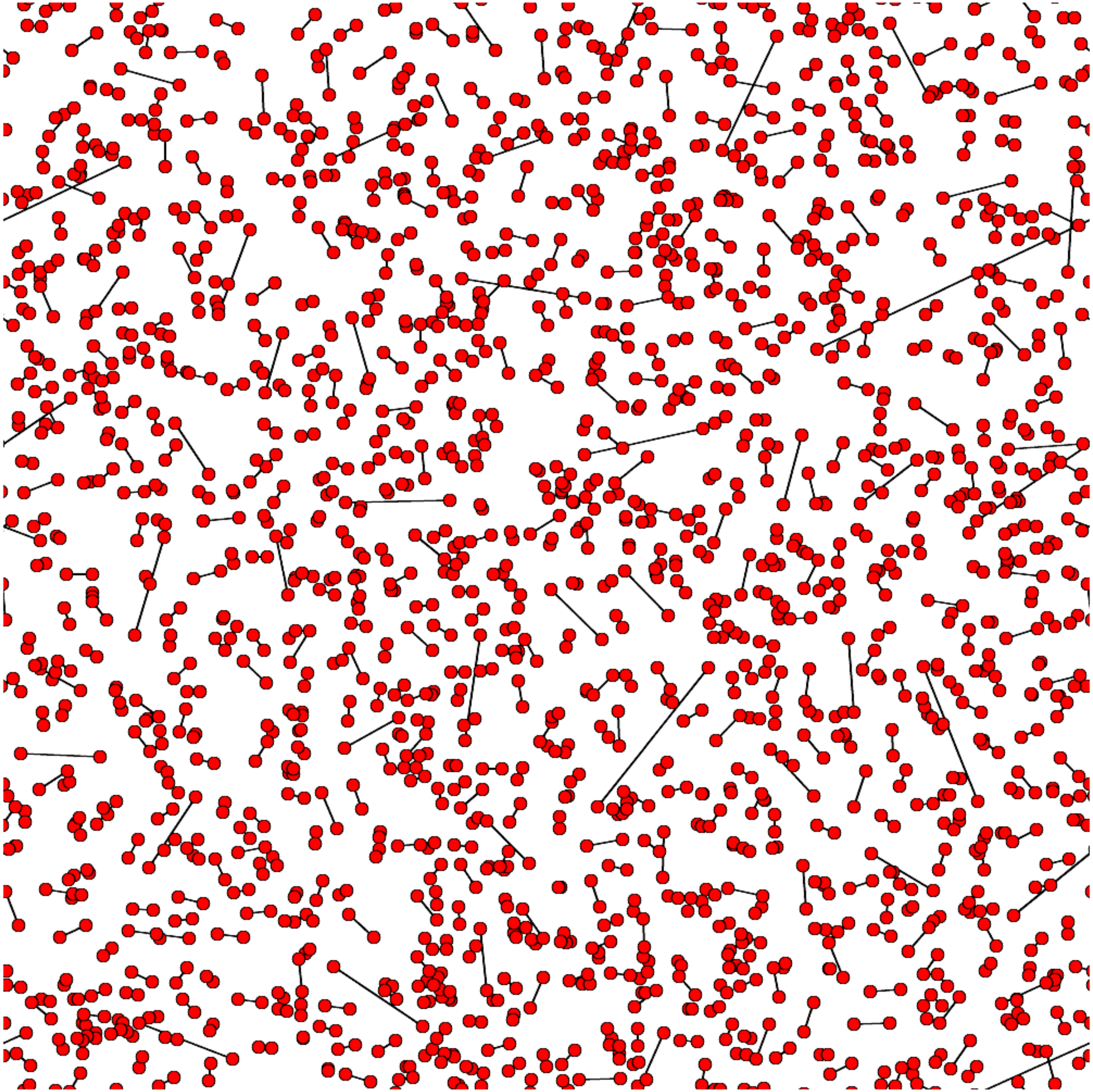
Two-colour stable matching

(on torus)



Two-colour minimum-length matching

(on torus)



One-colour stable matching

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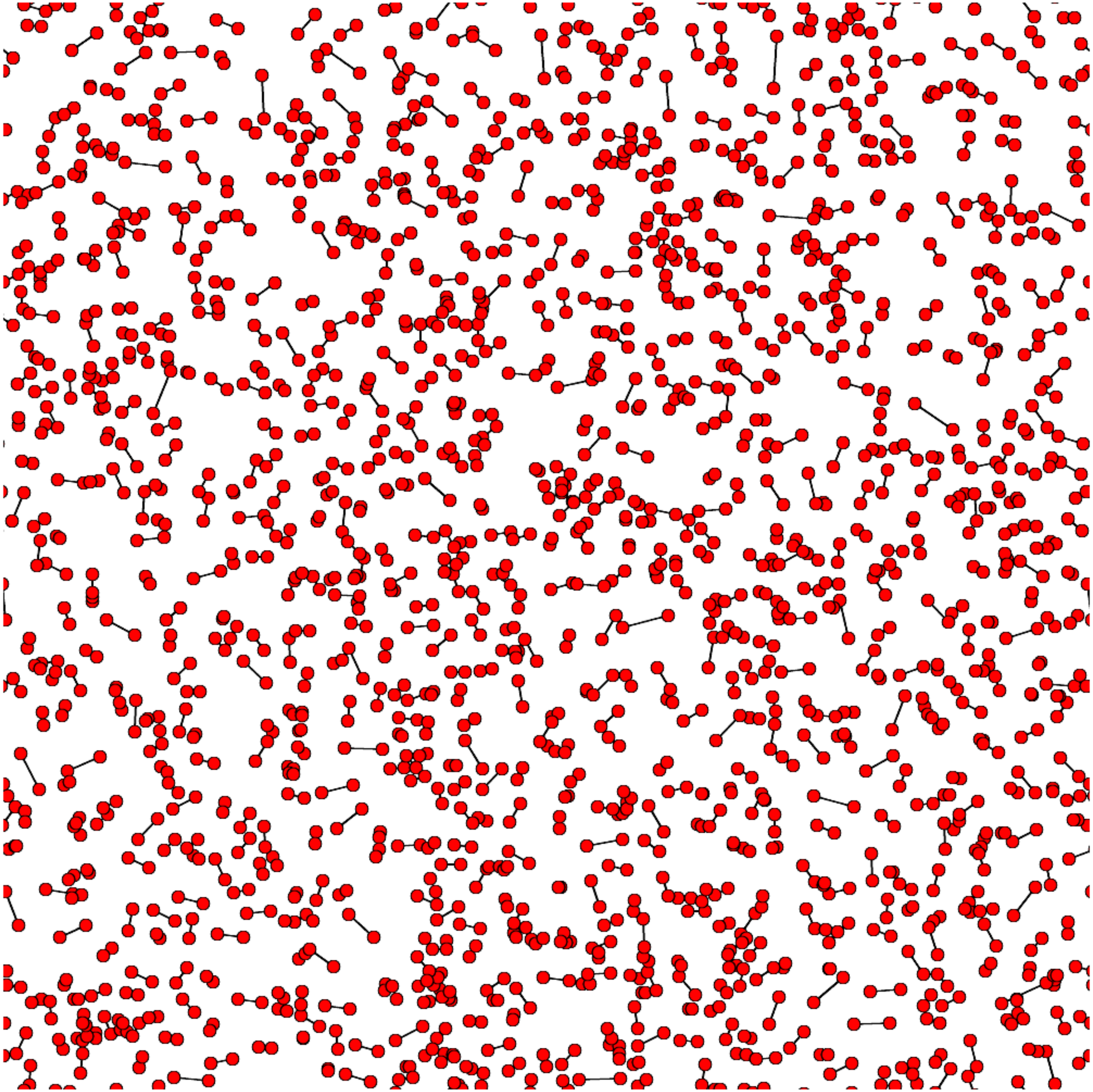
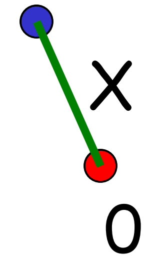


One-colour minimum-length matching

(on torus)

Given a matching scheme $\mathcal{M}$,



denote X = length of "typical edge"

= $|0-\mathcal{M}(0)|$ "conditioned" on {0 is red}

(i.e. under Palm measure P*

- for Poisson, equiv to adding pt at 0)

i.e. $P^*(X \leq r) :=$

$E\,\#\,\{\text{red points } z \in [0,1)^d \text{ with } |z-\mathcal{M}(z)| \leq r\}$

Question: how small can we make X ?

A trivial lower bound: for any matching,

$$P^*(X > r) \geq P^*(\exists \text{ no other point in } B(0,r)) \geq e^{-cr^d}$$

i.e. $E^* e^{cX^d} = \infty$

More results (H., Pemantle, Peres, Schramm 2008):

One color		Lower bound	Upper bound
Any matching	d=1 d≥2		
Stable	All d		

Two color		Lower bound	Upper bound
Any matching	d=1 d=2 d≥3		
Stable	d=1 d=2 d≥3		

One color		Lower bound	Upper bound
Any matching	$d=1$	$E^* e^{cX^d} = \infty$	
	$d\geq 2$	$E^* e^{cX^d} = \infty$	
Stable	All d	$E^* e^{cX^d} = \infty$	

Two color		Lower bound	Upper bound
Any matching	$d=1$	$E^* e^{cX^d} = \infty$	
	$d=2$	$E^* e^{cX^d} = \infty$	
	$d\geq 3$	$E^* e^{cX^d} = \infty$	
Stable	$d=1$	$E^* e^{cX^d} = \infty$	
	$d=2$	$E^* e^{cX^d} = \infty$	
	$d\geq 3$	$E^* e^{cX^d} = \infty$	

One color		Lower bound	Upper bound
Any matching	d=1 d≥2	$E^* e^{cX^d} = \infty$ $E^* e^{cX^d} = \infty$	
Stable	All d	$E^* e^{cX^d} = \infty$	

Two color		Lower bound	Upper bound
Any matching	d=1 d=2 d≥3	$E^* X^{1/2} = \infty$ $E^* X = \infty$ $E^* e^{cX^d} = \infty$	$E^* X^{1/2 - \epsilon} < \infty$ $E^* X^{1-\epsilon} < \infty$ $E^* e^{cX^d} < \infty$ [fromTalagrand 94]
Stable	d=1 d=2 d≥3	$E^* e^{cX^d} = \infty$ $E^* e^{cX^d} = \infty$ $E^* e^{cX^d} = \infty$	

