Monotonicity-based regularization of inverse coefficient problems

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EIT and the Calderón problem

Calderón problem

Can we recover $\sigma \in L^\infty_+(\Omega)$ in

$$\nabla \cdot (\sigma \nabla u) = 0, \quad x \in \Omega \qquad (1)$$

from all possible Dirichlet and Neumann boundary values

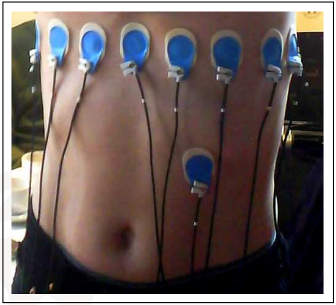
$$\{(u|_{\partial\Omega}, \sigma \partial_\nu u|_{\partial\Omega}) \quad : \quad u \text{ solves } (1)\}\,?$$

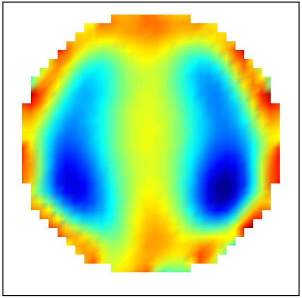
Equivalent: Recover σ from **Neumann-to-Dirichlet-Operator**

$$\Lambda(\sigma) : \; L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves (1) with $\sigma \partial_\nu u|_{\partial\Omega} = g$.

Application: Electrical impedance tomography (EIT)





- Apply electric currents on subject's boundary
- Measure necessary voltages

⇝ Reconstruct conductivity inside subject.

Generic approaches

Recover σ from Neumann-to-Dirichlet-Operator

$$\Lambda(\sigma):\ L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves (1) with $\sigma \partial_\nu u|_{\partial\Omega} = g$.

- Linearize and regularize:

  $$\Lambda_{\text{meas}} \approx \Lambda(\sigma) \approx \Lambda(\sigma_0) + \Lambda'(\sigma_0)(\sigma - \sigma_0).$$

  (σ_0: Initial guess or reference state e.g. exhaled state)

  $\rightsquigarrow$ Regularize linearized problem (& repeat for Newton-type algorithm.)

- Regularize and linearize:
  Consider non-linear Tikhonov functional, e.g.,

  $$\|\Lambda_{\text{meas}} - \Lambda(\sigma)\|^2 + \alpha \|\sigma - \sigma_0\|^2 \to \text{min!}$$

  and minimize by linearization (e.g., gradient-based or Newton-type methods)

Generic approaches

Advantages of generic optimization-based solvers:

- Very flexible, additional data/unknowns easily incorporated
- Problem-specific regularization can be applied
  (e.g., total variation penalization, stochastic priors, learning-based techniques, ...)

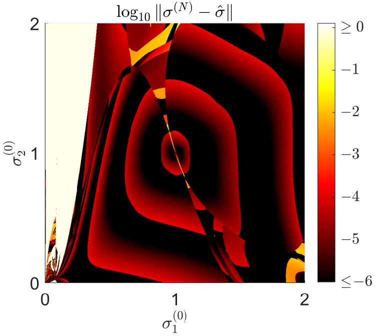
Problems with generic optimization-based solvers

- High computational cost (real-time imaging?)
  - Evaluations of $\Lambda(\cdot)$ and $\Lambda'(\cdot)$ require PDE solutions.
  - Iterative approaches typically require many evaluations
- Global convergence? Resolution?
  - Newton-type approaches highly depend on initial guess
  - Convergence of nonlinear Tikhonov requires global minimization
  - Resolution estimates & stability for realistic noise?

Problem of non-linearity / local convergence

Error of standard solver (Matlab's `lsqnonlin`) w.r.t. initial value:

(for simple 2D Calderón problem with 2 unknowns and 3 measurements)



Can we develop fast and globally convergent algorithms?

This talk: Fast and globally convergent method for inclusion detection

Monotonicity-based methods

Monotonicity w.r.t. Loewner order

For two conductivities $\sigma_0, \sigma_1 \in L^\infty(\Omega)$:

$$\sigma_0 \le \sigma_1 \quad \Longrightarrow \quad \Lambda(\sigma_0) \ge \Lambda(\sigma_1)$$