One color		Lower bound	Upper bound
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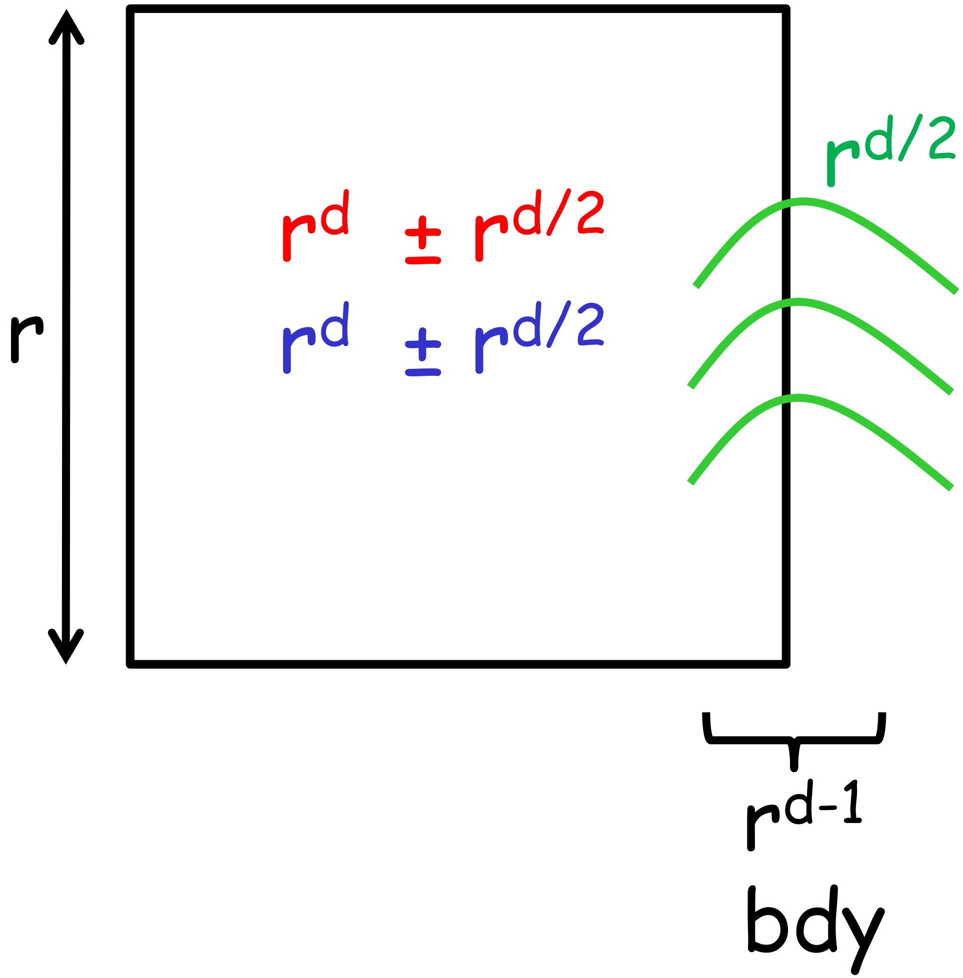
Stable	d=1 d=2 d≥3	$E^* e^{cX^d} = \infty$ $E^* e^{cX^d} = \infty$ $E^* e^{cX^d} = \infty$	

Heuristic reason:



$d \leq 2$: $r^{d/2} \geq r^{d-1}$

$P^*(X>r) \approx r^{d/2}/r^d$

$d \geq 3$: $r^{d/2} \ll r^{d-1}$

match "locally"

Call a matching scheme

- a *factor* if $\mathcal{M} = f(\mathcal{R}, \mathcal{B})$
  (e.g. stable matching)

- *randomized* if not

One color		Lower bound	Upper bound
Randomized	d=1	E* $e^{cX^d} = \infty$	
	$d \geq 2$	E* $e^{cX^d} = \infty$	
Factor	d=1	E* $e^{cX^d} = \infty$	
	$d \geq 2$	E* $e^{cX^d} = \infty$	
Stable	All d		

Two color		Lower bound	Upper bound
Randomized	d=1	E* $X^{1/2} = \infty$	E* $X^{1/2 - \epsilon} < \infty$
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	$d \geq 3$	E* $e^{cX^d} = \infty$	E* $e^{CX^d} < \infty$
Factor	d=1	E* $X^{1/2} = \infty$	E* $X^{1/2 - \epsilon} < \infty$
	d=2	E* $X = \infty$	E* $X^{2/3 - \epsilon} < \infty$ [Soo]
	$d \geq 3$	E* $e^{cX^d} = \infty$	E* $X^{2d/(d+4)-\varepsilon} < \infty$ [Soo]
Stable	d=1		
	d=2		
	$d \geq 3$		

One color		Lower bound	Upper bound
Randomized	d=1	$E^* e^{cX^d} = \infty$	
	d≥2	$E^* e^{cX^d} = \infty$	
Factor	d=1	$E^* e^{cX^d} = \infty$	
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Two color		Lower bound	Upper bound
Randomized	d=1	$E^* X^{1/2} = \infty$	$E^* X^{1/2 - \epsilon} < \infty$
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	d=2	$E^* X = \infty$	$E^* X^{1-\epsilon} < \infty$ [Timar]
	d≥3	$E^* e^{cX^d} = \infty$	$E^* e^{cX^{d-2}} < \infty$ [Timar]
Stable	d=1		
	d=2		
	d≥3		

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Stable	d=1		
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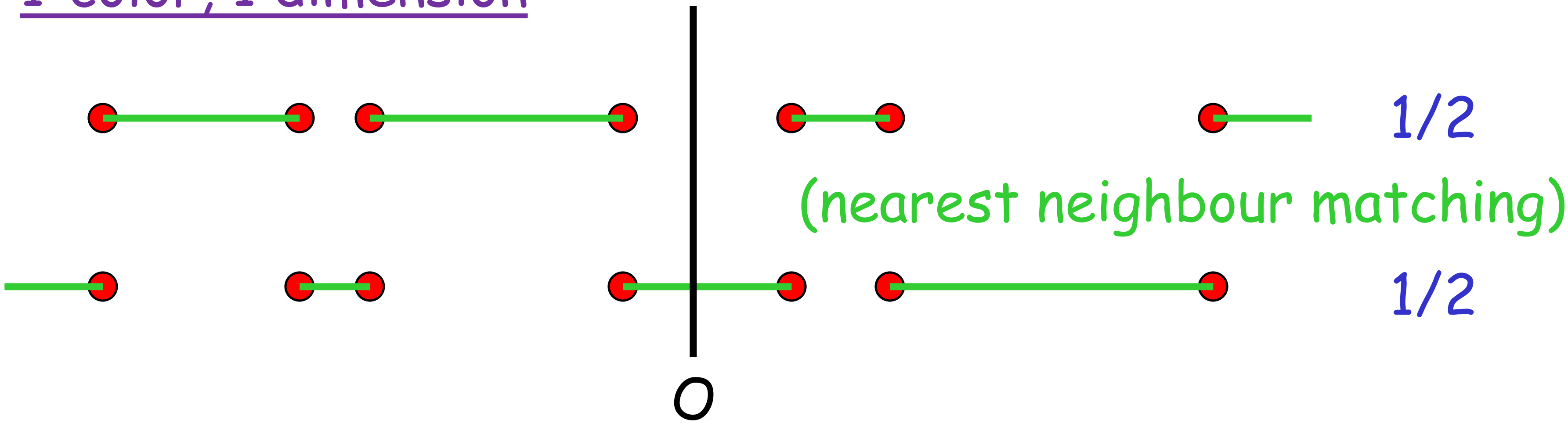
<u>1-color, 1 dimension</u>

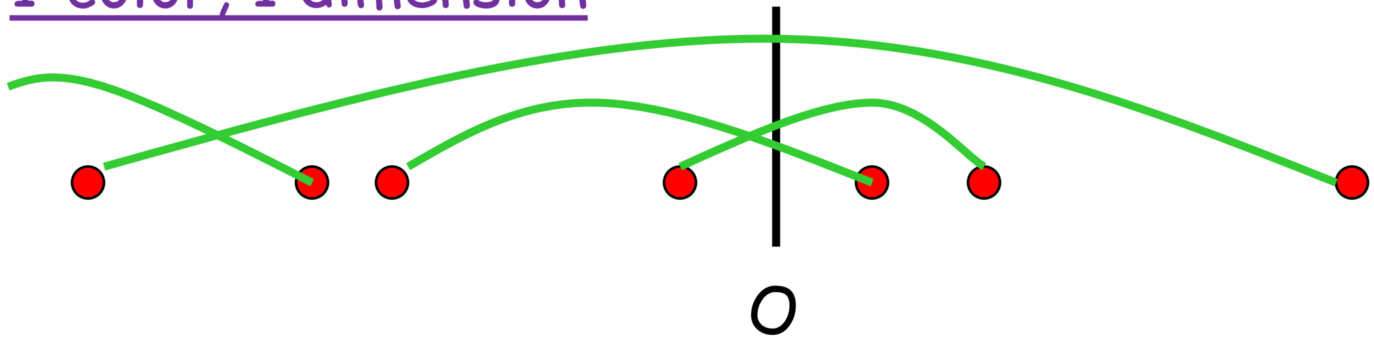


- a <u>randomized</u> matching with $P^*(X > r) = e^{-r}$

But ∄ a <u>factor</u> nearest neighbour matching

∴ any factor matching has $E^*X = \infty$. Proof:

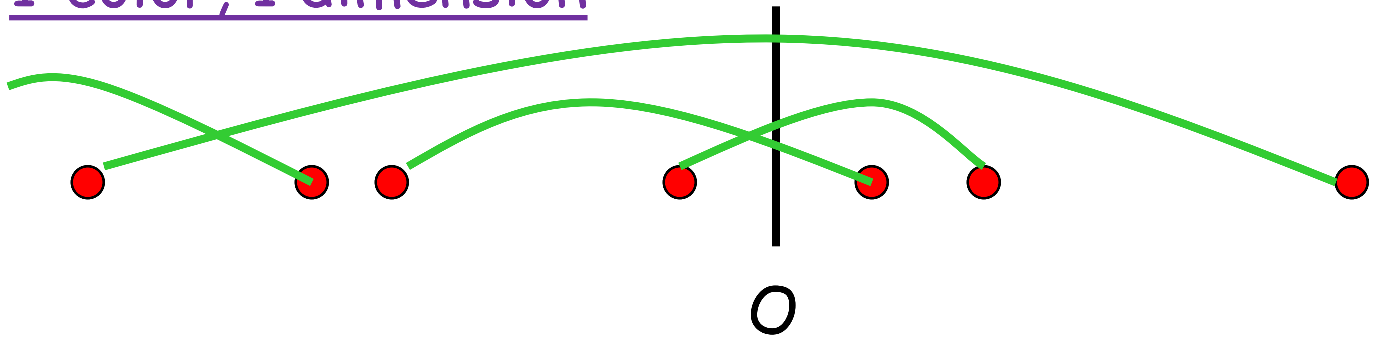
1-color, 1 dimension



Enough to show (by mass-transport):

$$E(\#\text{ edges crossing } O) = \infty$$

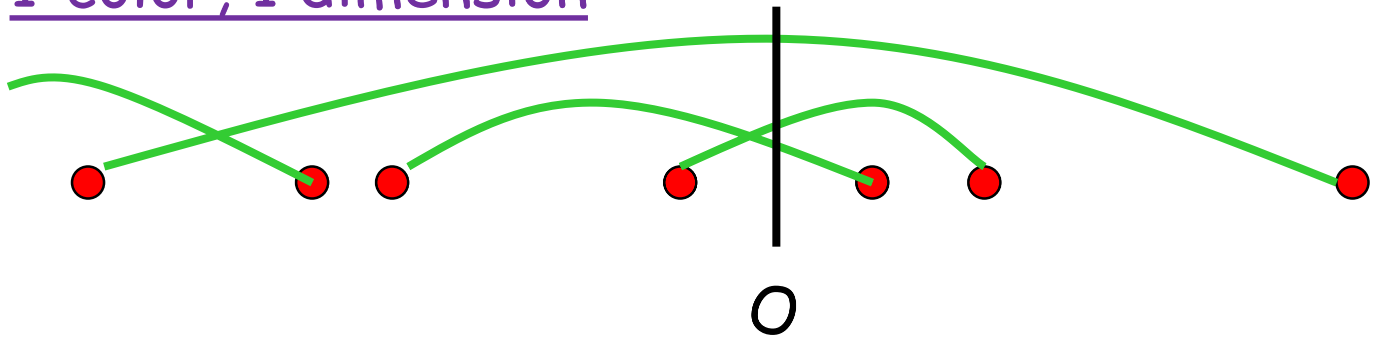
1-color, 1 dimension



Enough to show:

$$P(\# \text{ edges crossing } O = \infty) = 1$$

1-color, 1 dimension

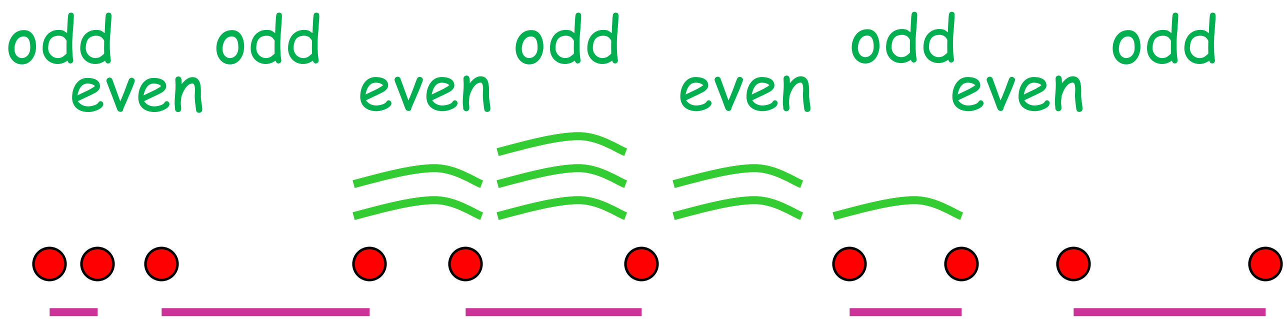


Suppose:

$$P(\#\text{ edges crossing } O < \infty) > 0$$

$$\Longrightarrow \quad P(< \infty \text{ edges crossing every site}) > 0$$

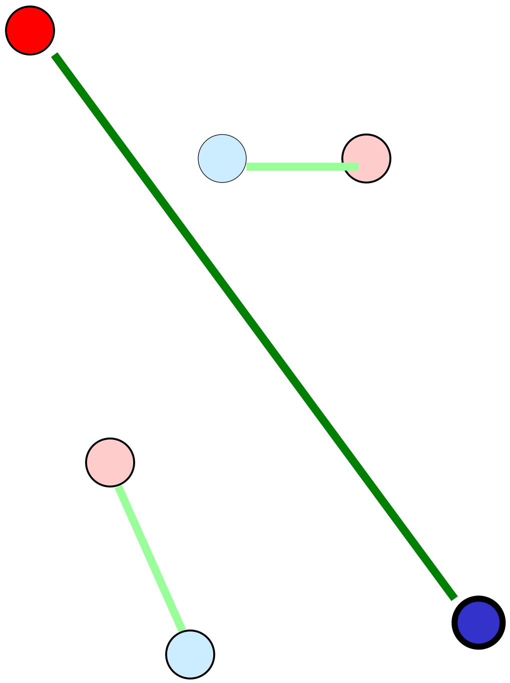
$$\therefore = 1 \text{ (ergodicity)}$$



Rematch $\Rightarrow$ factor nearest neighbour matching! #

Back to: Gale-Shapley stable matching.

- Match all *mutually closest* red/blue pairs.
- Remove them
- Repeat indefinitely



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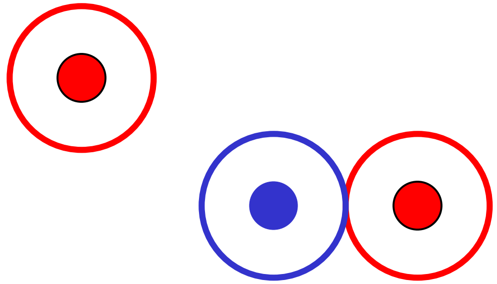
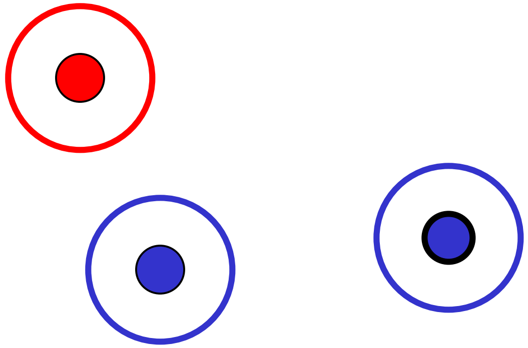


Alternative description:
ball-growing



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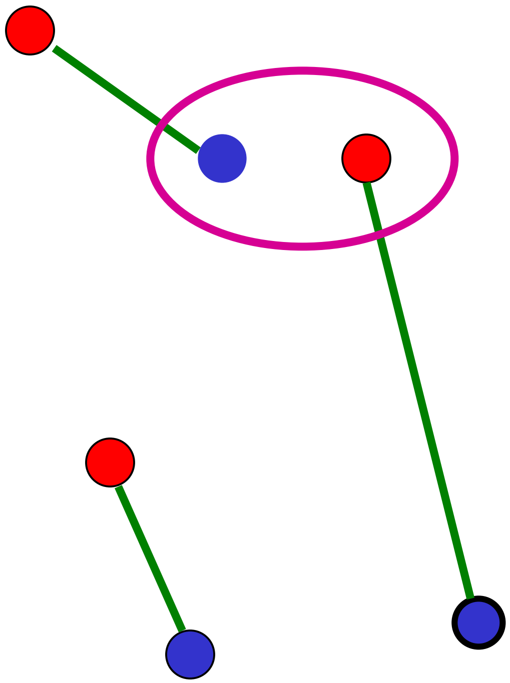
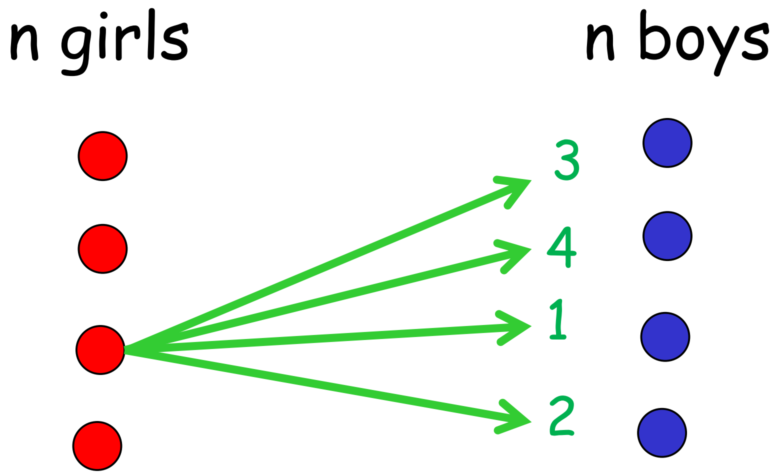


Alternative description:
ball-growing

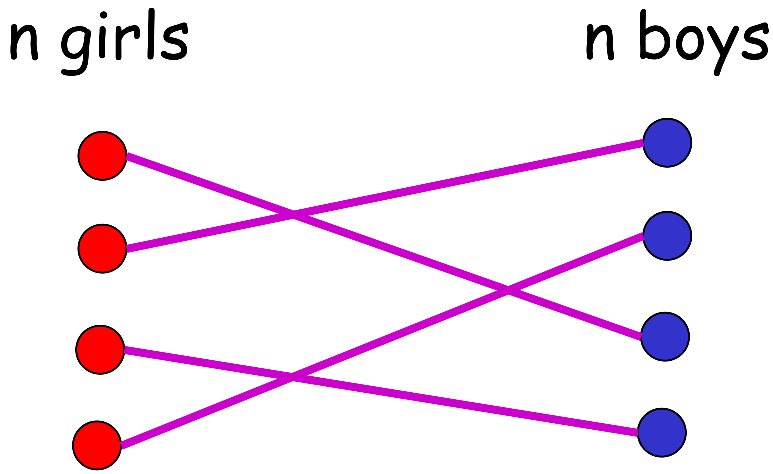
Alternative description:
unique matching with
no *unstable* pairs

Original formulation (Gale, Shapley, 1962)



Arbitrary preference orders

Original formulation (Gale, Shapley, 1962)



Theorem: ∃ a **stable** set of n heterosexual marriages (i.e. with no temptation for affairs).