This follows from *(Kang/Seo/Sheen 1997, Ikehata 1998)*

$$\int_\Omega (\sigma_1 - \sigma_0)|\nabla u_0|^2 \ge \int_{\partial\Omega} g\left(\Lambda(\sigma_0) - \Lambda(\sigma_1)\right) g \ge \int_\Omega \frac{\sigma_0}{\sigma_1}(\sigma_1 - \sigma_0)|\nabla u_0|^2$$

where $u_0 \in H^1_\diamond(\Omega)$ solves

$$\nabla \cdot (\sigma_0 \nabla u_0) = 0, \quad \sigma_0 \partial_\nu u_0|_{\partial\Omega} = g.$$

Converse monotonicity relations can be shown by controlling $|\nabla u_0|^2$

*(Localized Potentials: **H.**, Inverse Probl. Imaging 2008)*

Theoretical consequences

Monotonicity & localized potentials yield uniqueness results:

- Non-linear Calderón problem: (Kohn/Vogelius 1985, **H.**/Seo 2010)
  If $\sigma_1 \in L^\infty_+(\Omega)$ fulfills (UCP) and $\sigma_2 - \sigma_1$ is pcw. analytic then

  $$\Lambda(\sigma_1) = \Lambda(\sigma_2) \quad \text{implies} \quad \sigma_1 = \sigma_2.$$

- Linearized Calderón problem: (**H.**/Seo 2010)
  If $\sigma_1 \in L^\infty_+(\Omega)$ fulfills (UCP) and $\kappa \in L^\infty(\Omega)$ is pcw. analytic then

  $$\Lambda'(\sigma_1)\kappa = 0 \quad \text{implies} \quad \kappa = 0.$$

- Calderón problem with finitely many measurements:
  (Linearized: Lechleiter/Rieder 2008, Non-linear: **H.** 2019)
  Using sufficiently many electrodes (CEM) uniquely determines conductivity up to desired finite resolution (and Lipschitz stability holds).

Monotonicity method for inclusion detection

Simple inclusion detection problem (for ease of presentation)

- $\sigma_0 = 1$
- $\sigma_1 = 1 + \chi_D$
- D open, $\overline{D} \subseteq \Omega$, $\Omega \setminus \overline{D}$ connected

All of the following also holds for

- σ_0 pcw. analytic and known,
- $\sigma_1 = \sigma_0 + \kappa\chi_D$ with $\kappa \in L^\infty_+(D)$,
- in any dimension $n \geq 2$,
- for partial boundary data on open subset $\Gamma \subseteq \partial\Omega$.

Monotonicity method for inclusion detection

H./Ullrich, SIAM J. Math. Anal. 2013:

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(1+\chi_B) \geq \Lambda(\sigma)$$
$$\Longleftrightarrow \quad \Lambda(1) + \frac{1}{2}\Lambda'(1)\chi_B \geq \Lambda(\sigma)$$

- Yields theoretical uniqueness for inclusion detection
- Rigorously detects unknown shape for exact data
- Fast and simple, no PDE solutions! (Precalculate $\Lambda(1)$ and $\Lambda'(1)$)
- Convergence for noisy data $\Lambda^\delta_{\text{meas}} \to \Lambda(\sigma) - \Lambda(1)$:

$$R(\Lambda^\delta_{\text{meas}}, \delta, B) := \begin{cases} 1 & \text{if } \frac{1}{2}\Lambda'(1)\chi_B \geq \Lambda^\delta_{\text{meas}} - \delta I \\ 0 & \text{else.} \end{cases}$$

Then $R(\Lambda^\delta_{\text{meas}}, \delta, B) \to 1$ iff $B \subseteq D$.

Pixel-based implementation

Quantitative, pixel-based variant of monotonicity method:

- Pixel partition $\Omega = \bigcup_{k=1}^{m} P_k$
- Quantitative monotonicity tests

$$\beta_k \in [0,\infty] \text{ max. values s.t. } \beta_k \Lambda'(1)\chi_{P_k} \geq \Lambda(\sigma) - \Lambda(1)$$

$$\beta_k^\delta \in [0,\infty] \text{ max. values s.t. } \beta_k^\delta \Lambda'(1)\chi_{P_k} \geq \Lambda_{\text{meas}}^\delta - \delta I$$

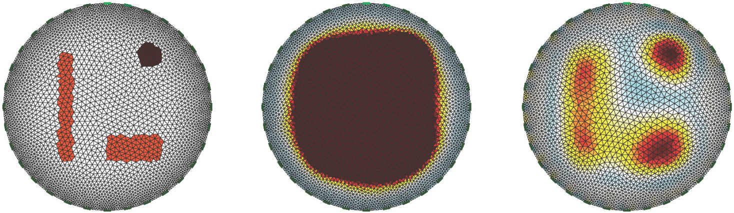
"Raise conductivity in each pixel until monotonicity test fails."