Not necessarily unique

Does not necessarily exist in same-sex ('roommates' version)

but both hold in our case owing to *symmetric* prefs.

2012 Nobel Prize in Economics: Stable allocations – from theory to practice

This year's Prize concerns a central economic problem: how to match different agents as well as possible. For example, students have to be matched with schools, and donors of human organs with patients in need of a transplant. How can such matching be accomplished as efficiently as possible? What methods are beneficial to what groups? The prize rewards two scholars who have answered these questions on a journey from abstract theory on stable allocations to practical design of market institutions.

Lloyd Shapley used so-called cooperative game theory to study and compare different matching methods. A key issue is to ensure that a matching is stable in the sense that two agents cannot be found who would prefer each other over their current counterparts. Shapley and his colleagues derived specific methods – in particular, the so-called Gale-Shapley algorithm – that always ensure a stable matching. These methods also limit agents' motives for manipulating the matching process. Shapley was able to show how the specific design of a method may systematically benefit one or the other side of the market.

Alvin Roth recognized that Shapley's theoretical results could clarify the functioning of important markets in practice. In a series of empirical studies, Roth and his colleagues demonstrated that stability is the key to understanding the success of particular market institutions. Roth was later able to substantiate this conclusion in systematic laboratory experiments. He also helped redesign existing institutions for matching new doctors with hospitals, students with schools, and organ donors with patients. These reforms are all based on the Gale-Shapley algorithm, along with modifications that take into account specific circumstances and ethical restrictions, such as the preclusion of side payments.

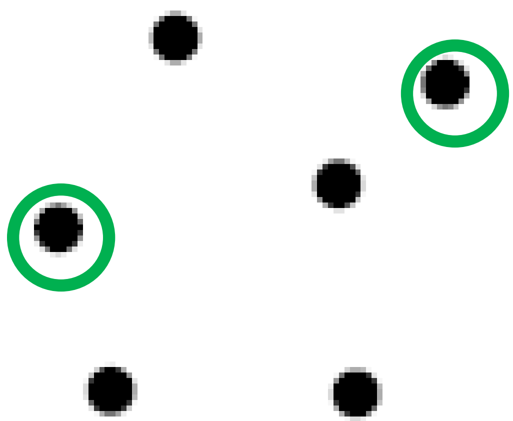
Even though these two researchers worked independently of one another, the combination of Shapley's basic theory and Roth's empirical investigations, experiments and practical design has generated a flourishing field of research and improved the performance of many markets. This year's prize is awarded for an outstanding example of economic engineering.

Yet another interpretation: two player game

Given a set of points.

Alice places a token on a point (of her choice).
Bob places a token on another point.

Taking turns starting with *Alice*, player moves either token to another point, *decreasing* the distance between the tokens



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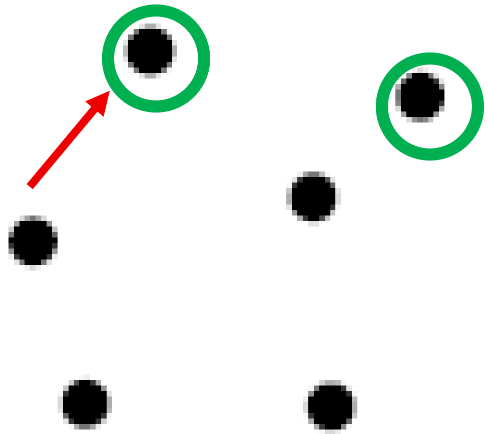

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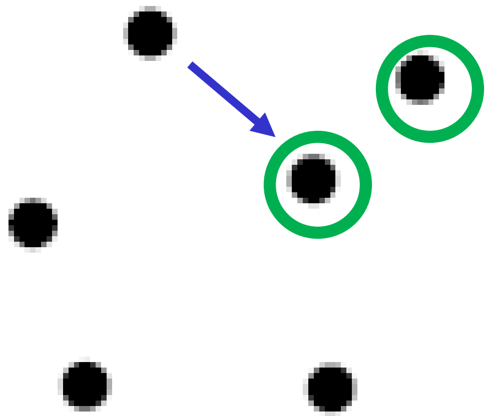

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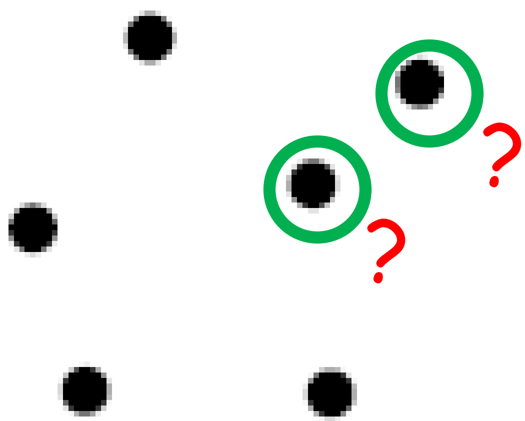
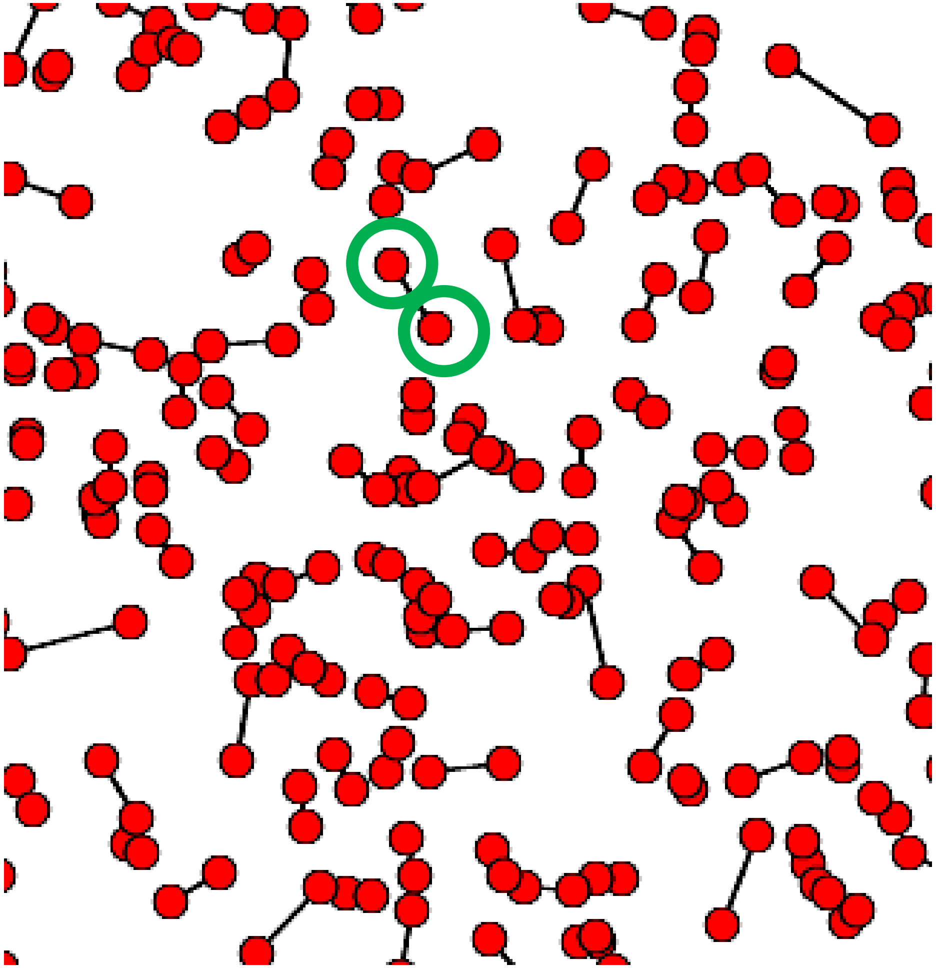


Player who cannot move loses.

Who wins starting with a Poisson process on R^d?

Solution: Bob wins, by always leaving Alice with a matched pair of the one-color stable matching!

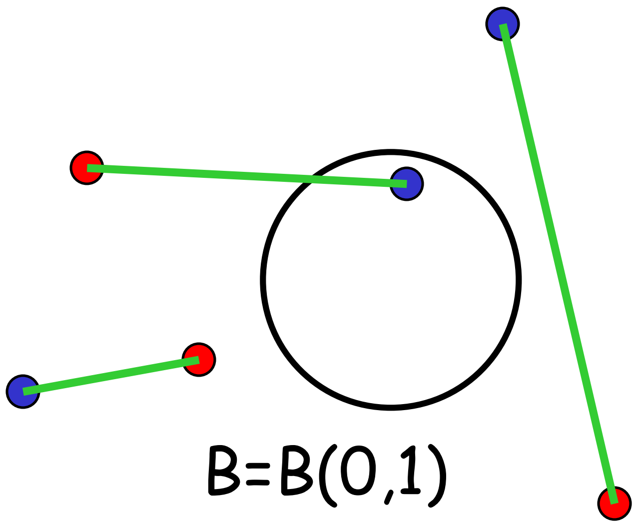


One color		Lower bound	Upper bound
Randomized	d=1	$E^* e^{cX} = \infty$	$E^* e^{CX} < \infty$
	$d \geq 2$	$E^* e^{cX^d} = \infty$	$E^* e^{CX^d} < \infty$
Factor	d=1	$E^* X = \infty$	$E^* X^{1-\epsilon} < \infty$
	$d \geq 2$	$E^* e^{cX^d} = \infty$	$E^* e^{CX^d} < \infty$
Stable	All d	$E^* X^d = \infty$	$E^* X^{d-\epsilon} < \infty$

Two color		Lower bound	Upper bound
Randomized	d=1	$E^* X^{1/2} = \infty$	$E^* X^{1/2 - \epsilon} < \infty$
	d=2	$E^* X = \infty$	$E^* X^{1-\epsilon} < \infty$
	$d \geq 3$	$E^* e^{cX^d} = \infty$	$E^* e^{CX^d} < \infty$
Factor	d=1	$E^* X^{1/2} = \infty$	$E^* X^{1/2 - \epsilon} < \infty$
	d=2	$E^* X = \infty$	$E^* X^{1-\epsilon} < \infty$
	$d \geq 3$	$E^* e^{cX^d} = \infty$	$E^* e^{CX^{d-2}} < \infty$
Stable	d=1	$E^* X^{1/2} = \infty$	$E^* X^{1/2 - \epsilon} < \infty$
	d=2	$E^* X = \infty$	$E^* X^{0.496} < \infty$
	$d \geq 3$	$E^* X^d = \infty$	$E^* X^{s(d)} < \infty$

Two-color stable matching - lower bound

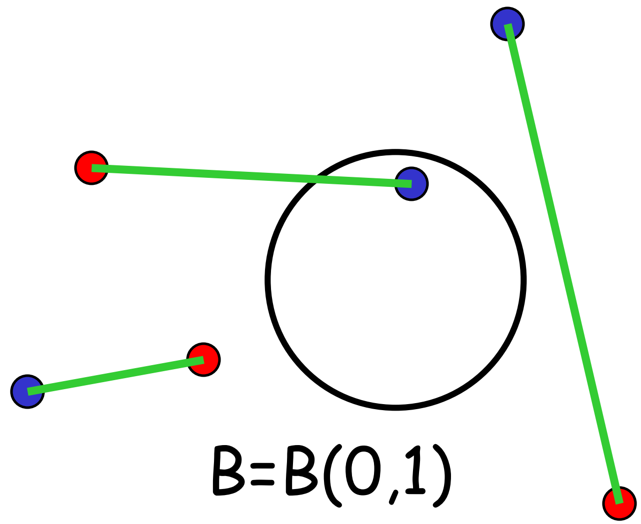
Claim: E(# red points that prefer some part of B) = ∞



Implies $E^*X^d = \infty$

Two-color stable matching - lower bound

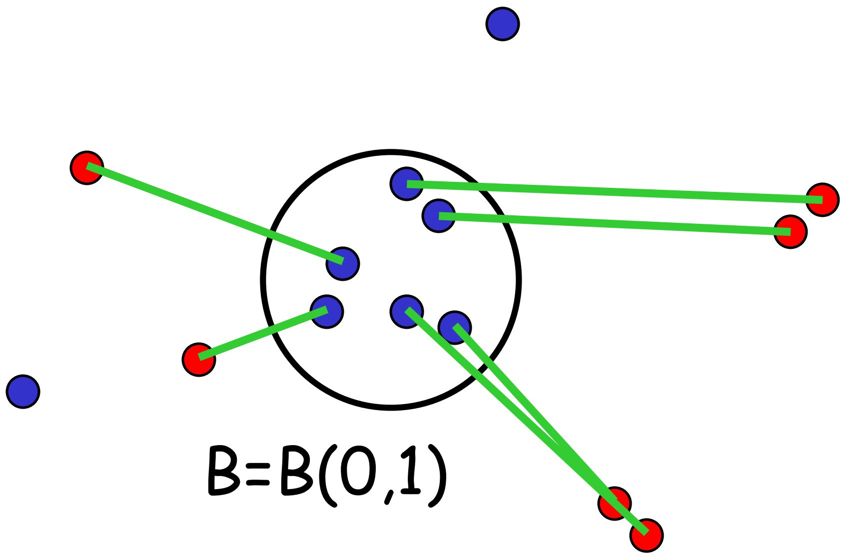
Prove: P($\geq$ k red points prefer some part of B) = 1



Implies $E^*X^d=\infty$

Two-color stable matching – lower bound

Prove: $P(\geq k$ red points prefer some part of $B) = 1$



Implies $E^* X^d = \infty$

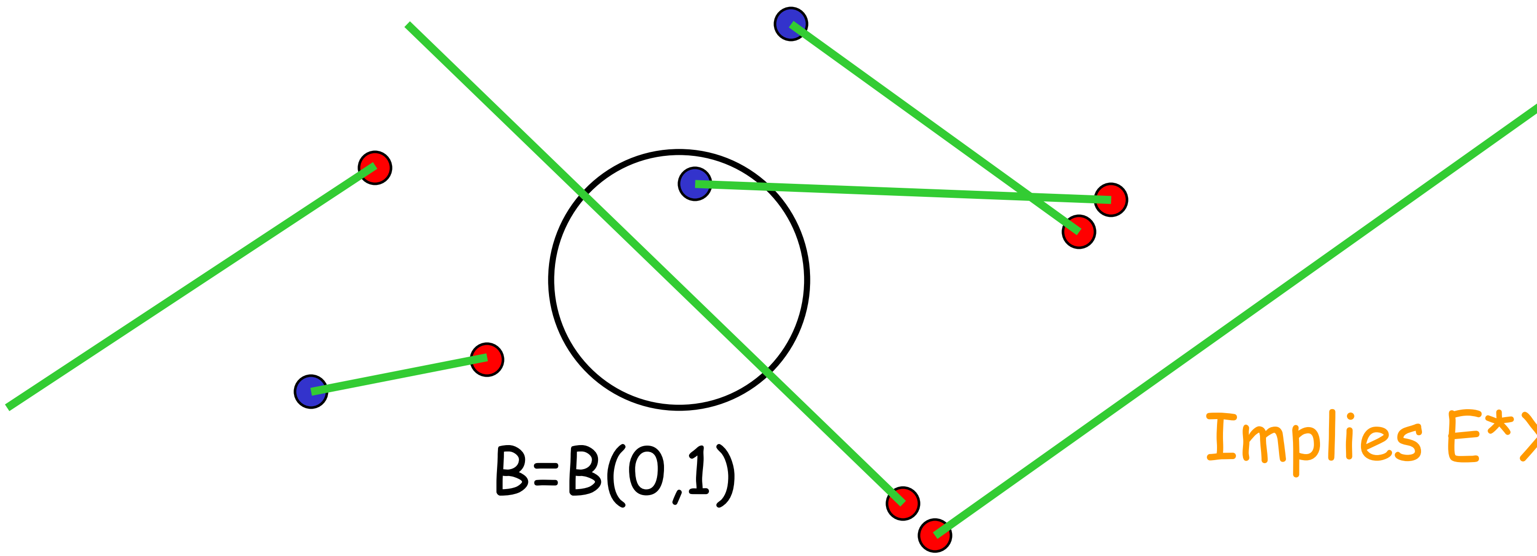
Add k extra blue points in B.   Law abs. cts. wrt Poisson.

So new points all get matched in the stable matching

Fact: adding blue points makes red points happier

Two-color stable matching – lower bound

Prove: P($\geq$ k red points prefer some part of B) = 1



Implies $E^*X^d=\infty$

Add k extra blue points in B. Law abs. cts. wrt Poisson.

So new points all get matched in the stable matching

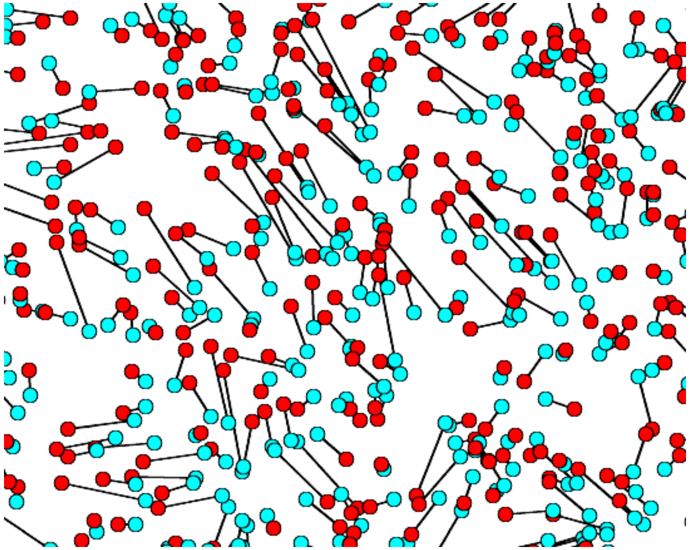
Fact: adding blue points makes red points happier

So k red partners preferred part of B before

(Works whenever point process is *insertion-tolerant* *or deletion-tolerant* – H.-Soo, 2010)

Geometric questions for matchings:

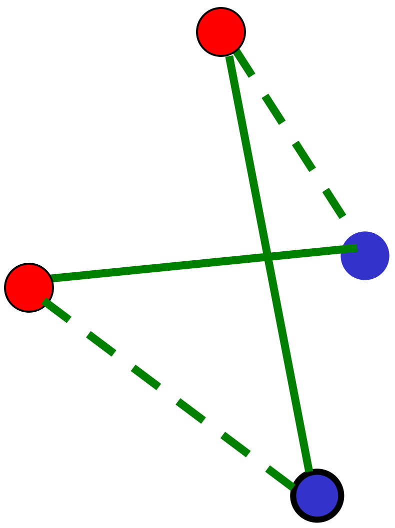
Open question (Peres, 2002):
For independent red and blue intensity-1 Poisson processes in R^2, does there exist a translation-invariant matching in which line segments joining matched pairs **do not cross**?



Proposition (H. 2009) Yes if we drop invariance,
or for one color, or allow partial matching, or curved edges!

Q: For independent red and blue intensity-1 Poisson processes in R^2, does there exist a **minimal** translation-invariant matching, i.e. s.t. every finite set of edges minimizes the total length?