- By theory of monotonicity method:

$$\beta_k^\delta \to \beta_k \quad \text{and} \quad \beta_k \text{ fulfills} \begin{cases} \beta_k = 0 & \text{if } P_k \nsubseteq D \\ \beta_k \geq \frac{1}{2} & \text{if } P_k \subseteq D \end{cases}$$

Plotting β_k^δ shows true inclusions up to pixel partition.

Realistic example (32 electrodes, 1% noise)



True image

Monotonicity method

Standard linearized method

- Monotonicity method rigorously converges for $\delta \to 0 \ldots$
- ...but the heuristic standard linearized method works much better for realistic scenarios.

Can we improve the monotonicity method without loosing convergence?

Monotonicity-based regularization

Monotonicity-based regularization

- Standard linearized methods for EIT: Minimize

$$\|\Lambda'(1)\kappa - (\Lambda(\sigma) - \Lambda(1))\|^2 + \alpha\|\kappa\|^2 \to \text{min!}$$

Choice of norms heuristic. No convergence theory!

- Monotonicity-based regularization: Minimize

$$\|\Lambda'(1)\kappa - (\Lambda(\sigma) - \Lambda(1))\|_{\mathrm{F}} \to \text{min!}$$

under the constraint $\kappa|_{P_k} = \text{const.}$, $0 \leq \kappa|_{P_k} \leq \min\{\frac{1}{2}, \beta_k\}$.

($\|\cdot\|_F$: Frobenius norm of Galerkin projektion to finite-dimensional space)

Theorem (**H.**/Mach, Inverse Problems 2016)

- There exists unique minimizer $\hat{\kappa}$ and

$$P_k \subseteq \operatorname{supp} \hat{\kappa} \quad \Longleftrightarrow \quad P_k \subseteq \operatorname{supp}(\sigma - 1).$$

- Minimizer fulfills $\hat{\kappa} = \sum_{k=1}^{m} \min\{1/2, \beta_k\} \chi_{P_k}$

Monotonicity-based regularization

For noisy measurements $\Lambda^\delta_{\text{meas}} \approx \Lambda(\sigma) - \Lambda(1)$:

- Use regularized monotonicity tests

$$\beta_k^\delta \in [0,\infty] \text{ max. values s.t. } \beta_k^\delta \Lambda'(1)\chi_{P_k} \geq \Lambda^\delta_{\text{meas}} - \delta I$$

($\delta > 0$: noise level in $\mathcal{L}(L^2_\diamond(\partial\Omega))$-norm)

- Minimize

$$\|\Lambda'(1)\kappa^\delta - \Lambda^\delta_{\text{meas}}\|_{\text{F}} \to \text{min!}$$

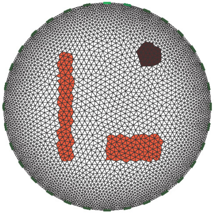
under the constraint $\kappa^\delta|_{P_k} = \text{const.}$, $0 \leq \kappa^\delta|_{P_k} \leq \min\{\frac{1}{2}, \beta_k^\delta\}$.

Theorem (**H.**/Mach, Inverse Problems 2016)

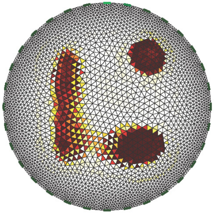
- There exist minimizers κ^δ and $\kappa^\delta \to \hat{\kappa}$ for $\delta \to 0$.

Monotonicity-regularized solutions converge against correct shape.

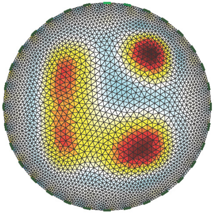
Realistic example (32 electrodes, 1% noise)



True image



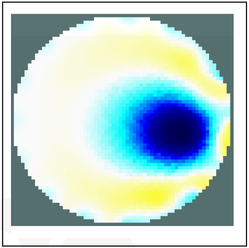
Monotonicity regularized method



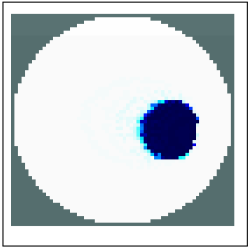
Standard linearized method

- Monotonicity regularized method rigorously converges and is up to par with (outperforms?) heuristic standard linearized method.