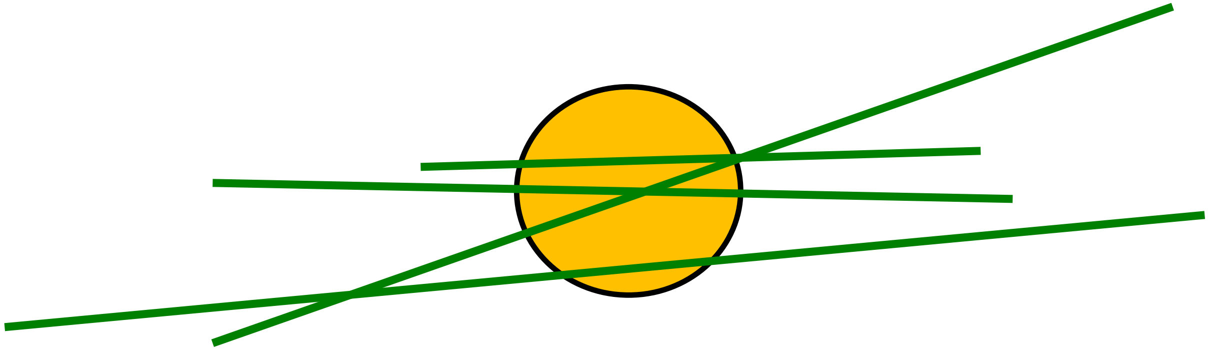
(If yes, then it would have no crossings)



Conjecture: No.

Theorem (H. 2009) Yes in R^d, d=1 and d≥3
No in strip R x [0,1]

For independent red and blue intensity-1 Poisson processes, does there exist a **locally finite** translation-invariant matching, i.e. s.t. B(0,1) meets only finitely many edges?



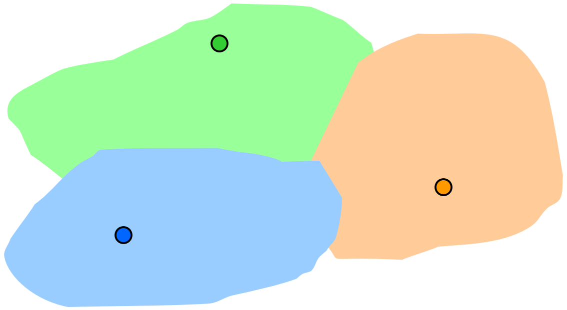
Theorem (H. 2009) Yes in R^d, $d \geq 2$
No in d=1, and strip

But:

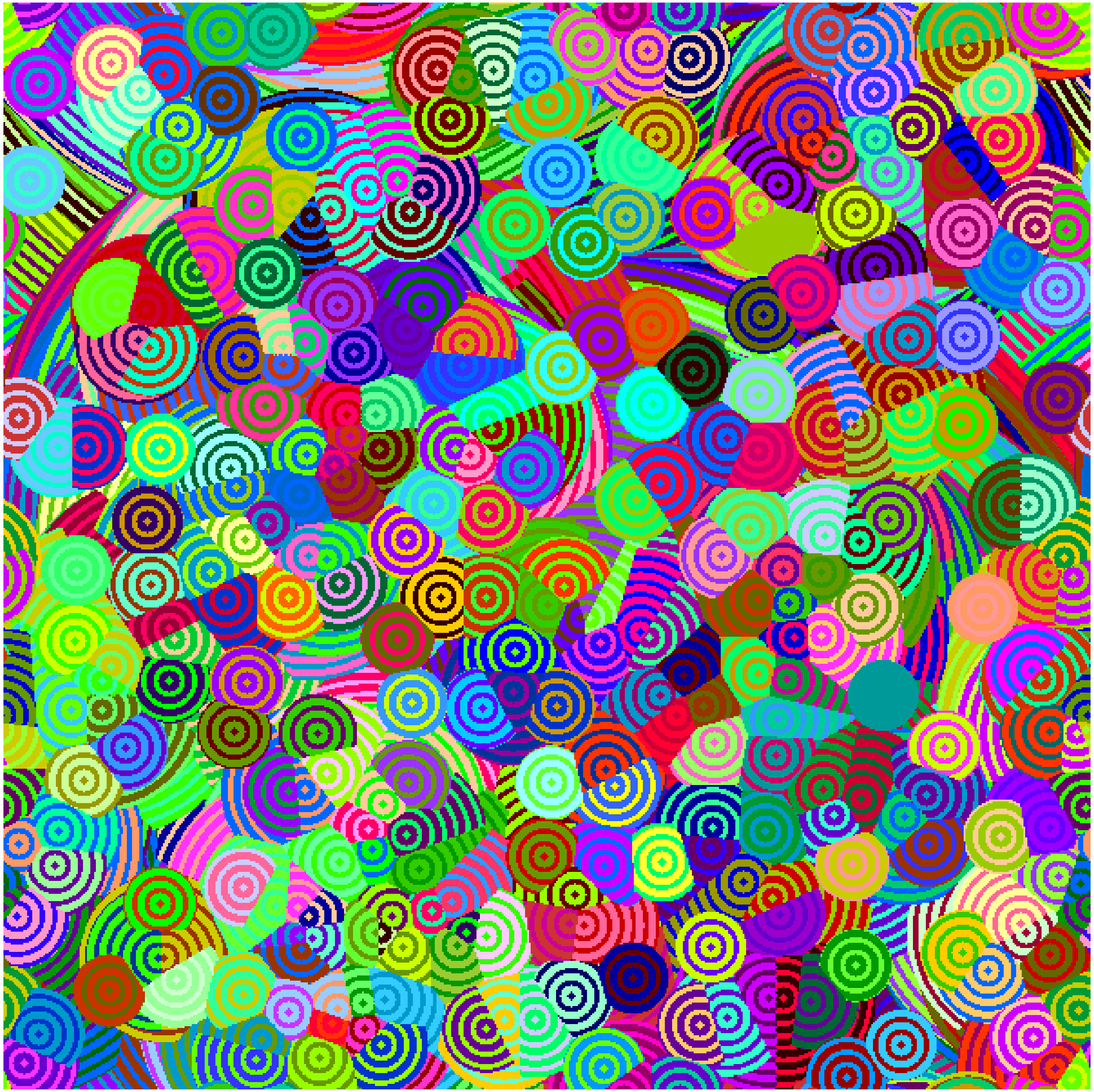
Theorem (HPPS) In any translation-invariant matching of independent red and blue Poisson processes in R^2,
$E[\#\text{ edges meeting } B(0,1)] = \infty$.

Variant problem: **allocation**

Given a point process of intensity 1 in R^d, partition space into cells of volume 1, with each cell allocated to a point, in a translation-invariant way.



Similar quantitative results, but richer structure…



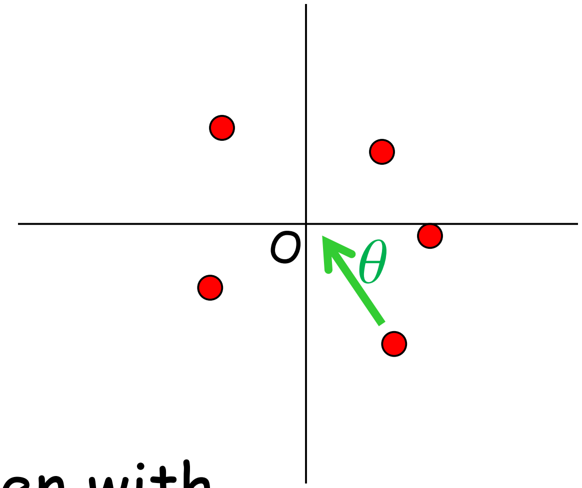
Stable allocation (Hoffman, H., Peres, 2005, 2009)

Application: let

Π = any translation-invariant ergodic point process
Π^* = associated Palm process: i.e. Π "conditioned" on $\{O \in \Pi\}$

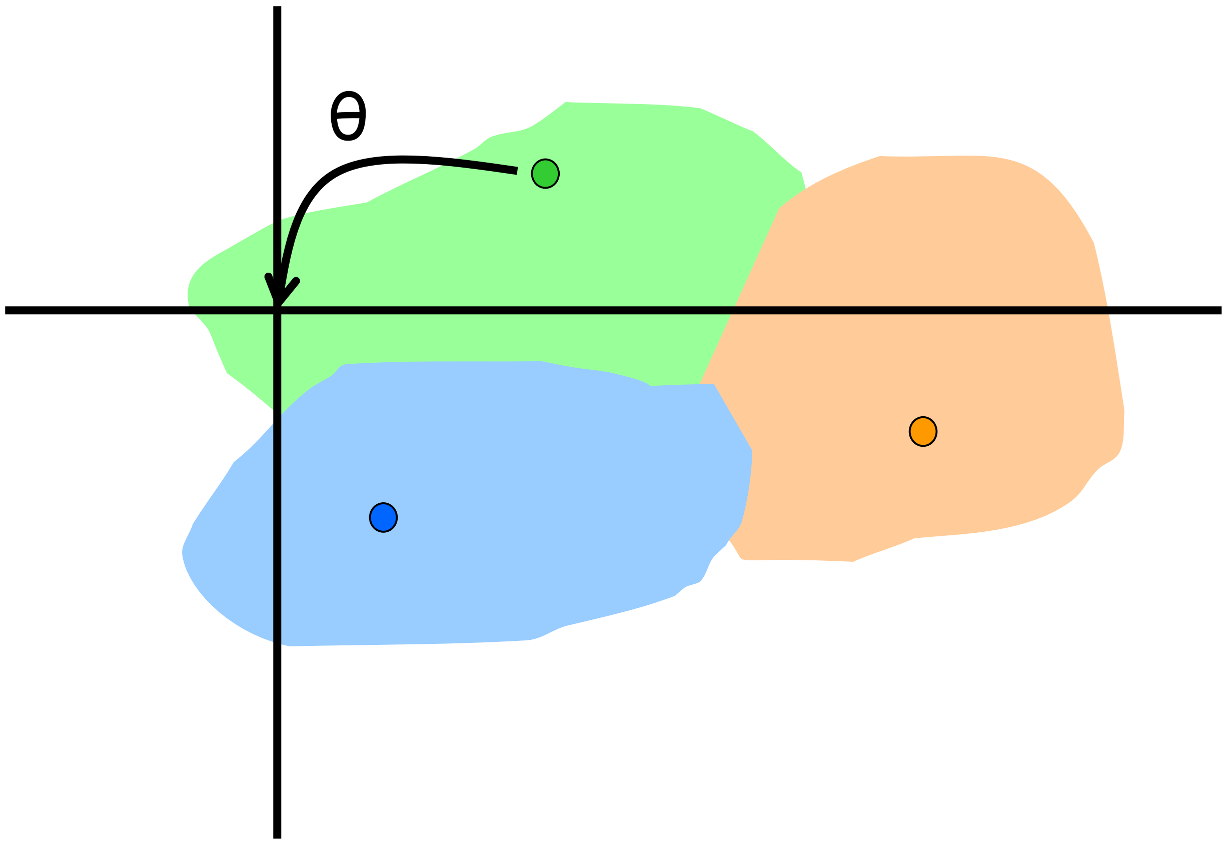
(E.g., if Π = Poisson process, then $\Pi^* = \Pi \cup O$)

Theorem (Thorisson, 2000):
Π and Π^* can be **shift-coupled**;
i.e. can define Π, Π^* and a random translation θ,
all on same prob. space, s.t. $\Pi^* = \theta \Pi$.