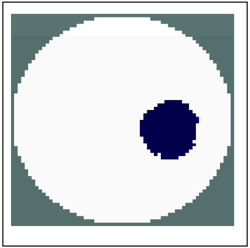
Phantom data example



standard



monoton.-regularized
(Matlab quadprog)



monoton.-regularized
(cvx package)

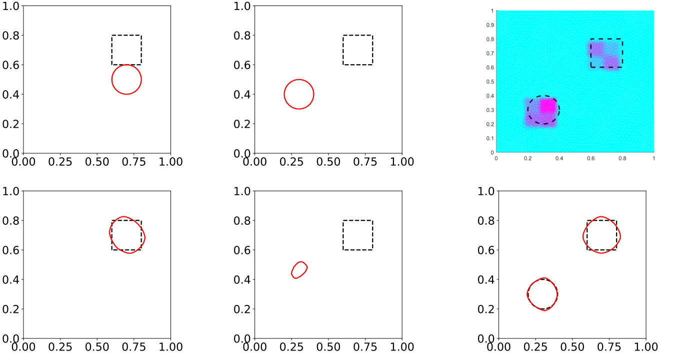
Monotonicity-regularization vs. community standard

(**H.**/Mach, Trends Math. 2018)

- EIDORS: `http://eidors3d.sourceforge.net` (Adler/Lionheart)
- EIDORS standard solver: linearized method with Tikhonov regularization
- Dataset: `iirc_data_2006` (Woo et al.): 2cm insulated inclusion in 20cm tank
  - using interpolated data on active electrodes (**H.**, Inverse Problems 2015)

Extensions and related recent results

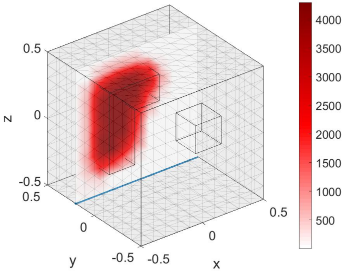
Monotonicity-based globalization of level-set methods

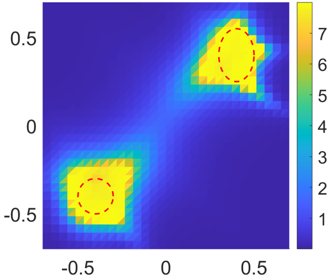


good initial guess | bad initial guess | monotonicity-based initial guess

- Monotonicity-based initialization yields faster & globally convergent level-set method (**H.**/Meftahi, arXiv:2501.15887)

Elasticity and Helmholtz equation





Recent significant extensions:

- Eberle/**H.**, *Comput. Mech.* 2022:
  Monotonicity-bas. regularization for elasticity (two Lamé parameters)
- Eberle/**H.**/Wang, 2025:
  Monotonicity-bas. regularization for Helmholtz (coercive + compact)

Lemma.

$$\int_{\partial\Omega} g(\Lambda(\sigma_1)-\Lambda(\sigma_2))g \, \mathrm{d}s \geq \int_\Omega (\sigma_2-\sigma_1)|\nabla u_{\sigma_2}^g|^2 \, \mathrm{d}x$$
$$= \int_{\partial\Omega} g\Lambda'(\sigma_2)(\sigma_1-\sigma_2)g \, \mathrm{d}s.$$

for all $\sigma_1, \sigma_2 \in L^\infty_+(\Omega)$, $g \in L^2_\diamond(\partial\Omega)$.

- $\rightsquigarrow$ For all $\sigma_1, \sigma_2 \in L^\infty_+(\Omega)$: $\Lambda(\sigma_1)-\Lambda(\sigma_2) \geq \Lambda'(\sigma_2)(\sigma_1-\sigma_2)$.
- $\rightsquigarrow$ Convexity: For all $\sigma_1, \sigma_2 \in L^\infty_+(\Omega)$, $t \in [0,1]$

$$\Lambda((1-t)\sigma_1 + t\sigma_2) \leq (1-t)\Lambda(\sigma_1) + t\Lambda(\sigma_2).$$

The "monotonicity lemma" also implies convexity.

$\rightsquigarrow$ Convex reformulation of Calderón problem (**H.**, SIAM J. Math. Anal. 2023)

Conclusions

Inverse coeff. problems such as EIT are highly ill-posed & non-linear.

- Global convergence of generic solvers seems out-of-reach.
- Often heuristic regularization without theor. justification is used.

Monotonicity and localized potentials yield

- theoretical uniqueness results,
- globally convergent inclusion detection methods,
- rigorous regularizers for noise-stable data fitting methods.

Monotonicity-based approaches

- work for partial boundary data, independently of dimension,
- extended to many other inverse elliptic PDE problems,
- can globalize iterative methods,
- connect inverse coeff. problems to convex optimization (SDP).