Theorem (H, Peres, 2005): can do this even with
$\theta = f(\Pi)$ (but not $\theta = g(\Pi^*)$).

Proof: Take any translation-invariant factor allocation (e.g. stable allocation).



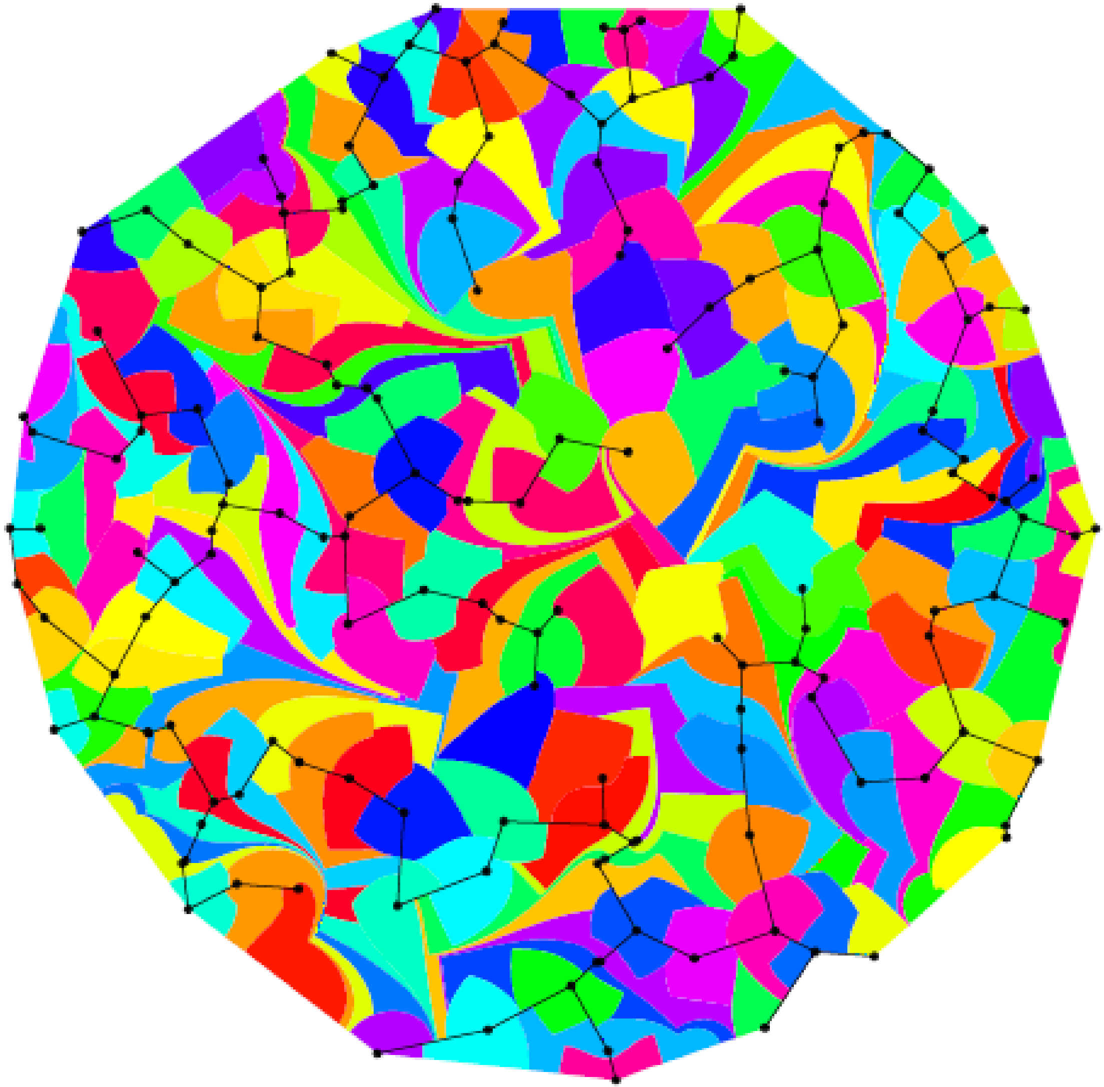
Let θ shift (point allocated to cell(O)) to O □

Many extensions (Last, Thorisson, 2009 …)

Gravtiational allocation, for Poisson process in d ≥ 3. (Chaterjee, Peled, Peres, Romik, 2010).

Cell diameters have exponential tail decay!

Connected allocation in R^2 (Krikun 2008)



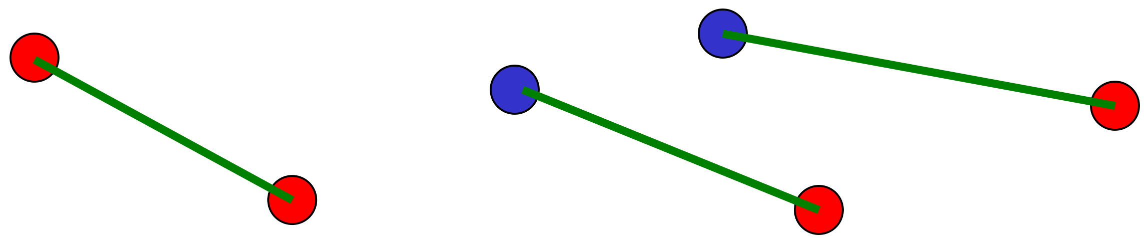
Huesmann, Sturm 2010: *optimal* allocation rule for any cost function with finite expectation

Marko,Timar, 2011: factor allocation with cell diameter R satisfying $E^* e^{c R^d} < \infty$

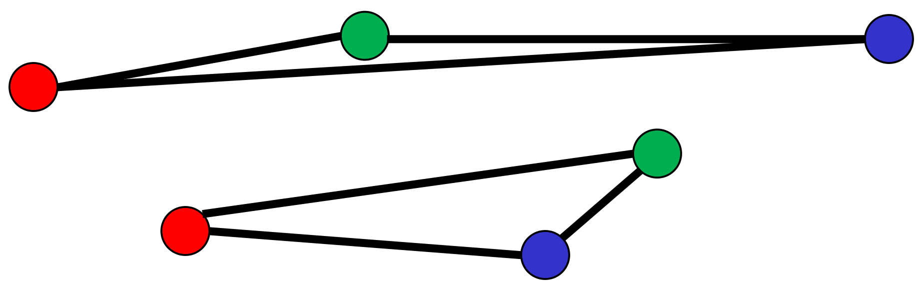
Multi-colour matching (Amir, Angel, H., in preparation)

Given independent Poisson processes of several colours. E.g.

may match red-blue or red-red



must match in red-green-blue families



families of RBB or RRRGB or GGY ....

X := diameter of "typical" family

$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_q)$ intensities of points each colour

Theorem

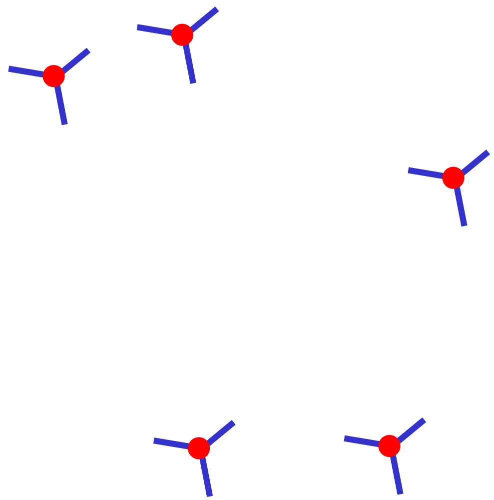
$\lambda \notin S$ $\Rightarrow$ no translation-invariant matching possible

$\lambda \in \partial S$ $\Rightarrow$ upper/lower bounds like 2-colour matching
(d/2 power in d≤2, exponential in volume in d≥3)

$\lambda \in S^o$ $\Rightarrow$ upper/lower bounds like 1-colour matching
(exponential in volume)

where $S \subset R^q$
is the cone generated by the allowed family vectors

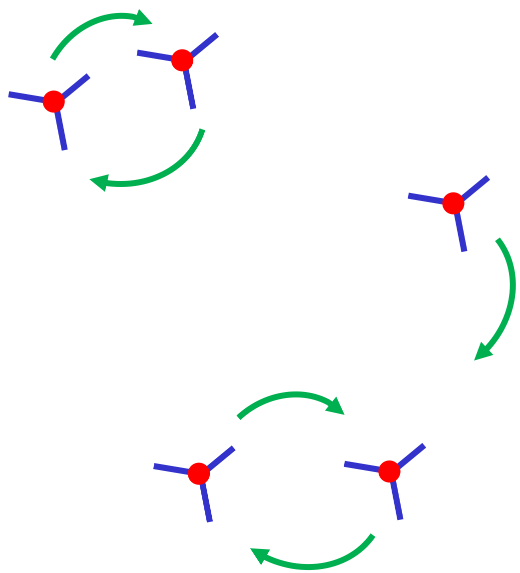
Stable Simple Graphs



D = 3

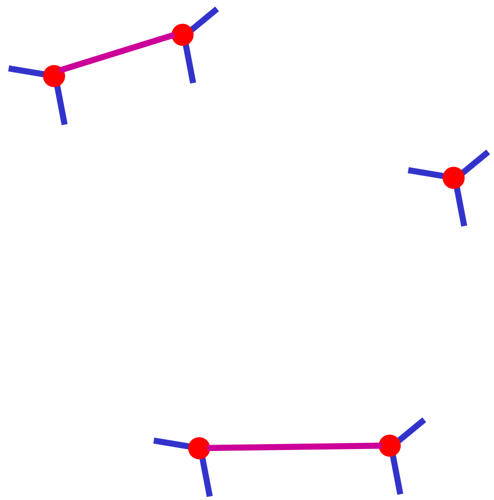
Assign each Poisson point x
a (deterministic or iid) degree D_x

Want a translation-invariant *simple* graph on the point process s.t. x has degree D_x



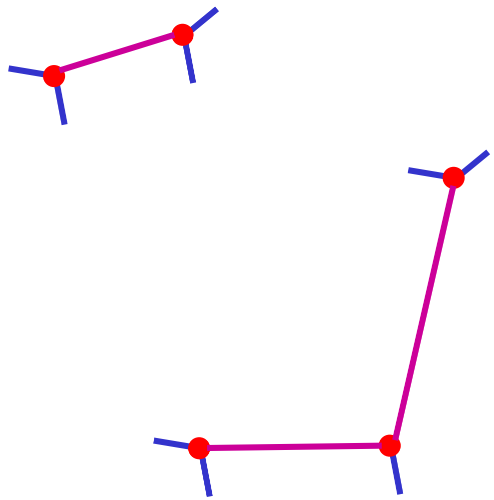
Stable matching :

- start with D_x stubs at each x
- each point x *looks at* closest other point with unused stubs and no edge to x already



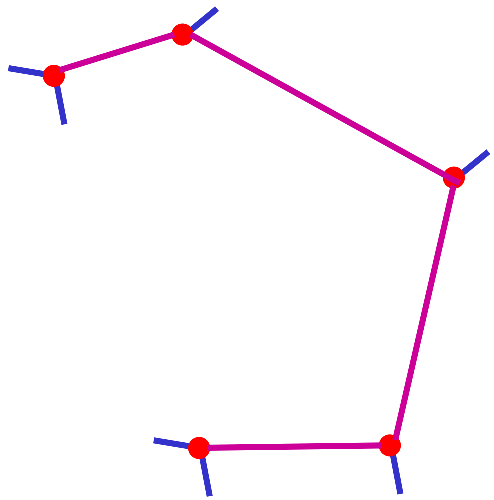
Stable matching:

- start with D stubs at every point
- each point x *looks at* closest other point with unused stubs and no edge to x already
- if x,y are looking at each other, match them, remove one stub from each



Stable matching:

- start with D stubs at every point
- each point x *looks at* closest other point with unused stubs and no edge to x already
- if x,y are looking at each other, match them, remove stubs
- iterate



Stable matching:

- start with D stubs at every point
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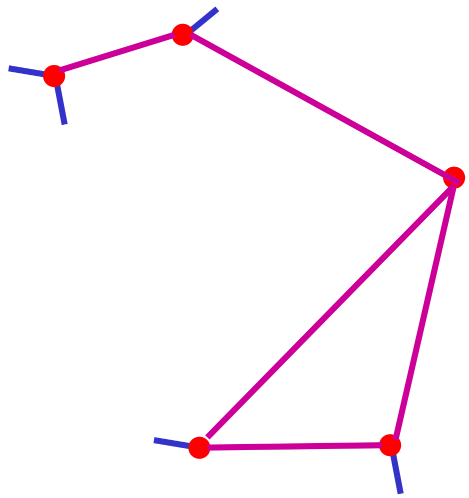
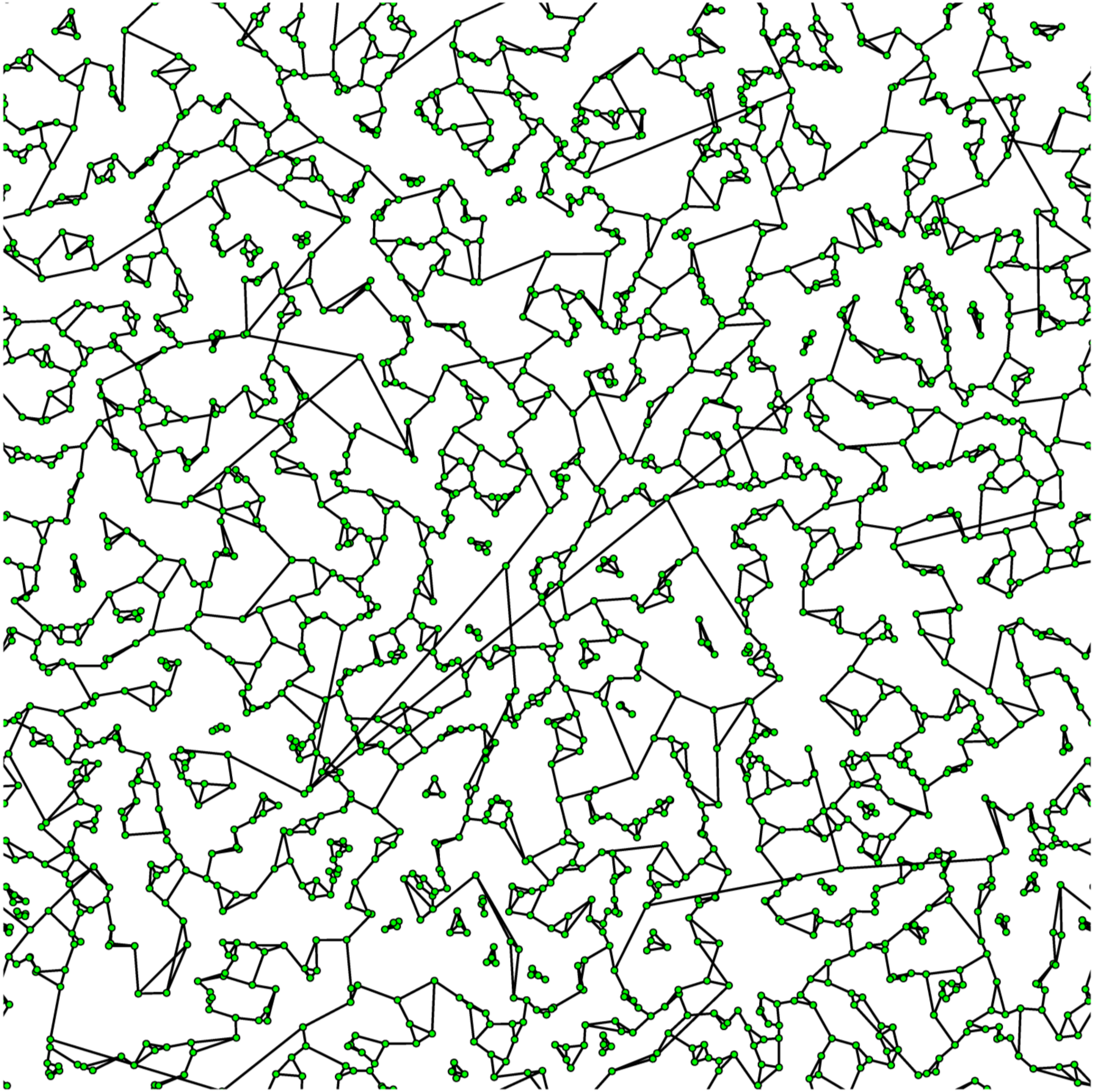


Stable matching:

- start with D stubs at every point
- each point x *looks at* closest other point with unused stubs and no edge to x already
- if x,y are looking at each other, match them, remove stubs
- iterate

E.g.:
R^2, D=3

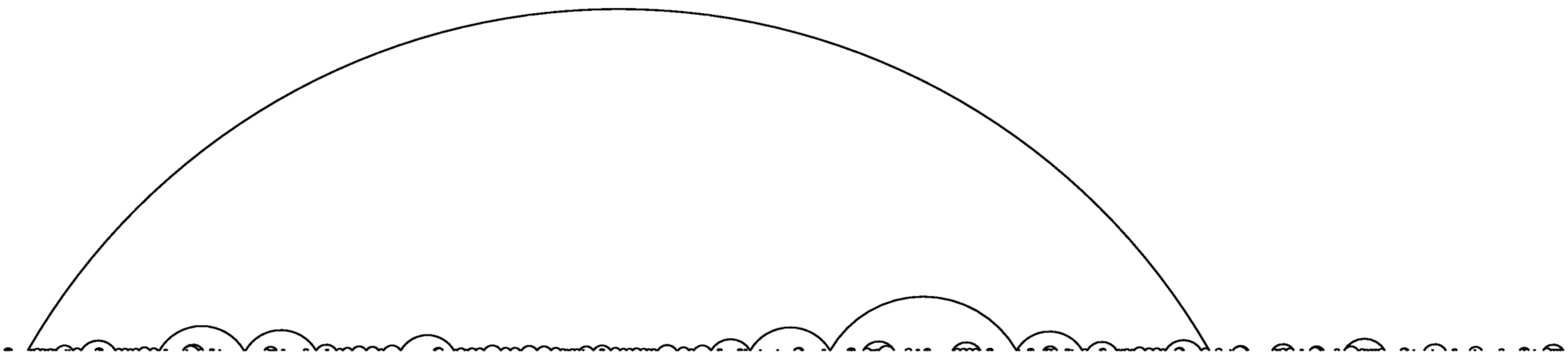


Central question: is there an infinite component?

Theorem (Deijfen, Häggstrom, H., 2010)

- Yes if $d \geq 2$ and $P(D > k(d)) = 1$
- No if $P(D \in \{1,2\}) = 1$ with $P(D=1) > 0$.

Basic case: R^1, D=2:



Is there an infinite path?

not rigorously known in the case, but…

Theorem (Deijfen, H., Peres, 2011)

R^1, D=2. For a certain event A_L, defined in terms of Poisson process on the *finite* interval [0,L],

if $P(A_L) > 0.97$ for some L, then

P(there is an infinite path) = 1.

Simulations support $P(A_{13000}) > 0.97$
at the 99.9999% confidence level

(subject to trusting the software and the pseudo-random number generator)

Possible (bold) conjectures:

For i.i.d. degrees D on R,

Infinite component $\Leftrightarrow$ P(D even) = 1

For degree D=2 on R^d,

Infinite component $\Leftrightarrow$ d=1 or d≥3

Theorem (Deijfen, Lopes, 2012):

Two-colour version with D=2 on R has no infinite component.

Open problems

Non-crossing 2-colour matching in R^2?

Better bounds for 2-colour stable matching?
e.g. d=2: $E^* X = \infty$ vs. $P^*(X > r) < C\, r^{-0.496\ldots}$

Nicer 2-colour matching with exponential tails
in $d \geq 3$?

Infinite component for stable Poisson graph
in R with degree D=2?

Stable multi-colour matching....?